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REGULARITY OF COMMUTATORS OF MULTILINEAR MAXIMAL OPERATORS WITH LIPSCHITZ SYMBOLS

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Abstract. We study the regularity properties for commutators of multilinear fractional maximal operators. More precisely, let $m \geqslant 1$, $0 \leqslant \alpha < mn$ and $\vec{b} = (b_1, \ldots, b_m)$ with each b_i belonging to the Lipschitz space $\mathrm{Lip}(\mathbb{R})$, we denote by $[\vec{b}, \mathfrak{M}_\alpha]$ (resp., $\mathfrak{M}_{\alpha,\vec{b}}$) the commutator of the multilinear fractional maximal operator $\mathfrak{M}_\alpha$ with $\vec{b}$ (resp., the multilinear fractional maximal commutators). When $\alpha = 0$, we denote $[\vec{b}, \mathfrak{M}_\alpha] = [\vec{b}, \mathfrak{M}]$ and $\mathfrak{M}_{\alpha,\vec{b}} = \mathfrak{M}_{\vec{b}}$. We show that for $0 < s < 1$, $1 < p_1, \ldots, p_m, p, q < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$, both $[\vec{b}, \mathfrak{M}]$ and $\mathfrak{M}_{\vec{b}}$ are bounded and continuous from $W^{s,p_1}(\mathbb{R}^n) \times \cdots \times W^{s,p_m}(\mathbb{R}^n)$ to $W^{s,p}(\mathbb{R}^n)$, from $F_s^{p_1,q}(\mathbb{R}^n) \times \cdots \times F_s^{p_m,q}(\mathbb{R}^n)$ to $F_s^{p,q}(\mathbb{R}^n)$ and from $B_s^{p_1,q}(\mathbb{R}^n) \times \cdots \times B_s^{p_m,q}(\mathbb{R}^n)$ to $B_s^{p,q}(\mathbb{R}^n)$. It was also shown that for $0 \leqslant \alpha < mn$, $1 < p_1, \ldots, p_m, q < \infty$ and $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$, both $[\vec{b}, \mathfrak{M}]$ and $\mathfrak{M}_{\vec{b}}$ are bounded from $W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n)$ to $W^{1,q}(\mathbb{R}^n)$.

1. Introduction

The primary purpose of this work is to investigate the regularity and continuity for commutators of multilinear fractional maximal operators on the Sobolev spaces, Triebel-Lizorkin spaces and Besov spaces. Let us recall some definitions.

DEFINITION 1. (Commutators of multilinear fractional maximal operators) Let $m \geqslant 1$, $0 \leqslant \alpha < mn$ and $\vec{b} = (b_1, \ldots, b_m)$ with each $b_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For $\vec{f} = (f_1, \ldots, f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the multilinear fractional maximal operator $\mathfrak{M}_\alpha$ is defined as

$$\mathfrak{M}_\alpha(\vec{f})(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{m-\alpha/n}} \prod_{j=1}^m \int_{B(x,r)} |f_j(y_j)| \mathrm{d}y_j, \quad x \in \mathbb{R}^n,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at x with radius r, and $|B(x,r)|$ is the volume of $B(x,r)$. The commutator of $\mathfrak{M}_\alpha$ and $\vec{b}$ is given by the formula

$$[\vec{b}, \mathfrak{M}_\alpha](\vec{f})(x) = \sum_{i=1}^m [\vec{b}, \mathfrak{M}_\alpha]_i(\vec{f})(x), \quad x \in \mathbb{R}^n,$$

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where

$$[\vec{b},\mathfrak{M}_\alpha]_i(\vec{f})(x) = b_i(x)\mathfrak{M}_\alpha(\vec{f})(x) - \mathfrak{M}_\alpha(f_1,\dots,f_{i-1},b_i f_i, f_{i+1},\dots,f_m)(x).$$

The multilinear fractional maximal commutator with $\vec{b}$ is defined by

$$\mathfrak{M}_{\alpha,\vec{b}}(\vec{f})(x) = \sum_{i=1}^{m} \mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})(x),$$

where

$$\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{m-\alpha/n}} \int_{B(x,r)^m} |b_i(x)-b_i(y_i)| \prod_{j=1}^{m} |f_j(y_j)| d\vec{y},$$

where $B(x,r)^m = \overbrace{B(x,r)\times\cdots\times B(x,r)}^{m}$ and $d\vec{y} = dy_1\cdots dy_m$.

When $\alpha = 0$, the operator $\mathfrak{M}_\alpha$ reduces to the usual multilinear maximal operator $\mathfrak{M}$, then $[\vec{b},\mathfrak{M}_\alpha]$ (resp., $\mathfrak{M}_{\alpha,\vec{b}}$) becomes the commutator of multilinear maximal operator $[\vec{b},\mathfrak{M}]$ (resp., multilinear maximal commutator $\mathfrak{M}_{\vec{b}}$). We also denote $\mathfrak{M}^i_{\alpha,\vec{b}} = \mathfrak{M}^i_{\vec{b}}$ for $\alpha = 0$ and $1 \leqslant i \leqslant m$. For the sake of simplicity, we denote $\mathfrak{M}_\alpha = \mathscr{M}_\alpha$, $[\vec{b},\mathfrak{M}_\alpha] = [b,\mathscr{M}_\alpha]$ and $\mathfrak{M}_{\alpha,\vec{b}} = \mathscr{M}_{\alpha,b}$ when $m = 1$. When $\alpha = 0$, we denote $\mathscr{M}_\alpha = \mathscr{M}$, $[\vec{b},\mathscr{M}_\alpha] = [b,\mathscr{M}]$ and $\mathscr{M}_{\alpha,b} = \mathscr{M}_b$. Clearly, the operator $\mathscr{M}$ is the usual centered Hardy-Littlewood maximal operator. The operator $[b,\mathscr{M}]$ (resp., $\mathscr{M}_b$) is the commutator of Hardy-Littlewood maximal operator (resp., maximal commutator).

The regularity theory of maximal operators has been an active topic of current research. The first work related to Sobolev regularity was due to Kinnunen [11] who established the boundedness of $\mathscr{M}$ on the first order Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leqslant \infty$, where

$$W^{1,p}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} :\ \|f\|_{W^{1,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\nabla f = (D_1 f,\dots,D_n f)$ is the weak gradient of f. Since then, Kinnunen's result was extended to various variants. For example, see [12] for the local case, [13] for the fractional case and [4, 19] for the multilinear case. Since we do not have sublinearity for the weak derivative of maximal operators, the continuity of $\mathscr{M} : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ is certainly a nontrivial issue, which was addressed in the affirmative by Luiro [24] and was later extended to a local version in [25] and a multilinear version in [4, 17]. Another way to extend the regularity theory of maximal operators is to study its behaviour on other smooth function spaces. Korry [14] firstly proved that $\mathscr{M}$ is bounded on the inhomogeneous Triebel-Lizorkin spaces $F^{p,q}_s(\mathbb{R}^n)$ and inhomogeneous Besov spaces $B^{p,q}_s(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p,q < \infty$. As an immediate result, we have that $\mathscr{M}$ is bounded on the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ for $0 < s < 1$ and $1 < p < \infty$ (see also [15]). Here $W^{s,p}(\mathbb{R}^n)$ is defined by the Bessel potentials and $F^{p,2}_s(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ for all $0 < s < 1$ and $1 < p < \infty$. In 2010, Luiro [25] established the continuity of $\mathscr{M}: F_s^{p,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ for $0<s<1$ and $1<p,q<\infty$. Later on, Liu and Wu [20] extended the above results to the maximal operators associated with polynomial mappings. Moreover, they obtained that $\mathscr{M}: B_s^{p,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ for $0<s<1$ and $1<p,q<\infty$. Other interesting works can be found in [1, 2, 3, 5, 9, 10].

We now formulate partial results of [17, 19].

THEOREM A. ([17, 19]) *Let $0 \leqslant \alpha < mn$, $1 < p_1,\dots,p_m,q<\infty$ and $1/q = 1/p_1+\cdots+1/p_m-\alpha/n$. Then $\mathfrak{M}_\alpha$ is bounded and continuous from $W^{1,p_1}(\mathbb{R}^n)\times \cdots \times W^{1,p_m}(\mathbb{R}^n)$ to $W^{1,q}(\mathbb{R}^n)$. Moreover, if $\vec{f}=(f_1,\dots,f_m)$ with each $f_i \in W^{1,p_i}(\mathbb{R}^n)$, then*

$$\|\mathfrak{M}_\alpha(\vec{f})\|_{W^{1,q}(\mathbb{R}^n)} \lesssim_{\alpha,m,n,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}.$$

It should be pointed out that Theorem A is based on the well known Lebesgue boundedness and continuity for $\mathfrak{M}_\alpha$. To be more precise, it was known that

$$\mathfrak{M}_\alpha : L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \tag{1}$$

is continuous and

$$\|\mathfrak{M}_\alpha(\vec{f})\|_{L^q(\mathbb{R}^n)} \lesssim_{\alpha,m,n,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}, \tag{2}$$

for $0\leqslant \alpha<mn$, $1<p_1,\dots,p_m<\infty$, $1\leqslant q<\infty$ and $1/q=1/p_1+\cdots+1/p_m-\alpha/n$. It is worth mentioning that the authors in [22] established the boundedness and continuity for the multilinear strong maximal operators on the Triebel-Lizorkin spaces and Besov spaces. Using similar arguments, we can obtain the following results. Here we only list these results without proofs, which are useful for our aim.

THEOREM B. *Let $1<p_1,\dots,p_m,p,q<\infty$, $0<s<1$ and $1/p=1/p_1+\cdots+1/p_m$. Then*

(i) *The map $\mathfrak{M}: F_s^{p_1,q}(\mathbb{R}^n)\times\cdots\times F_s^{p_m,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ is bounded and continuous. Moreover, if $\vec{f}=(f_1,\dots,f_m)$ with each $f_i\in F_s^{p_i,q}(\mathbb{R}^n)$, then*

$$\|\mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \lesssim_{m,n,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}.$$

(ii) *The map $\mathfrak{M}: B_s^{p_1,q}(\mathbb{R}^n)\times\cdots\times B_s^{p_m,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ is bounded and continuous. Moreover, if $\vec{f}=(f_1,\dots,f_m)$ with each $f_i\in B_s^{p_i,q}(\mathbb{R}^n)$, then*

$$\|\mathfrak{M}(\vec{f})\|_{B_s^{p,q}(\mathbb{R}^n)} \lesssim_{m,n,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}.$$

On the other hand, the regularity properties of the commutators of maximal operators have been studied by many authors. The first work in this direction was due to Liu et al. who [23] firstly investigated the regularity and continuity of commutators of Hardy-Littlewood maximal operators on the Sobolev spaces, Triebel-Lizorkin spaces and Besov spaces. Later on, the above Sobolev regularity results were extended to the fractional version by Liu and Xi in [21]. Very recently, Liu and Wang [18] studied the Sobolev regularity properties of the commutators of Hardy-Littlewood maximal operator and its fractional version with Lipschitz symbols. We now introduce the Lipschitz space.

DEFINITION 2. The *homogeneous* Lipschitz space $Lip(\mathbb{R}^n)$ is defined by

$$Lip(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \text{ continuous}: \ \|f\|_{Lip(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{Lip(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f(x+h)-f(x)|}{|h|} < \infty.$$

The *inhomogeneous* Lipschitz space $\mathrm{Lip}(\mathbb{R}^n)$ is given by

$$\mathrm{Lip}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \text{ continuous}: \ \|f\|_{\mathrm{Lip}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{Lip(\mathbb{R}^n)} < \infty.$$

REMARK 1. It was shown in [18] that if $b \in \mathrm{Lip}(\mathbb{R}^n)$, then the weak partial derivatives D_ib, $i = 1,\dots,n$, exist almost everywhere. Moreover, we have

$$D_ib(x) = \lim_{h\to 0} \frac{b(x+he_i)-b(x)}{h}$$

and

$$|D_ib(x)| \leqslant \|b\|_{Lip(\mathbb{R}^n)}$$

for almost every $x \in \mathbb{R}^n$. Here $e_i = (0,\dots,0,1,0,\dots,0)$ is the canonical i-th base vector in $\mathbb{R}^n$ for $i = 1,\dots,n$.

The partial result in [18] can be listed as follows:

THEOREM C. ([18]) *Let $1 < p < \infty$, $0 \leqslant \alpha < n/p$ and $1/q = 1/p - \alpha/n$. If $b \in \mathrm{Lip}(\mathbb{R}^n)$, then $[b, \mathscr{M}_\alpha]$ is bounded and continuous from $W^{1,p}(\mathbb{R}^n)$ to $W^{1,q}(\mathbb{R}^n)$. The same boundedness hold for $\mathscr{M}_{\alpha,b}$.*

Based on the above, a natural question is the following

QUESTION 1.1. *Let $\vec{b} = (b_1,\dots,b_m)$ with each $b_j \in \mathrm{Lip}(\mathbb{R}^n)$. Are the commutators $[\vec{b}, \mathfrak{M}]$ and $\mathfrak{M}_{\vec{b}}$ bounded and continuous on the Sobolev spaces, Triebel-Lizorkin spaces or Besov spaces?*

The main motivation of this work is to address the above question. It is well known that the commutator in multilinear setting was first studied by Pérez and Torres [26] and was later developed by many authors (see [16] et al.). Particularly, the commutators of multilinear maximal opeators associated to cubes were first introduced by Zhang [29] who investigated the multiple weighted estimates for these commutators. Here we focus on the regularity properties of the above commutators. Before presenting our main results, let us point out the following comments, which are very useful in our proofs.

REMARK 2. (i) Let $0 \leqslant \alpha < mn$, $1 < p_1,\ldots,p_m < \infty$, $1 \leqslant q < \infty$ and $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$. Let us fix $i = 1,\ldots,m$ and $\vec{b} = (b_1,\ldots,b_m)$ with each $b_j \in L^\infty(\mathbb{R}^n)$. By (1) and (2), one has

$$[\vec{b},\mathfrak{M}_\alpha]_i : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \tag{3}$$

is bounded and continuous. Moreover,

$$\|[\vec{b},\mathfrak{M}_\alpha]_i(\vec{f})\|_{L^q(\mathbb{R}^n)} \lesssim_{\alpha,m,n,p_1,\ldots,p_m} \|b_i\|_{L^\infty(\mathbb{R}^n)} \prod_{j=1}^m \|f\|_{L^{p_j}(\mathbb{R}^n)}. \tag{4}$$

(ii) Let $0 \leqslant \alpha < mn$, $1 < p_1,\ldots,p_m, q < \infty$ and $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$. Let us fix $i = 1,\ldots,m$ and $\vec{b} = (b_1,\ldots,b_m)$ with each $b_j \in L^\infty(\mathbb{R}^n)$. One can easily check that

$$\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})(x) \leqslant |b_i(x)|\mathfrak{M}_\alpha(\vec{f})(x) + \mathfrak{M}_\alpha(f_1,\ldots,f_{i-1},b_i f_i, f_{i+1},\ldots,f_m)(x). \tag{5}$$

By (2) and (5), we obtain

$$\|\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)} \lesssim_{\alpha,n,p_1,\ldots,p_m} \|b_i\|_{L^\infty(\mathbb{R}^n)} \prod_{j=1}^m \|f\|_{L^{p_j}(\mathbb{R}^n)}. \tag{6}$$

One the other hand, one can easily check that

$$|\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f}_j) - \mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})| \leqslant \sum_{l=1}^m \mathfrak{M}^i_{\alpha,\vec{b}}(\vec{F}_l),$$

where $\vec{f}_j = (f_{1,j},\ldots,f_{m,j})$ and $\vec{F}_l = (f_1,\ldots,f_{l-1},f_{l,j}-f_l,f_{l+1,j},\ldots,f_{m,j})$. This together with (6) implies that

$$\mathfrak{M}^i_{\alpha,\vec{b}} : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \tag{7}$$

is continuous.

(iii) For $y \in \mathbb{R}^n$, we define $f_y(x) = f(x+y)$. Let $\vec{f} = (f_1,\ldots,f_m)$, $\vec{b} = (b_1,\ldots,b_m)$, $\vec{f}_y = ((f_1)_y,\ldots,(f_m)_y)$ and $\vec{b}_y = ((b_1)_y,\ldots,(b_m)_y)$. Clearly, $(\mathfrak{M}_\alpha(\vec{f}))_y(x) = \mathfrak{M}_\alpha(\vec{f}_y)(x)$ and $(\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f}))_y(x) = \mathfrak{M}^i_{\alpha,\vec{b}_y}(\vec{f}_y)(x)$ for all $i = 1,\ldots,m$.

The main results of this paper are the following.

THEOREM 1. *Let $0 \leqslant \alpha < mn$, $1 < p_1,\ldots,p_m < \infty$, $1 \leqslant q < \infty$ and $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$. Let $\vec{b} = (b_1,\ldots,b_m)$ with each $b_j \in \mathrm{Lip}(\mathbb{R}^n)$, then*

$$[\vec{b},\mathfrak{M}_\alpha] : W^{1,p_1}(\mathbb{R}^n) \times \cdots \times W^{1,p_m}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $\vec{f} = (f_1,\dots,f_m)$ *with each* $f_i \in W^{1,p_i}(\mathbb{R}^n)$*, then*

$$\|[\vec{b},\mathfrak{M}_\alpha](\vec{f})\|_{W^{1,q}(\mathbb{R}^n)} \lesssim_{\alpha,m,n,p_1,\dots,p_m} \Big(\sum_{i=1}^m \|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)}\Big) \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}. \tag{8}$$

The above boundedness result holds for $\mathfrak{M}_{\alpha,\vec{b}}$*.*

THEOREM 2. *Let* $1 < p_1,\dots,p_m,p,q < \infty$*,* $0 < s < 1$ *and* $1/p = 1/p_1 + \dots + 1/p_m$*. Let* $\vec{b} = (b_1,\dots,b_m)$ *with each* $b_j \in \mathrm{Lip}(\mathbb{R}^n)$*, then*

$$[\vec{b},\mathfrak{M}] : F_s^{p_1,q}(\mathbb{R}^n) \times \dots \times F_s^{p_m,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $\vec{f} = (f_1,\dots,f_m)$ *with each* $f_i \in F_s^{p_i,q}(\mathbb{R}^n)$*, then*

$$\|[\vec{b},\mathfrak{M}](\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \lesssim_{m,n,p_1,\dots,p_m} \Big(\sum_{i=1}^m \|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)}\Big) \prod_{j=1}^m \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{9}$$

The same conclusions hold for $\mathfrak{M}_{\vec{b}}$*.*

THEOREM 3. *Let* $1 < p_1,\dots,p_m,p,q < \infty$*,* $0 < s < 1$ *and* $1/p = 1/p_1 + \dots + 1/p_m$*. Let* $\vec{b} = (b_1,\dots,b_m)$ *with each* $b_j \in \mathrm{Lip}(\mathbb{R}^n)$*, then*

$$[\vec{b},\mathfrak{M}] : B_s^{p_1,q}(\mathbb{R}^n) \times \dots \times B_s^{p_m,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $\vec{f} = (f_1,\dots,f_m)$ *with each* $f_i \in B_s^{p_i,q}(\mathbb{R}^n)$*, then*

$$\|[\vec{b},\mathfrak{M}](\vec{f})\|_{B_s^{p,q}(\mathbb{R}^n)} \lesssim_{m,n,p_1,\dots,p_m} \Big(\sum_{i=1}^m \|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)}\Big) \prod_{j=1}^m \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \tag{10}$$

The same conclusions hold for $\mathfrak{M}_{\vec{b}}$*.*

By the facts $W^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $W^{s,p}(\mathbb{R}^n) = F_s^{p,2}(\mathbb{R}^n)$ for any $s > 0$ and $1 < p < \infty$ and Theorems 1 and 2, we have

COROLLARY 1. *Let* $1 < p_1,\dots,p_m,p < \infty$*,* $0 \leqslant s \leqslant 1$ *and* $1/p = 1/p_1 + \dots + 1/p_m$*. Let* $\vec{b} = (b_1,\dots,b_m)$ *with each* $b_j \in \mathrm{Lip}(\mathbb{R}^n)$*, then*

$$[\vec{b},\mathfrak{M}] : W^{s,p_1}(\mathbb{R}^n) \times \dots \times W^{s,p_m}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $\vec{f} = (f_1,\dots,f_m)$ *with each* $f_i \in W^{s,p_i}(\mathbb{R}^n)$*, then*

$$\|[\vec{b},\mathfrak{M}](\vec{f})\|_{W^{s,p}(\mathbb{R}^n)} \lesssim_{m,n,p_1,\dots,p_m} \Big(\sum_{i=1}^m \|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)}\Big) \prod_{j=1}^m \|f_j\|_{W^{s,p_j}(\mathbb{R}^n)}.$$

The same result holds for $\mathfrak{M}_{\vec{b}}$*.*

REMARK 3. (i) It is unknown whether the map $\mathfrak{M}_{\alpha,\vec{b}}: W^{1,p_1}(\mathbb{R}^n)\times\cdots\times W^{1,p_m}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ is continuous under the conditions in Theorem 1, which is interesting, even in the special case $m=1$ and $\alpha=0$.

(ii) Theorem 1 implies Theorem C when $m=1$.

(iii) Theorems 2 and 3 and Corollary 1 are new, even in the special case $m=1$.

This paper will be organized as follows. Section 2 will be devoted to presenting the proof of Theorem 1. In Section 3 we shall prove Theorem 2. The proof of Theorem 3 will be given in Section 4. We would like to remark that the main ideas in the proofs of Theorems are motivated by [20, 22, 28].

Throughout the paper, the letter C or c, sometimes with certain parameters, will stand for positive constants not necessarily the same one at each occurrence, but are independent of the essential variables. If there exists a constant $c>0$ depending only on ϑ such that $A\leqslant cB$, we then write $A\lesssim_\vartheta B$ or $B\gtrsim_\vartheta A$; and if $A\lesssim_\vartheta$ and $B\lesssim_\vartheta A$, we then write $A\sim_\vartheta B$. In what follows, let $\mathfrak{R}_n=\{\zeta\in\mathbb{R}^n; 1/2<|\zeta|\leqslant 1\}$ and we denote by $\Delta_\zeta(f)$ the difference of f for an arbitrary function f defined on $\mathbb{R}^n$ and $\zeta\in\mathfrak{R}_n$, i.e., $\Delta_\zeta f(x)=f(x+\zeta)-f(x)$.

2. Proof of boundedness and continuity on Sobolev spaces

This section is devoted to proving Theorem 1. Let us present some notations and lemma, which play key roles in the proof of Theorem 1. Let $e_l=(0,\dots,0,1,0,\dots,0)$ be the canonical l-th base vector in $\mathbb{R}^n$ for $l=1,\dots,n$. Let $1<p<\infty$ and $f\in L^p(\mathbb{R}^n)$. For all $h\in\mathbb{R}$ with $|h|>0$, $y\in\mathbb{R}^n$ and $i=1,\dots,n$, we define the functions f_h^i and f_y by setting

$$f_h^i(x)=\frac{f(x+he_i)-f(x)}{h}\quad\text{and}\quad f_y(x)=f(x+y).$$

It is well known that

$$\|f_h^i-D_if\|_{L^p(\mathbb{R}^n)}\to 0 \text{ as } h\to 0 \tag{11}$$

if $f\in W^{1,p}(\mathbb{R}^n)$. For convenience, we set

$$G(f;p)=\limsup_{|h|\to 0}\frac{\|f_h-f\|_{L^p(\mathbb{R}^n)}}{|h|}.$$

According to [7, Section 7.11], we have

$$u\in W^{1,q}(\mathbb{R}^n),\ \ 1<q<\infty \Longleftrightarrow u\in L^q(\mathbb{R}^n)\text{ and } G(u;q)<\infty. \tag{12}$$

We now present a characterization of the product of a function in $W^{1,p}(\mathbb{R}^n)$ and a function in $\mathrm{Lip}(\mathbb{R}^n)$. which was proved in [18].

LEMMA 1. ([18]) *Let $1<p<\infty$. If $f\in W^{1,p}(\mathbb{R}^n)$ and $b\in\mathrm{Lip}(\mathbb{R}^n)$, then $bf\in W^{1,p}(\mathbb{R}^n)$. Moreover,*

$$D_i(bf)=bD_if+fD_ib,\quad i=1,\dots,n,$$

almost everywhere in $\mathbb{R}^n$. *Consequently,*

$$\nabla(bf) = b\nabla f + f\nabla b,$$

almost everywhere in $\mathbb{R}^n$. *In particular, it holds that*

$$\|bf\|_{W^{1,p}(\mathbb{R}^n)} \leqslant \sqrt{n}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{W^{1,p}(\mathbb{R}^n)}.$$

Now we turn to present the proof of Theorem 1.

Proof of Theorem 1. In what follows, we let $0 \leqslant \alpha < mn$, $1 < p_1,\dots,p_m < \infty$, $1 \leqslant q < \infty$ and $1/q = 1/p_1 + \cdots + 1/p_m - \alpha/n$. Let $\vec{f} = (f_1,\dots,f_m)$ with each $f_i \in W^{1,p_i}(\mathbb{R}^n)$ and $\vec{b} = (b_1,\dots,b_m)$ with each $b_i \in \mathrm{Lip}(\mathbb{R}^n)$. We divide the proof into two steps:

Step 1: Proof of Theorem 1 *for* $[\vec{b},\mathfrak{M}_\alpha]$

By the definition of $[\vec{b},\mathfrak{M}_\alpha]$, to prove (8), it suffices to prove that

$$\|[\vec{b},\mathfrak{M}_\alpha]_i(\vec{f})\|_{W^{1,q}(\mathbb{R}^n)} \leqslant C\|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)} \tag{13}$$

for each $i = 1,\dots,m$. We now prove (13) for $i = 1$ and other cases are analogous. By Theorem A and invoking Lemma 1, we have

$$\begin{aligned}\|b_1\mathfrak{M}_\alpha(\vec{f})\|_{W^{1,q}(\mathbb{R}^n)} &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\|\mathfrak{M}_\alpha(\vec{f})\|_{W^{1,q}(\mathbb{R}^n)}\\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}.\end{aligned} \tag{14}$$

For convenience, we set $\vec{f}_{1,b_1} = (b_1 f_1, f_2,\dots,f_m)$. Invoking Lemma 1, we have that $b_1 f_1 \in W^{1,p_1}(\mathbb{R}^n)$ and

$$\|b_1 f_1\|_{W^{1,p_1}(\mathbb{R}^n)} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f_1\|_{W^{1,p_1}(\mathbb{R}^n)},$$

which together with Theorem A leads to

$$\|\mathfrak{M}_\alpha(\vec{f}_{1,b_1})\|_{W^{1,q}(\mathbb{R}^n)} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}. \tag{15}$$

Combining (15) with (14) leads to (13) with $i = 1$.

We now prove the continuity result. Let $\vec{f}_j = (f_{1,j},\dots,f_{m,j})$ with $f_{i,j} \to f_i$ in $W^{1,p_i}(\mathbb{R}^n)$ as $j \to \infty$ for all $i = 1,\dots,m$. We want to show that

$$\|[\vec{b},\mathfrak{M}_\alpha]_i(\vec{f}_j) - [\vec{b},\mathfrak{M}_\alpha]_i(\vec{f})\|_{W^{1,q}(\mathbb{R}^n)} \to 0 \ \text{ as } \ j \to \infty \tag{16}$$

for all $i = 1,\dots,m$. We only work with the case $i = 1$ and other cases are analogous. By Theorem A and Lemma 1, we have

$$\begin{aligned}&\|b_1\mathfrak{M}_\alpha(\vec{f}_j) - b_1\mathfrak{M}_\alpha(\vec{f})\|_{W^{1,q}(\mathbb{R}^n)}\\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\|\mathfrak{M}_\alpha(\vec{f}_j) - \mathfrak{M}_\alpha(\vec{f})\|_{W^{1,q}(\mathbb{R}^n)} \to 0 \ \text{ as} j \to \infty.\end{aligned} \tag{17}$$

By Lemma 1 again, we have that $b_1 f_1 \in W^{1,p_1}(\mathbb{R}^n)$, $b_1 f_{1,j} \in W^{1,p_1}(\mathbb{R}^n)$ and

$$\begin{aligned}\|b_1 f_{1,j} - b_1 f_1\|_{W^{1,p_1}(\mathbb{R}^n)} &= \|b_1(f_{1,j} - f_1)\|_{W^{1,p_1}(\mathbb{R}^n)} \\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f_{1,j} - f_1\|_{W^{1,p_1}(\mathbb{R}^n)} \to 0 \ \text{ as } j \to \infty,\end{aligned}$$

which together with Theorem A implies

$$\|\mathfrak{M}_\alpha(b_1 f_{1,j}, f_{2,j}, \dots, f_{m,j}) - \mathfrak{M}_\alpha(b_1 f_1, f_2, \dots, f_m)\|_{W^{1,q}(\mathbb{R}^n)} \to 0 \ \text{ as } j \to \infty.$$

This together with (17) implies (16) with $i = 1$.

Step 2: Proof of Theorem 1 for $\mathfrak{M}_{\alpha,\vec{b}}$

By the definition of $\mathfrak{M}_{\alpha,\vec{b}}$, it suffices to show that

$$\|\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})\|_{W^{1,q}(\mathbb{R}^n)} \leqslant C\|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)} \tag{18}$$

for each $i = 1, \dots, m$. By (6), to prove (18), it suffices to show that

$$\|\nabla\mathfrak{M}^i_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)} \leqslant C\|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{W^{1,p_j}(\mathbb{R}^n)} \tag{19}$$

for each $i = 1, \dots, m$. We only prove (19) for $i = 1$ and other cases are analogous. Fix $y \in \mathbb{R}^n$. By Remark 2 (iii), we have

$$\begin{aligned}
&|(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}))_y(x) - \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)| \\
&= |\mathfrak{M}^1_{\alpha,\vec{b}_y}(\vec{f}_y)(x) - \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)| \\
&\leqslant \sup_{r>0} \frac{1}{|B(x,r)|^{m-\alpha/n}} \int_{B(x,r)^m} \Big| |(b_1)_y(x) - (b_1)_y(z_1)| \Big| \prod_{j=1}^m (f_j)_y(z_j) \Big| \\
&\quad - |b_1(x) - b_1(z_1)| \Big| \prod_{j=1}^m f_j(z_j) \Big| \Big| dz_1 dz_2 \cdots dz_m \\
&\leqslant \sup_{r>0} \frac{1}{|B(x,r)|^{m-\alpha/n}} \int_{B(x,r)^m} |(b_1)_y(x) - (b_1)_y(z_1) - b_1(x) + b_1(z_1)| \\
&\qquad \times \Big| \prod_{j=1}^m (f_j)_y(z_j) \Big| dz_1 dz_2 \cdots dz_m \\
&\quad + \sup_{r>0} \frac{1}{|B(x,r)|^{m-\alpha/n}} \int_{B(x,r)^m} |b_1(x) - b_1(z_1)| \\
&\qquad \times \Big| \prod_{j=1}^m (f_j)_y(z_j) - \prod_{j=1}^m f_j(z_j) \Big| dz_1 dz_2 \cdots dz_m.
\end{aligned} \tag{20}$$

Observe that

$$\prod_{j=1}^m (f_j)_y(z_j) - \prod_{j=1}^m f_j(z_j) = \sum_{l=1}^m ((f_l)_y(z_l) - f_l(z_l)) \Big(\prod_{\mu=1}^{l-1} f_\mu(z_\mu) \Big) \Big(\prod_{\nu=l+1}^m (f_\nu)_y(z_\nu) \Big),$$

which combine with (20) yields

$$
\begin{aligned}
&|(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}))_y(x)-\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})(x)| \\
&\leqslant |(b_1)_y(x)-b_1(x)|\mathfrak{M}_\alpha(\vec{f}_y)(x)+\mathfrak{M}_\alpha(\vec{f}_{1,b_1,y})(x)+\sum_{l=1}^m \mathfrak{M}^1_{\alpha,\vec{b}}(\vec{F}_{l,y})(x),
\end{aligned}
\tag{21}
$$

where

$$
\vec{f}_{1,b_1,y}=(((b_1)_y-b_1)(f_1)_y,(f_2)_y,\dots,(f_m)_y),
$$
$$
\vec{F}_{l,y}=(f_1,\dots,f_{l-1},(f_l)_y-f_l,(f_{l+1})_y,\dots,(f_m)_y).
$$

By (2), (6), (21), Hölder's inequality and Minkowski's inequality, we have

$$
\begin{aligned}
&\|(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}))_y-\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)} \\
&\leqslant \|((b_1)_y-b_1)\mathfrak{M}_\alpha(\vec{f}_y)\|_{L^q(\mathbb{R}^n)}+\|\mathfrak{M}_\alpha(\vec{f}_{1,b_1,y})\|_{L^q(\mathbb{R}^n)}+\sum_{l=1}^m\|\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{F}_{l,y})\|_{L^q(\mathbb{R}^n)} \\
&\leqslant C\|(b_1)_y-b_1\|_{L^\infty(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)} \\
&\quad +C\sum_{l=1}^m\|b_1\|_{L^\infty(\mathbb{R}^n)}\|(f_l)_y-f_l\|_{L^{p_l}(\mathbb{R}^n)}\prod_{\mu=1}^{l-1}\|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)}\prod_{\nu=l+1}^m\|(f_\nu)_y\|_{L^{p_\nu}(\mathbb{R}^n)}.
\end{aligned}
\tag{22}
$$

Since $f_j\in W^{1,p_j}(\mathbb{R}^n)$ for $1\leqslant j\leqslant m$, then by (12), we have that $G(f_j,p_j)<\infty$. Therefore, we get from (22) and the property of b_1 that

$$
\begin{aligned}
G(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f});q)&=\limsup_{|y|\to 0}\frac{\|(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}))_y-\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)}}{|y|} \\
&\leqslant C\|b_1\|_{Lip(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)} \\
&\quad +C\sum_{l=1}^m\|b_1\|_{L^\infty(\mathbb{R}^n)}G(f_l,p_l)\prod_{\substack{1\leqslant\mu\leqslant m\\ \mu\neq l}}\|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)}<\infty.
\end{aligned}
\tag{23}
$$

Combining (23), (12) and (8) lead to $\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})\in W^{1,q}(\mathbb{R}^n)$.

Fix $l\in\{1,\dots,n\}$. From (11) and (22) we see that

$$
\begin{aligned}
&\|D_l\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f})\|_{L^q(\mathbb{R}^n)} \\
&\leqslant \liminf_{h\to 0}\left\|(\mathfrak{M}^1_{\alpha,\vec{b}}(\vec{f}))^l_h\right\|_{L^q(\mathbb{R}^n)} \\
&\leqslant C\liminf_{h\to 0}\left\|(b_1)^l_h\right\|_{L^\infty(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^n)} \\
&\quad +C\sum_{j=1}^m\|b_1\|_{L^\infty(\mathbb{R}^n)}\liminf_{h\to 0}\left\|(f_j)^l_h\right\|_{L^{p_j}(\mathbb{R}^n)}\prod_{\substack{1\leqslant\mu\leqslant m\\ \mu\neq j}}\|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)}
\end{aligned}
$$

$$\leqslant C\|b_1\|_{Lip(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}+C\sum_{j=1}^{m}\|b_1\|_{L^\infty(\mathbb{R}^n)}\|D_l f_j\|_{L^{p_j}(\mathbb{R}^n)}\prod_{\substack{1\leqslant\mu\leqslant m\\ \mu\neq l}}\|f_\mu\|_{L^{p_\mu}(\mathbb{R}^n)}$$

$$\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{W^{1,p_j}(\mathbb{R}^n)}.$$

This gives (19) for $i=1$ and completes the proof of Theorem 1. □

3. Proof of boundedness and continuity on Triebel-Lizorkin spaces

In this section we shall prove Theorem 2. At first, let us introduce some properties of Triebel-Lizorkin spaces and lemmas, which are the main ingredients of proof.

3.1. Properties on Triebel-Lizorkin spaces and some lemmas

Let $\dot{F}_s^{p,q}(\mathbb{R}^n)$ be the homogeneous Triebel-Lizorkin spaces. Let $s>0$ and $1<p<\infty$, $1<q\leqslant\infty$, $1\leqslant r<\infty$. We denote by $E^s_{p,q,r}$ the mixed norm of three variable functions $g(x,k,\zeta)$ by

$$\|g\|_{E^s_{p,q,r}}:=\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|g(x,k,\zeta)|^r\mathrm{d}\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

It was shown by Yabuta [28] that

$$\|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)}\sim\|\Delta_{2^{-k}\zeta}f\|_{E^s_{p,q,r}},\ \text{for } 0<s<1,\ 1<p<\infty,\ 1<q\leqslant\infty,\ 1\leqslant r<\min\{p,q\}. \tag{24}$$

Moreover, it was pointed out in [6, 8, 27] that

$$\|f\|_{F_s^{p,q}(\mathbb{R}^n)}\sim\|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)}+\|f\|_{L^p(\mathbb{R}^n)},\ \text{ for } s>0,\ 1<p,q<\infty, \tag{25}$$

$$\|f\|_{F_{s_1}^{p,q}(\mathbb{R}^n)}\leqslant\|f\|_{F_{s_2}^{p,q}(\mathbb{R}^n)},\ \text{ for } s_1\leqslant s_2,\ 1<p,\ q<\infty, \tag{26}$$

$$\|f\|_{F_s^{p,q_2}(\mathbb{R}^n)}\leqslant\|f\|_{F_s^{p,q_1}(\mathbb{R}^n)},\quad\text{for } s\in\mathbb{R},\ 1<p<\infty,\ 1<q_1\leqslant q_2<\infty. \tag{27}$$

The following presents a characterization of the product of a function in $F_s^{p,q}(\mathbb{R}^n)$ and a function in $\mathrm{Lip}(\mathbb{R}^n)$.

LEMMA 2. *Let* $0<s<1$ *and* $1<p,q<\infty$. *If* $f\in F_s^{p,q}(\mathbb{R}^n)$ *and* $g\in\mathrm{Lip}(\mathbb{R}^n)$, *then* $fg\in F_s^{p,q}(\mathbb{R}^n)$. *Moreover,*

$$\|fg\|_{F_s^{p,q}(\mathbb{R}^n)}\lesssim_{s,q}\|g\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p,q}(\mathbb{R}^n)}. \tag{28}$$

Proof. Note that

$$\Delta_{2^{-k}\zeta}(fg)(x)=\Delta_{2^{-k}\zeta}f(x)\Delta_{2^{-k}\zeta}g(x)+g(x)\Delta_{2^{-k}\zeta}f(x)+f(x)\Delta_{2^{-k}\zeta}g(x), \tag{29}$$

for all $x\in\mathbb{R}^n$, $\zeta\in\mathfrak{R}_n$ and $k\in\mathbb{Z}$. Combining (29) with (24) and Minkowski's inequality implies that

$$\begin{aligned}\|fg\|_{\dot F_s^{p,q}(\mathbb{R}^n)} &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}(fg)|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}f\Delta_{2^{-k}\zeta}g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\quad+C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|g\Delta_{2^{-k}\zeta}f|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\quad+C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|f\Delta_{2^{-k}\zeta}g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\|g\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}f|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\quad+C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|f\Delta_{2^{-k}\zeta}g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\|g\|_{L^\infty(\mathbb{R}^n)}\|f\|_{\dot F_s^{p,q}(\mathbb{R}^n)}\\ &\quad+C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|f\Delta_{2^{-k}\zeta}g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Note that $0<s<1$. By Minkowski's inequality and the property of g, one has

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|f\Delta_{2^{-k}\zeta}g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant\Big\|\Big(\sum_{k=-\infty}^{0}2^{ksq}\Big(\int_{\mathfrak{R}_n}|f\Delta_{2^{-k}\zeta}g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\quad+\Big\|\Big(\sum_{k=1}^{\infty}2^{ksq}\Big(\int_{\mathfrak{R}_n}|f\Delta_{2^{-k}\zeta}g|\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\ &\leqslant\Big(2\|g\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0}2^{ksq}\Big)^{1/q}+\|g\|_{Lip(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty}2^{kq(s-1)}\Big)^{1/q}\Big)|\mathfrak{R}_n|\|f\|_{L^p(\mathbb{R}^n)}\\ &\leqslant 2\Big(\Big(\frac{1}{1-2^{-sq}}\Big)^{1/q}+\Big(\frac{1}{1-2^{-q(1-s)}}\Big)^{1/q}\Big)|\mathfrak{R}_n|\|g\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Therefore, we get from (25) that

$$\|fg\|_{\dot F_s^{p,q}(\mathbb{R}^n)}\leqslant C\|g\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p,q}(\mathbb{R}^n)}.$$

This together with (25) and the trivial estimate $\|fg\|_{L^p(\mathbb{R}^n)}\leqslant\|g\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}$ implies (28). □

The following result will play a key role in the proof of Theorem 2.

LEMMA 3. ([28]). *For any* $1<p,q,r<\infty$, *we have*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\|\mathscr{M}(f_{k,\zeta})\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\leqslant C_{p,q,r}\Big\|\Big(\sum_{k\in\mathbb{Z}}\|f_{k,\zeta}\|_{L^r(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

3.2. Proof of Theorem 2

In what follows, we fix $1 < p_1, \dots, p_m, p, q < \infty$, $0 < s < 1$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Let $\vec{b} = (b_1, \dots, b_m)$ with each $b_j \in \mathrm{Lip}(\mathbb{R}^n)$. The proof of Theorem 2 will be divided into two steps:

Step 1: Proof of Theorem 2 for $[\vec{b}, \mathfrak{M}]$

By Minkowski's inequality, to prove (9), it suffices to show that

$$\|[\vec{b}, \mathfrak{M}]_i(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)} \tag{30}$$

for each $i = 1, \dots, m$.

We only prove (30) for the case $i = 1$ and other cases are analogous. By Theorem B (i) and invoking Lemma 2, we have

$$\begin{aligned} \|b_1\mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \|\mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}, \end{aligned} \tag{31}$$

$$\begin{aligned} \|\mathfrak{M}(b_1 f_1, f_2, \dots, f_m)\|_{F_s^{p,q}(\mathbb{R}^n)} &\leqslant C\|b_1 f_1\|_{F_s^{p_1,q}(\mathbb{R}^n)} \prod_{j=2}^{m} \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)} \\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \end{aligned} \tag{32}$$

Then (30) with $i = 1$ follows from (31) and (32).

Let $\vec{f}_j = (f_{1,j}, \dots, f_{m,j})$ with each $f_{i,j} \to f_i$ in $F_s^{p_i,q}(\mathbb{R}^n)$ as $j \to \infty$ for all $i \in \{1, \dots, m\}$. It suffices to show that

$$\|[\vec{b}, \mathfrak{M}]_i(\vec{f}_j) - [\vec{b}, \mathfrak{M}]_i(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to \infty \tag{33}$$

for all $i = 1, \dots, m$. We only prove (33) for $i = 1$ since other cases can be proved similarly. Invoking Lemma 2, we have

$$\|b_1 f_{1,j} - b_1 f_1\|_{F_s^{p_1,q}(\mathbb{R}^n)} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \|f_{1,j} - f_1\|_{F_s^{p_1,q}(\mathbb{R}^n)},$$

which together with the continuity result in Theorem B (i) yields that

$$\|\mathfrak{M}(b_1 f_{1,j}, f_{2,j}, \dots, f_{m,j}) - \mathfrak{M}(b_1 f_1, f_2, \dots, f_m)\|_{F_s^{p,q}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to \infty. \tag{34}$$

On the other hand, by invoking Lemma 2 and Theorem B (i) again,

$$\|b_1\mathfrak{M}(\vec{f}_j) - b_1\mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \|\mathfrak{M}(\vec{f}_j) - \mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j \to \infty,$$

which together with (34) leads to (33) with $i = 1$.

Step 2: Proof of Theorem 2 for $\mathfrak{M}_{\vec{b}}$

At first, we shall prove the boundedness part. We want to show that

$$\|\mathfrak{M}_{\vec{b}}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\Big(\sum_{i=1}^{m} \|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)}\Big) \prod_{j=1}^{m} \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{35}$$

To prove (35), it suffices to prove that

$$\|\mathfrak{M}^i_{\vec{b}}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)} \tag{36}$$

for each $i = 1,\dots,m$.

Without loss of generality we only prove (36) for the case $i = 1$ and other cases are analogous. By (21) and (5), we have that, for any $(x,k,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n$,

$$\begin{aligned} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)| &\leqslant |\Delta_{2^{-k}\zeta} b_1(x)|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})(x) + \mathfrak{M}(\vec{f}_{1,b_1,2^{-k}\zeta})(x) \\ &\quad + |b_1(x)| \sum_{l=1}^m \mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})(x) + \sum_{l=1}^m \mathfrak{M}(\vec{G}_{b_1,l,2^{-k}\zeta})(x) \\ &=: \Gamma(x,k,\zeta), \end{aligned} \tag{37}$$

where

$$\vec{f}_{1,b_1,2^{-k}\zeta} = (\Delta_{2^{-k}\zeta} b_1 (f_1)_{2^{-k}\zeta}, (f_2)_{2^{-k}\zeta}, \dots, (f_m)_{2^{-k}\zeta}),$$

$$\vec{G}_{l,2^{-k}\zeta} = (f_1,\dots,f_{l-1}, \Delta_{2^{-k}\zeta} f_l, (f_{l+1})_{2^{-k}\zeta}, \dots, (f_m)_{2^{-k}\zeta}),$$

$$\vec{G}_{b_1,1,2^{-k}\zeta} = (b_1 \Delta_{2^{-k}\zeta} f_1, (f_2)_{2^{-k}\zeta}, \dots, (f_m)_{2^{-k}\zeta}),$$

$$\vec{G}_{b_1,l,2^{-k}\zeta} = (b_1 f_1, f_2, \dots, f_{l-1}, \Delta_{2^{-k}\zeta} f_l, (f_{l+1})_{2^{-k}\zeta}, \dots, (f_m)_{2^{-k}\zeta}), \quad l = 2,\dots,m.$$

In light of (24) and (37) we would have

$$\begin{aligned} \|\mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} b_1|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\quad + C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathfrak{M}(\vec{f}_{1,b_1,2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\quad + C\Big\||b_1|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \sum_{l=1}^m \mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\quad + C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \sum_{l=1}^m \mathfrak{M}(\vec{G}_{b_1,l,2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned} \tag{38}$$

By the property of b_1 and (2), we get

$$\begin{aligned} A_1 &\leqslant C\Big\|\Big(\sum_{k=-\infty}^0 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} b_1|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\quad + C\Big\|\Big(\sum_{k=1}^\infty 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} b_1|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C\Big(\|b_1\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^0 2^{ksq}\Big)^{1/q} \\ &\quad + \|b_1\|_{Lip(\mathbb{R}^n)}\Big(\sum_{k=1}^\infty 2^{-kq(1-s)}\Big)^{1/q}\Big)\|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^n)}. \end{aligned} \tag{39}$$

By Hölder's inequality, we get

$$\begin{aligned} A_2 &\leqslant \Big\| \prod_{j=2}^{m} \mathscr{M}((f_j)_{2^{-k}\zeta}) \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathscr{M}(\Delta_{2^{-k}\zeta} b_1 (f_1)_{2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C \prod_{j=2}^{m} \|\mathscr{M}((f_j)_{2^{-k}\zeta})\|_{L^{p_j}(\mathbb{R}^n)} \\ &\quad \times \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathscr{M}(\Delta_{2^{-k}\zeta} b_1 (f_1)_{2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^{p_1}(\mathbb{R}^n)}. \end{aligned} \tag{40}$$

By the property of b_1, we have

$$\begin{aligned} &\Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathscr{M}(\Delta_{2^{-k}\zeta} b_1 (f_1)_{2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^{p_1}(\mathbb{R}^n)} \\ &\leqslant \Big\| \Big(\sum_{k=-\infty}^{0} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathscr{M}(\Delta_{2^{-k}\zeta} b_1 (f_1)_{2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^{p_1}(\mathbb{R}^n)} \\ &\quad + \Big\| \Big(\sum_{k=1}^{\infty} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathscr{M}(\Delta_{2^{-k}\zeta} b_1 (f_1)_{2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^{p_1}(\mathbb{R}^n)} \\ &\leqslant C\|b_1\|_{L^\infty(\mathbb{R}^n)} \Big\| \Big(\sum_{k=-\infty}^{0} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathscr{M}((f_1)_{2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^{p_1}(\mathbb{R}^n)} \\ &\quad + C\|b_1\|_{Lip(\mathbb{R}^n)} \Big\| \Big(\sum_{k=1}^{\infty} 2^{ksq} \Big(\int_{\mathfrak{R}_n} 2^{-k}|\zeta| \mathscr{M}((f_1)_{2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^{p_1}(\mathbb{R}^n)} \\ &\leqslant C \Big(\|b_1\|_{L^\infty(\mathbb{R}^n)} \Big(\sum_{k=-\infty}^{0} 2^{ksq} \Big)^{1/q} \\ &\quad + \|b_1\|_{Lip(\mathbb{R}^n)} \Big(\sum_{k=1}^{\infty} 2^{-kq(1-s)} \Big)^{1/q} \Big) \|\mathscr{M}((f_1)_{2^{-k}\zeta})\|_{L^{p_1}(\mathbb{R}^n)} \\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \|f_1\|_{L^{p_1}(\mathbb{R}^n)}. \end{aligned} \tag{41}$$

It follows from (40) and (41) that

$$A_2 \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)}. \tag{42}$$

For A_3, by Minkowski's inequality, we get

$$A_3 \leqslant C \sum_{l=1}^{m} \|b_1\|_{L^\infty(\mathbb{R}^n)} \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} \mathfrak{M}(\vec{G}_{l,2^{-k}\zeta}) \mathrm{d}\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}. \tag{43}$$

Let us fix $l \in \{1,\dots,m\}$. Note that

$$\mathfrak{M}(\vec{G}_{l,2^{-k}\zeta}) \leqslant \sum_{\tau\subset E_l} \prod_{\mu\in\tau\cup\{l\}} \mathscr{M}(\Delta_{2^{-k}\zeta} f_\mu) \prod_{\nu\in\tau'} \mathscr{M}(f_\nu), \tag{44}$$

where $E_l = \{l+1,\dots,m\}$ and $\tau' = \{1,\dots,m\} \setminus (\tau\cup\{l\})$. Let α_τ be such that $1/\alpha_\tau = \sum_{\ell\in\tau} 1/p_\ell + 1/p_l$. It is clear that $p < \alpha_\tau < p_l$ and $1/p = 1/\alpha_\tau + \sum_{\kappa\in\tau'} 1/p_\kappa$. By (44), Minkowski's inequality, Hölder's inequality and the bounds for $\mathcal{M}$, we have

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant\sum_{\tau\subset E_l}\Big\|\prod_{\nu\in\tau'}\mathcal{M}(f_\nu)\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\mathcal{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\sum_{\tau\subset E_l}\prod_{\nu\in\tau'}\|\mathcal{M}(f_\nu)\|_{L^{p_\nu}(\mathbb{R}^n)}\\
&\quad\times\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\mathcal{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\leqslant C\sum_{\tau\subset E_l}\prod_{\nu\in\tau'}\|f_\nu\|_{L^{p_\nu}(\mathbb{R}^n)}\\
&\quad\times\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\mathcal{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}.
\end{aligned}\tag{45}$$

Fix $\tau\subset E_l$. By Hölder's inequality, Lemma 3 and (24)–(27), we have

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\prod_{\mu\in\tau\cup\{l\}}\mathcal{M}(\Delta_{2^{-k}\zeta}f_\mu)\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\leqslant\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\prod_{\mu\in\tau\cup\{l\}}\|\mathcal{M}(\Delta_{2^{-k}\zeta}f_\mu)\|^q_{L^{p_\mu/\alpha_\tau}(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\leqslant\Big\|\prod_{\mu\in\tau\cup\{l\}}\Big(\sum_{k\in\mathbb{Z}}(2^{ks\alpha_\tau/p_\mu}\|\mathcal{M}(\Delta_{2^{-k}\zeta}f_\mu)\|_{L^{p_\mu/\alpha_\tau}(\mathfrak{R}_n)})^{p_\mu q/\alpha_\tau}\Big)^{\alpha_\tau/(qp_\mu)}\Big\|_{L^{\alpha_\tau}(\mathbb{R}^n)}\\
&\leqslant\prod_{\mu\in\tau\cup\{l\}}\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ks\alpha_\tau/p_\mu}\|\mathcal{M}(\Delta_{2^{-k}\zeta}f_\mu)\|_{L^{p_\mu/\alpha_\tau}(\mathfrak{R}_n)})^{p_\mu q/\alpha_\tau}\Big)^{\alpha_\tau/(qp_\mu)}\Big\|_{L^{p_\mu}(\mathbb{R}^n)}\\
&\leqslant C\prod_{\mu\in\tau\cup\{l\}}\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ks\alpha_\tau/p_\mu}\|\Delta_{2^{-k}\zeta}f_\mu\|_{L^{p_\mu/\alpha_\tau}(\mathfrak{R}_n)})^{p_\mu q/\alpha_\tau}\Big)^{\alpha_\tau/(qp_\mu)}\Big\|_{L^{p_\mu}(\mathbb{R}^n)}\\
&\leqslant C\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{\dot{F}^{p_\mu,p_\mu q/\alpha_\tau}_{\alpha_\tau s/p_\mu}(\mathbb{R}^n)}\\
&\leqslant C\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{F^{p_\mu,p_\mu q/\alpha_\tau}_{\alpha_\tau s/p_\mu}(\mathbb{R}^n)}\\
&\leqslant C\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{F^{p_\mu,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{46}$$

Combining (46) with (45) and (25) implies that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^\alpha(\mathbb{R}^n)}\leqslant C\prod_{j=1}^m\|f_j\|_{F^{p_j,q}_s(\mathbb{R}^n)}.\tag{47}$$

It follows from (43) and (47) that

$$A_3\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F^{p_j,q}_s(\mathbb{R}^n)}.\tag{48}$$

Invoking Lemma 2, we have $b_1f_1\in F^{p_1,q}_s(\mathbb{R}^n)$ and

$$\|b_1f_1\|_{F^{p_1,q}_s(\mathbb{R}^n)}\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f_1\|_{F^{p_1,q}_s(\mathbb{R}^n)}.\tag{49}$$

By (49) and the arguments similar to those used to derive (47), we have

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\vec{G}_{b_1,l,2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)} \quad (50)$$

for each $l=2,\dots,m$. Observe that

$$\begin{aligned}\mathfrak{M}(\vec{G}_{b_1,1,2^{-k}\zeta}) &\leqslant \mathscr{M}(b_1\Delta_{2^{-k}\zeta}f_1)\sum_{\tau\subset E_1}\prod_{\mu\in\tau}\mathscr{M}(\Delta_{2^{-k}\zeta}f_\mu)\prod_{\nu\in\tau'}\mathscr{M}(f_\nu)\\ &\leqslant \|b_1\|_{L^\infty(\mathbb{R}^n)}\sum_{\tau\subset E_1}\prod_{\mu\in\tau\cup\{1\}}\mathscr{M}(\Delta_{2^{-k}\zeta}f_\mu)\prod_{\nu\in\tau'}\mathscr{M}(f_\nu).\end{aligned} \quad (51)$$

By (51) and the arguments similar to those used to derive (47), one gets

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\vec{G}_{b_1,1,2^{-k}\zeta})\mathrm{d}\zeta\Big)^q\Big)^{1/q}\Big\|_{L^\alpha(\mathbb{R}^n)} \leqslant C\|b_1\|_{L^\infty(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \quad (52)$$

By (50), (52) and Minkowski's inequality, we obtain

$$A_4 \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^m\|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \quad (53)$$

Combining (53) with (38), (39), (42) and (48) implies (36) with $i=1$.

Now we prove the continuity result for $\mathfrak{M}_{\vec{b}}$. Let $\vec{f}_j=(f_{1,j},\dots,f_{m,j})$ with each $f_{i,j}\to f_i$ in $F_s^{p_i,q}(\mathbb{R}^n)$ as $j\to\infty$ for all $i\in\{1,\dots,m\}$. It is enough to show that

$$\|\mathfrak{M}_{\vec{b}}^i(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^i(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)}\to 0 \quad\text{as}\quad j\to\infty \quad (54)$$

for all $i=1,\dots,m$.

We only prove (54) for $i=1$ since other cases are analogous. By (25) we have that $f_{i,j}\to f_i$ in $\dot{F}_s^{p_i,q}(\mathbb{R}^n)$ and in $L^{p_i}(\mathbb{R}^n)$ as $j\to\infty$ for $i=1,\dots,m$. By (25) and (7), to prove (54) with $i=1$, it suffices to show that

$$\|\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^1(\vec{f})\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)}\to 0 \;\text{ as }\; j\to\infty. \quad (55)$$

We shall prove (55) by contradiction. Assume that (55) doesn't hold. Without loss of generality we may assume that there exists a constant $c>0$ such that

$$\|\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^1(\vec{f})\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)}>c,\quad \forall\; j\geqslant 1. \quad (56)$$

By (7), we may assume without loss of generality, by extracting a subsequence that $\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)(x)-\mathfrak{M}_{\vec{b}}^1(\vec{f})(x)\to 0$ as $j\to\infty$ for almost every $x\in\mathbb{R}^n$. Hence

$$\Delta_{2^{-k}\zeta}(\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^1(\vec{f}))(x)\to 0 \quad\text{as}\quad j\to\infty \quad (57)$$

for every $(k,\zeta)\in\mathbb{Z}\times\mathfrak{R}_n$ and almost every $x\in\mathbb{R}^n$. By (37) we have that, for $(x,k,\zeta)\in\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n$,

$$|\Delta_{2^{-k}\zeta}(\mathfrak{M}_{\vec{b}}^1(\vec{f}_j)-\mathfrak{M}_{\vec{b}}^1(\vec{f}))(x)|\leqslant\Gamma_j(x,k,\zeta)+\Gamma(x,k,\zeta), \quad (58)$$

where Γ is given as in (37) and

$$\begin{aligned}
&\Gamma_j(x,k,\zeta)\\
&:=\Big||\Delta_{2^{-k}\zeta}b_1(x)|\mathfrak{M}((\vec{f}_j)_{2^{-k}\zeta})(x)+\mathfrak{M}(\vec{f}_{1,j,b_1,2^{-k}\zeta})(x)+|b_1(x)|\sum_{l=1}^{m}\mathfrak{M}(\vec{G}_{l,j,2^{-k}\zeta})(x)\\
&\quad+\sum_{l=1}^{m}\mathfrak{M}(\vec{G}_{b_1,l,j,2^{-k}\zeta})(x)-\Gamma(x,k,\zeta)\Big|\\
&\leqslant\sum_{i=1}^{4}\varphi_{i,j}(x,k,\zeta),
\end{aligned}\tag{59}$$

where

$$\varphi_{1,j}(x,k,\zeta):=|\Delta_{2^{-k}\zeta}b_1(x)||\mathfrak{M}((\vec{f}_j)_{2^{-k}\zeta})(x)-\mathfrak{M}(\vec{f}_{2^{-k}\zeta})(x)|,$$

$$\varphi_{2,j}(x,k,\zeta):=|\mathfrak{M}(\vec{f}_{1,j,b_1,2^{-k}\zeta})(x)-\mathfrak{M}(\vec{f}_{1,b_1,2^{-k}\zeta})(x)|,$$

$$\varphi_{3,j}(x,k,\zeta):=|b_1(x)|\sum_{l=1}^{m}|\mathfrak{M}(\vec{G}_{l,j,2^{-k}\zeta})(x)-\mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})(x)|,$$

$$\varphi_{4,j}(x,k,\zeta)=\sum_{l=1}^{m}|\mathfrak{M}(\vec{G}_{b_1,l,j,2^{-k}\zeta})(x)-\mathfrak{M}(\vec{G}_{b_1,l,2^{-k}\zeta})(x)|,$$

$$\vec{f}_{1,j,b_1,2^{-k}\zeta}:=(\Delta_{2^{-k}\zeta}b_1(f_{1,j})_{2^{-k}\zeta},(f_{2,j})_{2^{-k}\zeta},\dots,(f_{m,j})_{2^{-k}\zeta}),$$

$$\vec{G}_{b_1,1,j,2^{-k}\zeta}:=(b_1\Delta_{2^{-k}\zeta}f_{1,j},(f_{2,j})_{2^{-k}\zeta},\dots,(f_{m,j})_{2^{-k}\zeta}),$$

$$\vec{G}_{l,j,2^{-k}\zeta}:=(f_{1,j},\dots,f_{l-1,j},\Delta_{2^{-k}\zeta}f_{l,j},(f_{l+1,j})_{2^{-k}\zeta},\dots,(f_{m,j})_{2^{-k}\zeta}),$$

$$\vec{G}_{b_1,l,j,2^{-k}\zeta}:=(b_1f_{1,j},f_{2,j},\dots,f_{l-1,j},\Delta_{2^{-k}\zeta}f_{l,j},(f_{l+1,j})_{2^{-k}\zeta},\dots,(f_{m,j})_{2^{-k}\zeta}),\ \ l=2,\dots,m.$$

By the arguments similar to those used to derive (39), one has

$$\|\varphi_{1,j}\|_{E^s_{p,q,1}}\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\|\mathfrak{M}(\vec{f}_j)-\mathfrak{M}(\vec{f})\|_{F_s^{p,q}(\mathbb{R}^n)},$$

which together with Theorem B (i) leads to

$$\|\varphi_{1,j}\|_{E^s_{p,q,1}}\to 0\quad\text{as}\quad j\to\infty.\tag{60}$$

On the other hand, by the sublinearity for $\mathfrak{M}$ and similar arguments as in deriving (48) and (50), we can obtain

$$\begin{aligned}
\|\varphi_{i,j}\|_{E^s_{p,q,1}}\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}&\sum_{l=1}^{m}\|f_{l,j}-f_l\|_{F_s^{p_l,q}(\mathbb{R}^n)}\\
&\times\prod_{\substack{1\leqslant\mu\leqslant m\\ \mu\neq l}}(\|f_{\mu,j}-f_\mu\|_{F_s^{p_\mu,q}(\mathbb{R}^n)}+\|f_\mu\|_{F_s^{p_\mu,q}(\mathbb{R}^n)}),\ \ i=2,3,4.
\end{aligned}\tag{61}$$

It follows from (59)–(61) that

$$\|\Gamma_j\|_{E^s_{p,q,1}}\to 0\ \text{as}\ \ j\to\infty.\tag{62}$$

By (62), there exists a subsequence $\{j_\ell\}_{\ell=1}^\infty \subset \{j\}_{j=1}^\infty$ such that

$$\sum_{\ell=1}^{\infty} \|\Gamma_{j_\ell}\|_{E^s_{p,q,1}} < \infty. \tag{63}$$

We get from (39), (42), (48), (50) and Minkowski's inequality that

$$\|\Gamma\|_{E^s_{p,q,1}} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{F_s^{p_j,q}(\mathbb{R}^n)}. \tag{64}$$

On the other hand, we get from (58) that

$$|\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)| \leqslant \sum_{\ell=1}^{\infty} \Gamma_{j_\ell}(x,k,\zeta) + \Gamma(x,k,\zeta) =: \Phi(x,k,\zeta), \tag{65}$$

for all $(x,k,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n$. By (64), (65) and Minkowski's inequality, we have that $\|\Phi\|_{E^s_{p,q,1}} < \infty$. Hence, $\int_{\mathfrak{R}_n} \Phi(x,k,\zeta)\mathrm{d}\zeta < \infty$ for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$. By (57), (65) and the dominated convergence theorem, we have

$$\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\mathrm{d}\zeta \to 0 \text{ as } \ell \to \infty \tag{66}$$

for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$. By the fact $\|\Phi\|_{E^s_{p,q,1}} < \infty$, one has

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \Phi(x,k,\zeta)\mathrm{d}\zeta\Big)^q\Big)^{1/q} < \infty \tag{67}$$

for almost every $x \in \mathbb{R}^n$. From (65) we see that

$$\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\mathrm{d}\zeta \leqslant \int_{\mathfrak{R}_n} \Phi(x,k,\zeta)\mathrm{d}\zeta, \tag{68}$$

for all $(x,k,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n$ and $\ell \geqslant 1$. By (66)–(68) and the dominated convergence theorem, one finds that

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\mathrm{d}\zeta\Big)^q\Big)^{1/q} \to 0 \text{ as } \ell \to \infty. \tag{69}$$

Using (65) and the fact that $\|\Phi\|_{E^s_{p,q,1}} < \infty$ again,

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))(x)|\mathrm{d}\zeta\Big)^q\Big)^{1/q} \\ &\leqslant \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Phi(x,k,\zeta)|\mathrm{d}\zeta\Big)^q\Big)^{1/q} < \infty\end{aligned} \tag{70}$$

for almost every $x \in \mathbb{R}^n$. It follows from (69), (70) and the dominated convergence theorem that

$$\|\Delta_{2^{-k}\zeta}(\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f}))\|_{E^s_{p,q,1}} \to 0 \text{ as } \ell \to \infty,$$

which combining with (24) leads to $\|\mathfrak{M}^1_{\vec{b}}(\vec{f}_{j_\ell}) - \mathfrak{M}^1_{\vec{b}}(\vec{f})\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} \to 0$ as $\ell \to \infty$. This is in contradiction with (56). This finishes the proof of Theorem 2. □

4. Proof of boundedness and continuity on Besov spaces

In this section we shall present the proof of Theorem 3. Let us begin with some properties of Besov spaces.

4.1. Properties on Besov spaces

We denote by $\dot{B}_s^{p,q}(\mathbb{R}^n)$ the homogeneous Besov spaces. It was proved by Yabuta [28] that if $0<s<1$, $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$ and $1\leqslant r\leqslant p$, then

$$\|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}\sim\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big\|\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}f|^r\mathrm{d}\zeta\Big)^{1/r}\Big\|_{L^p(\mathbb{R}^n)}^q\Big)^{1/q}. \tag{71}$$

For a measurable function $g:\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n\to\mathbb{R}$, we define

$$\|g\|_{p,q,s}:=\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|g(x,k,\zeta)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}.$$

Then, by (71) and Fubini's theorem, we have

$$\|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}\sim\|\Delta_{2^{-k}\zeta}f\|_{p,q,s}. \tag{72}$$

It is well known that (see [6, 8, 27])

$$\|f\|_{B_s^{p,q}(\mathbb{R}^n)}\sim\|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}+\|f\|_{L^p(\mathbb{R}^n)},\quad \text{for } s>0,\ 1<p,q<\infty, \tag{73}$$

$$\|f\|_{B_{s_1}^{p,q}(\mathbb{R}^n)}\leqslant\|f\|_{B_{s_2}^{p,q}(\mathbb{R}^n)},\quad \text{for } s_1\leqslant s_2,\ 1<p,\ q<\infty, \tag{74}$$

$$\|f\|_{B_s^{p,q_2}(\mathbb{R}^n)}\leqslant\|f\|_{B_s^{p,q_1}(\mathbb{R}^n)},\quad \text{for } s\in\mathbb{R},\ 1<p<\infty,\ 1<q_1\leqslant q_2<\infty. \tag{75}$$

The following presents a characterization of of the product of a function in $B_s^{p,q}(\mathbb{R}^n)$ and a function in $\mathrm{Lip}(\mathbb{R}^n)$.

LEMMA 4. *Let $0<s<1$ and $1<p,q<\infty$. If $f\in B_s^{p,q}(\mathbb{R}^n)$ and $g\in\mathrm{Lip}(\mathbb{R}^n)$, then $fg\in B_s^{p,q}(\mathbb{R}^n)$. Moreover,*

$$\|fg\|_{B_s^{p,q}(\mathbb{R}^n)}\leqslant C\|g\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p,q}(\mathbb{R}^n)}. \tag{76}$$

Proof. By (73) and the trivial estimate $\|fg\|_{L^p(\mathbb{R}^n)}\leqslant\|g\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}$, to prove (76), it suffices to show that

$$\|fg\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}\leqslant C\|g\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p,q}(\mathbb{R}^n)}. \tag{77}$$

By (29), (72) and Minkowski's inequality, we have

$$\begin{aligned}\|fg\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} &\leqslant C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}(fg)(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leqslant C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}f(x)\Delta_{2^{-k}\zeta}g(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|g(x)\Delta_{2^{-k}\zeta}f(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|f(x)\Delta_{2^{-k}\zeta}g(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leqslant C\|g\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}f(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|f(x)\Delta_{2^{-k}\zeta}g(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leqslant C\|g\|_{L^\infty(\mathbb{R}^n)}\|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}\\ &\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|f(x)\Delta_{2^{-k}\zeta}g(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}.\end{aligned} \tag{78}$$

By the property of g, we have

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|f(x)\Delta_{2^{-k}\zeta}g(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leqslant\Big(\sum_{k=-\infty}^{0}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|f(x)\Delta_{2^{-k}\zeta}g(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\quad+\Big(\sum_{k=1}^{\infty}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|f(x)\Delta_{2^{-k}\zeta}g(x)|^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leqslant C|\mathfrak{R}_n|\|g\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0}2^{ksq}\Big)^{1/q}\\ &\quad+C|\mathfrak{R}_n|\|g\|_{Lip(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty}2^{-kq(1-s)}\Big)^{1/q}\\ &\leqslant C\|g\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{79}$$

Then (77) follows from (73), (78) and (79). □

4.2. Proof of Theorem 3

In this subsection we shall prove Theorem 3. Applying Lemma 4, Theorem B (ii) and the arguments similar to those used in deriving the Triebel-Lizorkin space boundedness and continuity for $[\vec{b},\mathfrak{M}]$, one can get (10) and the continuity for $[\vec{b},\mathfrak{M}]:B_s^{p_1,q}(\mathbb{R}^n)\times\cdots\times B_s^{p_m,q}(\mathbb{R}^n)\to B_s^{p,q}(\mathbb{R}^n)$.

Next we prove Theorem 3 for $\mathfrak{M}_{\vec{b}}$. In what follows, we fix $0<s<1$, $1<p_1,\dots,p_m,p,q<\infty$ and $1/p=1/p_1+\cdots+1/p_m$. At first, we shall prove that

$$\|\mathfrak{M}_{\vec{b}}(\vec{f})\|_{B_s^{p,q}(\mathbb{R}^n)}\leqslant C\Big(\sum_{i=1}^{m}\|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)}\Big)\prod_{j=1}^{m}\|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \tag{80}$$

By Minkowski's inequality, to prove (80), it is enough to show that

$$\|\mathfrak{M}_{\vec{b}}^{i}(\vec{f})\|_{B_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b_i\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)} \tag{81}$$

for each $i = 1,\dots,m$.

We only work with (81) for the case $i = 1$ since other cases are analogous. By (72) and (37), we have

$$\begin{aligned}
&\|\mathfrak{M}_{\vec{b}}^{1}(\vec{f})\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)}\\
&\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(|\Delta_{2^{-k}\zeta}(\mathfrak{M}_{\vec{b}}^{1}(\vec{f}))(x)|)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(|\Delta_{2^{-k}\zeta}b_1(x)|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})(x))^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad +C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\vec{f}_{1,b_1,2^{-k}\zeta})(x))^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad +C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\sum_{l=1}^{m}|b_1(x)|\mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})(x)\Big)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad +C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\sum_{l=1}^{m}\mathfrak{M}(\vec{G}_{b_1,l,2^{-k}\zeta})(x)\Big)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&=: B_1+B_2+B_3+B_4.
\end{aligned} \tag{82}$$

By (2) and the property of b_1, we have

$$\begin{aligned}
B_1 &\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_d}\int_{\mathbb{R}^n}(|\Delta_{2^{-k}\zeta}b_1(x)|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})(x))^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k=-\infty}^{0} 2^{ksq}\Big(\int_{\mathfrak{R}_d}\int_{\mathbb{R}^n}(|\Delta_{2^{-k}\zeta}b_1(x)|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})(x))^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad +C\Big(\sum_{k=1}^{\infty} 2^{ksq}\Big(\int_{\mathfrak{R}_d}\int_{\mathbb{R}^n}(|\Delta_{2^{-k}\zeta}b_1(x)|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})(x))^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\Big(\|b_1\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0} 2^{ksq}\Big)^{1/q} + \|b\|_{Lip(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty} 2^{-kq(1-s)}\Big)^{1/q}\Big)\\
&\quad \times\|\mathfrak{M}(\vec{f}_{2^{-k}\zeta})\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}.
\end{aligned} \tag{83}$$

$$\begin{aligned}
B_2 &\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\|\Delta_{2^{-k}\zeta}b_1(f_1)_{2^{-k}\zeta}\|_{L^{p_1}(\mathbb{R}^n\times\mathfrak{R}_n)}\prod_{j=2}\|(f_j)_{2^{-k}\zeta}\|_{L^{p_j}(\mathbb{R}^n)}\Big)^q\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k=-\infty}^{0} 2^{ksq}\Big(\|\Delta_{2^{-k}\zeta}b_1(f_1)_{2^{-k}\zeta}\|_{L^{p_1}(\mathbb{R}^n\times\mathfrak{R}_n)}\prod_{j=2}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\Big)^q\Big)^{1/q}\\
&\quad +C\Big(\sum_{k=1}^{\infty} 2^{ksq}\Big(\|\Delta_{2^{-k}\zeta}b_1(f_1)_{2^{-k}\zeta}\|_{L^{p_1}(\mathbb{R}^n\times\mathfrak{R}_n)}\prod_{j=2}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}\Big)^q\Big)^{1/q}
\end{aligned} \tag{84}$$

$$\leqslant C\Big(\|b_1\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0}2^{ksq}\Big)^{1/q}+\|b\|_{Lip(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty}2^{-kq(1-s)}\Big)^{1/q}\Big)$$
$$\times\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$
$$\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

By Minkowski's inequality and Hölder's inequality, one has

$$B_3\leqslant C\|b_1\|_{L^\infty(\mathbb{R}^n)}\sum_{l=1}^{m}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})(x)\Big)^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}. \tag{85}$$

Fix $l\in\{1,\dots,m\}$. Let $E_l=\{l+1,\dots,m\}$ and τ',α_τ be given as in the proof of Theorem 2. By Minkowski's inequality, Hölder's inequality, (44) and the bounds for $\mathscr{M}$, one finds that

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})(x))^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant\sum_{\tau\subset E_l}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\prod_{\mu\in\tau\cup\{l\}}\mathscr{M}(\Delta_{2^{-k}\zeta}f_\mu)\prod_{\nu\in\tau'}\mathscr{M}(f_\nu)\Big)^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\sum_{\tau\subset E_l}\prod_{\nu\in\tau'}\|\mathscr{M}(f_\nu)\|_{L^{p_\nu}(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big\|\prod_{\mu\in\tau\cup\{l\}}\mathscr{M}(\Delta_{2^{-k}\zeta}f_\mu)\Big\|^q_{L^{\alpha_\tau}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^{1/q}\\
&\leqslant C\sum_{\tau\subset E_l}\prod_{\nu\in\tau'}\|f_\nu\|_{L^{p_\nu}(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big\|\prod_{\mu\in\tau\cup\{l\}}\mathscr{M}(\Delta_{2^{-k}\zeta}f_\mu)\Big\|^q_{L^{\alpha_\tau}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^{1/q}.
\end{aligned}\tag{86}$$

By the bounds for $\mathscr{M}$ and (72)–(75), we have

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big\|\prod_{\mu\in\tau\cup\{l\}}\mathscr{M}(\Delta_{2^{-k}\zeta}f_\mu)\Big\|^q_{L^{\alpha_\tau}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^{1/q}\\
&\leqslant\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\prod_{\mu\in\tau\cup\{l\}}\|\mathscr{M}(\Delta_{2^{-k}\zeta}f_\mu)\|_{L^{p_\mu}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^q\Big)^{1/q}\\
&\leqslant\prod_{\mu\in\tau\cup\{l\}}\Big(\sum_{k\in\mathbb{Z}}(2^{ks\alpha_\tau/p_\mu}\|\mathscr{M}(\Delta_{2^{-k}\zeta}f_\mu)\|_{L^{p_\mu}(\mathbb{R}^n\times\mathfrak{R}_n)})^{p_\mu q/\alpha_\tau}\Big)^{\alpha_\tau/(p_\mu q)}\\
&\leqslant\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{\dot{B}^{p_\mu,p_\mu q/\alpha_\tau}_{s\alpha_\tau/p_\mu}(\mathbb{R}^n)}\\
&\leqslant\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{B^{p_\mu,p_\mu q/\alpha_\tau}_{s\alpha_\tau/p_\mu}(\mathbb{R}^n)}\\
&\leqslant\prod_{\mu\in\tau\cup\{l\}}\|f_\mu\|_{B^{p_\mu,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{87}$$

It follows from (73), (86) and (87) that

$$\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\vec{G}_{l,2^{-k}\zeta})(x))^p\mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\leqslant C\prod_{j=1}^{m}\|f_j\|_{B^{p_j,q}_s(\mathbb{R}^n)}. \tag{88}$$

Combining (85) with (88) and the property of b_1 implies

$$B_3 \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \tag{89}$$

By (51) and the arguments similar to those used to derive (88), one has

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\mathfrak{M}(\vec{G}_{b_1,1,2^{-k}\zeta})(x)\Big)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}.\end{aligned} \tag{90}$$

On the other hand, by Lemma 4, we have that $b_1 f_1 \in B_s^{p_1,q}(\mathbb{R}^n)$ and

$$\|b_1 f_1\|_{B_s^{p_1,q}(\mathbb{R}^n)} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f_1\|_{B_s^{p_1,q}(\mathbb{R}^n)},$$

which together with the arguments similar to those used to derive (88) implies

$$\begin{aligned}&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}\Big(\mathfrak{M}(\vec{G}_{b_1,l,2^{-k}\zeta})(x)\Big)^p \mathrm{d}x\mathrm{d}\zeta\Big)^{q/p}\Big)^{1/q}\\ &\leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}\end{aligned} \tag{91}$$

for each $l = 1,\dots,m$. Then by (90), (91) and Minkowski's inequality, one has

$$B_4 \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \tag{92}$$

Combining (92) with (82)–(84) and (89) implies (81) for $i=1$.

It remains to prove the continuity result for $\mathfrak{M}_{\vec{b}}$. The proof is similar as in the proof of the continuity part for $\mathfrak{M}_{\vec{b}}$ in Theorem 2. Let $\vec{f}_j = (f_{1,j},\dots,f_{m,j})$ with each $f_{i,j} \to f_i$ in $B_s^{p_i,q}(\mathbb{R}^n)$ as $j\to\infty$ for all $i\in\{1,\dots,m\}$. It suffices to show that

$$\|\mathfrak{M}_{\vec{b}}^i(\vec{f}_j) - \mathfrak{M}_{\vec{b}}^i(\vec{f})\|_{B_s^{p,q}(\mathbb{R}^n)} \to 0 \quad \text{as} \quad j\to\infty \tag{93}$$

for all $i = 1,\dots,m$.

We only prove (93) for $i=1$ since other cases are analogous. By (73), we have that, $f_{i,j}\to f_i$ in $\dot{B}_s^{p_i,q}(\mathbb{R}^n)$ and in $L^{p_i}(\mathbb{R}^n)$ as $j\to\infty$ for all $i\in\{1,\dots,m\}$. By (7), to conclude (93) with $i=1$, it suffices to prove that

$$\|\mathfrak{M}_{\vec{b}}^1(\vec{f}_j) - \mathfrak{M}_{\vec{b}}^1(\vec{f})\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \to 0 \ \text{ as } j\to\infty. \tag{94}$$

We shall prove (94) by contradiction. Assume that (94) doesn't hold. We may assume, without loss of generality that, there exists a constant $c>0$ such that

$$\|\mathfrak{M}_{\vec{b}}^1(\vec{f}_j) - \mathfrak{M}_{\vec{b}}^1(\vec{f})\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} > c, \ \text{ for all } j\geqslant 1.$$

Let $\{\varphi_{i,j}\}_{i=1}^4$, and Γ_j, Γ be given as in the proof of Theorem 2. Using arguments similar to those used in deriving (90) and (91), one obtains

$$\|\varphi_{i,j}\|_{p,q,s} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \sum_{l=1}^{m} \|f_{l,j} - f_l\|_{B_s^{p_l,q}(\mathbb{R}^n)} \times \prod_{\substack{1\leqslant\mu\leqslant m \\ \mu\neq l}} (\|f_{\mu,j} - f_\mu\|_{B_s^{p_\mu,q}(\mathbb{R}^n)} + \|f_\mu\|_{B_s^{p_\mu,q}(\mathbb{R}^n)}),$$

for $i = 1,2,3,4$. It follows that

$$\|\Gamma_j\|_{p,q,s} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \sum_{l=1}^{m} \|f_{l,j} - f_l\|_{B_s^{p_l,q}(\mathbb{R}^n)} \times \prod_{\substack{1\leqslant\mu\leqslant m \\ \mu\neq l}} (\|f_{\mu,j} - f_\mu\|_{B_s^{p_\mu,q}(\mathbb{R}^n)} + \|f_\mu\|_{B_s^{p_\mu,q}(\mathbb{R}^n)}). \tag{95}$$

By (37), (83), (84), (89), (92) and Minkowski's inequality, we have

$$\|\Gamma\|_{p,q,s} \leqslant C\|b_1\|_{\mathrm{Lip}(\mathbb{R}^n)} \prod_{j=1}^{m} \|f_j\|_{B_s^{p_j,q}(\mathbb{R}^n)}. \tag{96}$$

The rest of proof follows from (95), (96) and the arguments similar to the proof of Theorem 2. We omit the details. □

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