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MAJORIZATION REFINEMENTS OF KY FAN'S EIGENVALUE INEQUALITY AND RELATED RESULTS

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Abstract. In this note we show the anti-isotonicity of a certain operator function induced by the eigenvalue map $\lambda(\cdot)$ on the space of Hermitian matrices. In consequence, we obtain some majorization refinements of the Ky Fan's eigenvalue inequality. Thus we extend Maligranda's inequalities from a norm to the eigenvalue map, and from the usual order on $\mathbb{R}$ to the majorization preorder on $\mathbb{R}^n$.

1. Motivation and preliminaries

In [7] Maligranda obtained the following result.

THEOREM A. [7, Theorem 1] *For nonzero vectors of x and y in a normed space with norm $\|\cdot\|$ it holds that*

$$\|x+y\| \leqslant \|x\|+\|y\|-\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\min\{\|x\|,\|y\|\} \tag{1}$$

and

$$\|x+y\| \geqslant \|x\|+\|y\|-\left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\max\{\|x\|,\|y\|\}. \tag{2}$$

See [3, 7, 9, 10] for related results.

The purpose of the present note is to extend Maligranda's inequalities (1)–(2) from a norm $\|\cdot\|$ to the eigenvalue map $\lambda(\cdot):\mathbb{H}_n\to\mathbb{R}^n$, where $\mathbb{H}_n$ stands for the set of Hermitian matrices of size $n\times n$. Furthermore, we replace the usual order $\leqslant$ on $\mathbb{R}$ in (1)–(2) by the majorization preorder $\prec$ on $\mathbb{R}^n$ [8]. To this end, we prove a theorem on anti-isotonicity of a certain operator function induced by the map $\lambda(\cdot)$ (see Theorem 1). In particular, we obtain some majorization refinements of the eigenvalue inequalities due to Ky Fan (see Theorem 2 and Corollaries 2–3). Norm versions of the results presented in the paper have the potential to be applied in studying uniformly non-square Banach spaces and James type constants [13].

In the rest of this preliminary section we demonstrate some basics of the majorization theory [8].

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For a vector $z=(z_1,z_2,\dots,z_n)\in\mathbb{R}^n$, the symbols $z_{[1]},z_{[2]},\dots,z_{[n]}$ stand for the entries of z stated in decreasing order, i.e., $z_{[1]}\geqslant z_{[2]}\geqslant\dots\geqslant z_{[n]}$.

We say that a vector $y=(y_1,y_2,\dots,y_n)$ in $\mathbb{R}^n$ is *majorized* by a vector $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ (written as $y\prec x$), if the sum of k largest entries of y does not exceed the sum of k largest entries of x for all $k=1,2,\dots,n$ with equality for $k=n$, that is

$$\sum_{i=1}^{k} y_{[i]}\leqslant\sum_{i=1}^{k} x_{[i]}\quad\text{for all } k=1,2,\dots,n,\quad\text{and}\quad\sum_{i=1}^{n} y_i=\sum_{k=1}^{n} x_i$$

(see [8, p. 8]).

We use the notation

$$\mathbb{R}^n_{\downarrow}=\{(x_1,x_2,\dots,x_n)\in\mathbb{R}^n : x_1\geqslant x_2\geqslant\dots\geqslant x_n\}.$$

We equip the real linear space $\mathbb{H}_n$ of $n\times n$ Hermitian matrices with the trace inner product $\langle x,y\rangle=\operatorname{tr}xy$ for $x,y\in\mathbb{H}_n$. By $\mathbb{U}_n$ we mean the group of $n\times n$ unitary matrices. In addition, we take G to be the group of all unitary similarities of the form $x\to uxu^*$, $x\in\mathbb{H}_n$, where u runs over the group $\mathbb{U}_n$.

We denote by $\lambda(x)=(\lambda_1(x),\lambda_2(x),\dots,\lambda_n(x))$ the vector of the eigenvalues of $x\in\mathbb{H}_n$ arranged so that $\lambda_1(x)\geqslant\lambda_2(x)\geqslant\dots\geqslant\lambda_n(x)$. Evidently, $\lambda(x)\in\mathbb{R}^n_{\downarrow}$ for each $x\in\mathbb{H}_n$, and $\lambda(\mathbb{H}_n)=\mathbb{R}^n_{\downarrow}$.

The symbol $\operatorname{diag}z$ stands for the $n\times n$ diagonal matrix with the entries of the vector $z\in\mathbb{R}^n$ on the main diagonal.

It is known by Spectral Theorem that for each $x\in\mathbb{H}_n$ there exists a $u\in\mathbb{U}_n$ such that

$$x=u(\operatorname{diag}\lambda(x))u^*.$$

We denote

$$D=\{\operatorname{diag}(\lambda_1,\lambda_2,\dots,\lambda_n):\lambda_1\geqslant\lambda_2\geqslant\dots\geqslant\lambda_n\}.$$

Clearly, $D=\operatorname{diag}\mathbb{R}^n_{\downarrow}$.

The Ky Fan's eigenvalue inequality [4] (see also [1]) says that

$$\lambda(x+y)\prec\lambda(x)+\lambda(y)\quad\text{for } x,y\in\mathbb{H}_n. \tag{3}$$

A related result is the following Lidskii-Wielandt's inequality

$$\lambda(x)-\lambda(y)\prec\lambda(x-y)\quad\text{for } x,y\in\mathbb{H}_n \tag{4}$$

(see [6, 12], [1, p. 69, p. 98]).

Let V be a real linear space endowed with a (real) inner product $\langle\cdot,\cdot\rangle$. We say that a (nonempty) subset $C\subset V$ is a *convex cone* if

$$a,b\in C\quad\text{implies}\quad a+b\in C,$$

and

$$a\in C\quad\text{and}\quad 0\leqslant t\in\mathbb{R}\quad\text{imply}\quad ta\in C.$$

The relation $\leqslant_C$ on V defined by

$$z \leqslant_C y \quad \text{iff} \quad y - z \in C \quad (y, z \in V) \tag{5}$$

is called the *cone preorder* induced by C.

Let $C \subset V$ be a convex cone. The *dual cone* of C is defined by

$$\operatorname{dual} C = \{z \in V : \langle z, c \rangle \geqslant 0 \quad \text{for all } c \in C\}.$$

It is known that the preorder of majorization $\prec$ restricted to the convex cone $\mathbb{R}^n_\downarrow$ is the cone preorder induced by the dual cone $\operatorname{dual} \mathbb{R}^n_\downarrow$. That is,

$$a \prec b \quad \text{iff} \quad b - a \in \operatorname{dual} \mathbb{R}^n_\downarrow \quad (\text{for } a, b \in \mathbb{R}^n_\downarrow). \tag{6}$$

However, $\prec$ is not a cone preorder on the whole space $\mathbb{R}^n$.

LEMMA 1. *Assume* $a, b \in \mathbb{R}^n_\downarrow$ *and*

$$c \in \mathbb{R}^n \text{ is such that } a + c \in \mathbb{R}^n_\downarrow \text{ and } b + c \in \mathbb{R}^n_\downarrow. \tag{7}$$

Then

$$a \prec b \quad \text{iff} \quad a + c \prec b + c. \tag{8}$$

In particular, if $c \in \mathbb{R}^n_\downarrow$ *then* (7) *and* (8) *hold.*

Proof. Denote $C = \operatorname{dual} \mathbb{R}^n_\downarrow$. Then (5) is met. Hence, by (6) and (7),

$$a \prec b \quad \text{iff} \quad b - a \in C \quad \text{iff} \quad (b + c) - (a + c) \in C \quad \text{iff} \quad a + c \prec b + c,$$

completing the proof. $\square$

2. Anti-isotonicity of some eigenvalue maps

We begin this section with some needed definitions.

Hermitian matrices z and w of size $n \times n$ are said to be *simultaneously diagonalizable* (abbreviated as, *SD*), if there exists a unitary matrix $u \in \mathbb{U}_n$ such that

$$uzu^* = \operatorname{diag} \lambda(z) \quad \text{and} \quad uwu^* = \operatorname{diag} \lambda(w). \tag{9}$$

Let $z_t \in \mathbb{H}_n$ for $t \in T$ with an index set T. We say that all z_t are *mutually simultaneously diagonalizable* (abbreviated as, *MSD*), if there exists a unitary matrix $u \in \mathbb{U}_n$ such that

$$uz_t u^* = \operatorname{diag} \lambda(z_t) \quad \text{for all } t \in T. \tag{10}$$

LEMMA 2. *Let* $z, w \in \mathbb{H}_n$. *The following statements* (i), (ii), (iii) *and* (iv) *are equivalent.*

(i) *Matrices* z *and* w *are SD.*

(ii) *Matrices z and w satisfy*

$$\lambda(z+w) = \lambda(z) + \lambda(w). \tag{11}$$

(iii) *Matrices z and w commute.*

(iv) *Matrices z and w satisfy*

$$\langle z, w\rangle = \langle \operatorname{diag}\lambda(z), \operatorname{diag}\lambda(w)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the trace inner product on $\mathbb{H}_n$.

Proof. $(i) \Rightarrow (ii)$. There is a unitary matrix $u \in \mathbb{U}_n$ such that (9) is fulfilled, so that

$$z+w = u^*\operatorname{diag}\lambda(z)u + u^*\operatorname{diag}\lambda(w)u = u^*(\operatorname{diag}(\lambda(z)+\lambda(w)))u.$$

Consequently,

$$\lambda(z+w) = \lambda(u^*(\operatorname{diag}(\lambda(z)+\lambda(w)))u)$$
$$= \lambda(\operatorname{diag}(\lambda(z)+\lambda(w))) = \lambda(z)+\lambda(w),$$

as wanted.

$(ii) \Rightarrow (i)$. It follows from (11) that

$$\operatorname{diag}\lambda(z+w) = \operatorname{diag}\lambda(z) + \operatorname{diag}\lambda(w).$$

Hence,

$$\langle \operatorname{diag}\lambda(z+w), \operatorname{diag}\lambda(z+w)\rangle = \langle \operatorname{diag}\lambda(z)+\operatorname{diag}\lambda(w), \operatorname{diag}\lambda(z)+\operatorname{diag}\lambda(w)\rangle,$$

and further, by Spectral Theorem,

$$\langle z+w, z+w\rangle = \langle \operatorname{diag}\lambda(z)+\operatorname{diag}\lambda(w), \operatorname{diag}\lambda(z)+\operatorname{diag}\lambda(w)\rangle.$$

A simple algebra gives

$$\langle z, w\rangle = \langle \operatorname{diag}\lambda(z), \operatorname{diag}\lambda(w)\rangle.$$

We now deduce from [5, Theorem 2.2] applied to the space $\mathbb{H}_n$ that matrices z and w are SD.

$(i) \Leftrightarrow (iii)$. This equivalence is well-known.

$(i) \Leftrightarrow (iv)$. It is sufficient to apply [5, Theorem 2.2]. □

LEMMA 3. *Let $y, z \in \mathbb{H}_n$. If z and $y-z$ are SD then*

$$\lambda(y-z) = \lambda(y) - \lambda(z). \tag{12}$$

Proof. Use (11) for the matrices z and $w = y-z$. □

REMARK 1. In general, the condition that z and $y-z$ are SD is *not equivalent* to the condition that z and y are SD. (Compare (11) and (12).)

Additionally, if z and y are SD, then $uzu^* = \operatorname{diag}\lambda(z)$ and $uyu^* = \operatorname{diag}\lambda(y)$ for some $u \in \mathbb{U}_n$. Hence

$$u(z-y)u^* = \operatorname{diag}(\lambda(z)-\lambda(y)). \tag{13}$$

However, by Lidskii-Wielandt inequality (see (4)) we have $\lambda(z)-\lambda(y) \prec \lambda(z-y)$, so $\operatorname{diag}(\lambda(z)-\lambda(y)) \prec_G \operatorname{diag}\lambda(z-y)$, which means that

$$\operatorname{diag}(\lambda(z)-\lambda(y)) = \sum_{i=1}^{k} t_i u_i (\operatorname{diag}(\lambda(z-y)))u_i^* \tag{14}$$

for some $t_1,\ldots,t_k \geqslant 0$ with $\sum_{i=1}^{k} t_i = 1$ and unitary $u_1,\ldots,u_k$.

It follows from (13) and (14) that

$$u(z-y)u^* = \sum_{i=1}^{k} t_i u_i (\operatorname{diag}(\lambda(z-y)))u_i^*,$$

which need not imply that $u(z-y)u^* = \operatorname{diag}\lambda(z-y)$.

LEMMA 4. *Let $a,b,c \in \mathbb{H}_n$. If*

(i) *c and $a-b-c$ are SD,*

(ii) *b and c are SD,*

then

$$\lambda(a)-\lambda(b)-\lambda(c) \prec \lambda(a-b)-\lambda(c). \tag{15}$$

Proof. By applying Lemma 2 and the above condition (ii), we obtain

$$\lambda(b+c) = \lambda(b)+\lambda(c).$$

Likewise, from condition (i) we derive

$$\lambda(a-b) = \lambda(a-b-c)+\lambda(c).$$

Therefore, by Lidskii-Wielandt inequality (4),

$$\lambda(a)-\lambda(b)-\lambda(c) = \lambda(a)-\lambda(b+c) \prec \lambda(a-b-c) = \lambda(a-b)-\lambda(c),$$

which was to be proved. □

REMARK 2. Inequality (15) generalizes Lidskii-Wielandt inequality (4). In fact, the requirements (i)–(ii) in Lemma 4 with $c=0$ hold trivially, and (15) reduces to (4).

THEOREM 1. *Let $x, y, z_1, z_2 \in \mathbb{H}_n$. Then the inequality*

$$\lambda(x+z_1)+\lambda(y)-\lambda(z_1) \prec \lambda(x+z_2)+\lambda(y)-\lambda(z_2) \tag{16}$$

holds, provided that one of the following conditions (I), (II) or (III) is satisfied:

(I) *z_1 and $y-z_1$ are SD, z_2 and $y-z_2$ are SD, z_2 and z_1-z_2 are SD.*

(II) *y and z_1-y are SD, z_2 and $y-z_2$ are SD.*

(III) *y and z_1-y are SD, y and z_2-y are SD, z_2 and z_1-z_2 are SD, $x+y$ and z_2-y are SD.*

REMARK 3. In Theorem 1, the assumption (I) can be replaced by the stronger condition that

$$z_2, z_1-z_2, y-z_1 \text{ are MSD},$$

which means that

$$0 \leqslant_C z_2 \leqslant_C z_1 \leqslant_C y,$$

where $\leqslant_C$ is the cone preorder on $\mathbb{H}_n$ induced by $C = gD$ for some $g = u^*(\cdot)u$ with $u \in \mathbb{U}_n$ and $D = \operatorname{diag}\mathbb{R}^n_\downarrow$.

In consequence, the assertion (16) says that

$$\text{the map } z \longrightarrow \varphi(z) = \lambda(x+z)+\lambda(y)-\lambda(z) \text{ is } (\leqslant_C, \prec)\text{-anti-isotone on } [0,y]_C,$$

where $[0,y]_C = \{z \in \mathbb{H}_n : 0 \leqslant_C z \leqslant_C y\}$.

Likewise, with the property of MSD, assumptions of type (II) and (III) imply that

$$0 \leqslant_C z_2 \leqslant_C y \leqslant_C z_1 \quad \text{and} \quad y \leqslant_C z_2 \leqslant_C z_1,$$

respectively.

Proof of Theorem 1. We denote

$$l_i = \lambda(x+z_i)+\lambda(y)-\lambda(z_i) \quad \text{for } i = 1, 2. \tag{17}$$

We have to prove that $l_1 \leqslant l_2$.

Case (I). On account of assumption (I), it follows from Lemma 3 that

$$\lambda(y-z_1) = \lambda(y)-\lambda(z_1), \tag{18}$$

$$\lambda(y-z_2) = \lambda(y)-\lambda(z_2), \tag{19}$$

$$\lambda(z_1-z_2) = \lambda(z_1)-\lambda(z_2). \tag{20}$$

In light of (17) and (18))-(19) we see that

$$l_i = \lambda(x+z_i)+\lambda(y-z_i) \in \mathbb{R}^n_\downarrow + \mathbb{R}^n_\downarrow \subset \mathbb{R}^n_\downarrow \quad \text{for } i = 1, 2.$$

In conclusion,

$$l_i \in \mathbb{R}^n_\downarrow \quad \text{for } i = 1,2. \tag{21}$$

On the other hand, by Ky Fan's inequality (3), we obtain

$$\lambda(x+z_2+(z_1-z_2)) \prec \lambda(x+z_2)+\lambda(z_1-z_2). \tag{22}$$

We consider the vectors $a=\lambda(x+z_1)$ and $b=\lambda(x+z_2)+\lambda(z_1-z_2)$. So, (22) says that $a \prec b$. By making use of Lemma 1 with the aid of (21)) and (18)–(20), and by adding the vector $c=\lambda(y)-\lambda(z_1)=\lambda(y-z_1)$ to the both sides of the inequality $a \prec b$, we establish the inequality

$$\begin{aligned} l_1 &= \lambda(x+z_1)+\lambda(y)-\lambda(z_1) \\ &\prec \lambda(x+z_2)+\lambda(z_1-z_2)+\lambda(y)-\lambda(z_1) \\ &= \lambda(x+z_2)+\lambda(z_1)-\lambda(z_2)+\lambda(y)-\lambda(z_1) = l_2, \end{aligned}$$

completing the proof of inequality (16) in case (I).

Case (II). From condition (II) by Lemma 3, we get

$$\lambda(z_1-y)=\lambda(z_1)-\lambda(y), \tag{23}$$

$$\lambda(y-z_2)=\lambda(y)-\lambda(z_2). \tag{24}$$

Therefore, by Lidskii-Wielandt and Ky-Fan inequalities (4) and (3), we derive

$$\begin{aligned} l_1 &= \lambda(x+z_1)+\lambda(y)-\lambda(z_1) = \lambda(x+z_1)-\lambda(z_1-y) \\ &\prec \lambda((x+z_1)-(z_1-y)) = \lambda((x+z_2)+(y-z_2)) \\ &\prec \lambda(x+z_2)+\lambda(y-z_2) = \lambda(x+z_2)+\lambda(y)-\lambda(z_2) = l_2. \end{aligned}$$

This completes the proof of (16) in case (II).

Case (III). According to Lemma 3 condition (III) implies that

$$\lambda(z_1-y)=\lambda(z_1)-\lambda(y), \tag{25}$$

$$\lambda(z_2-y)=\lambda(z_2)-\lambda(y). \tag{26}$$

$$\lambda(z_1-z_2)=\lambda(z_1)-\lambda(z_2). \tag{27}$$

We consider the matrices $a=x+z_1$, $b=z_1-z_2$ and $c=z_2-y$.

The requirements (i)–(ii) of Lemma 4 are fulfilled, because, by (III), the matrices

$$a-b-c=(x+z_1)-(z_1-z_2)-(z_2-y)=x+y \quad \text{and} \quad c=z_2-y \quad \text{are SD.}$$

On the other hand, by (25)–(27),

$$\begin{aligned} \lambda(b+c) &= \lambda(z_1-z_2+z_2-y) = \lambda(z_1-y) = \lambda(z_1)-\lambda(y) \\ &= \lambda(z_1)-\lambda(z_2)+\lambda(z_2)-\lambda(y) = \lambda(z_1-z_2)+\lambda(z_2-y) = \lambda(b)+\lambda(c). \end{aligned}$$

Thus matrices b and c are SD (see Lemma 2).

Therefore we are allowed to apply Lemma 4. In consequence, from (15) we obtain the following majorization inequality

$$\lambda(x+z_1)-\lambda(z_1-z_2)-\lambda(z_2-y)\prec\lambda((x+z_1)-(z_1-z_2))-\lambda(z_2-y).$$

For this reason, by (25)–(27), we can write

$$\begin{aligned} l_1 &= \lambda(x+z_1)+\lambda(y)-\lambda(z_1)=\lambda(x+z_1)-\lambda(z_1)+\lambda(z_2)-\lambda(z_2)+\lambda(y)\\ &= \lambda(x+z_1)-\lambda(z_1-z_2)-\lambda(z_2-y)\prec\lambda((x+z_1)-(z_1-z_2))-\lambda(z_2-y)\\ &= \lambda(x+z_2)-\lambda(z_2-y)=\lambda(x+z_2)+\lambda(y)-\lambda(z_2)=l_2. \end{aligned}$$

This finishes the proof of Theorem 1. $\square$

THEOREM 2. *Let* $x,y,z\in\mathbb{H}_n$.

(i) *If* y *and* $z-y$ *are SD, then*

$$\lambda(x+z)+\lambda(y)-\lambda(z)\prec\lambda(x+y). \tag{28}$$

(ii) *If* z *and* $y-z$ *are SD then*

$$\lambda(x+y)\prec\lambda(x+z)+\lambda(y)-\lambda(z)\prec\lambda(x)+\lambda(y). \tag{29}$$

Proof. (i). With the substitutions $z_1=z$ and $z_2=y$ in Theorem 1 condition (II) is satisfied. It is now sufficient to apply inequality (16) to get (28).

(ii). It is not hard to verify that assumption (I) of Theorem 1 is satisfied for $z_1=y$ and $z_2=z$. So, in order to obtain the left-hand side of (29), it is enough to use (16).

Also, it is sufficient to apply (16) to $z_1=z$ and $z_2=0$ to get the right-hand-side of inequality (29). Indeed, in this case condition (I) is satisfied, too. $\square$

REMARK 4. The assertion (29) in Theorem 2 means that

$$\varphi(y)\prec\varphi(z)\prec\varphi(0)\quad\text{for } z\in[0,y]_C,$$

where $\varphi(z)=\lambda(x+z)+\lambda(y)-\lambda(z)$ (see Remark 3).

COROLLARY 1. *If* $x,y\in\mathbb{H}_n$ *and* $z=y-I_n$, *where* I_n *is the* $n\times n$ *identity matrix, then the following inequalities hold:*

$$\lambda(x+y)\prec\lambda(x+z)+(1,1,\dots,1)^T\prec\lambda(x)+\lambda(y).$$

Proof. Observe that z and $y-z=I_n$ are SD. Use Theorem 2, item (ii), eq. (29). $\square$

The next result can be compared to Maligranda inequalities for norms (see Theorem A in Section 1).

COROLLARY 2. (Cf. [11, Cor. 4.1]) *Let* $x, y \in \mathbb{H}_n$. *Let* $0 \leqslant t \leqslant 1 \leqslant s$ *be given scalars. Then*

$$\begin{aligned}\lambda(x+sy)+(1-s)\lambda(y) &\prec \lambda(x+y)\\ &\prec \lambda(x+ty)+(1-t)\lambda(y) \prec \lambda(x)+\lambda(y).\end{aligned} \tag{30}$$

Proof. It is clearly seen that for $z = sy$ the matrices y and $z-y=(s-1)y$ are SD. Applying Theorem 2, eq. (28), gives the left-hand side inequality of (30).

To see the remaining inequalities of (30), we utilize (29) for $z = ty$. It is permitted, because $z = ty$ and $y - z = (1-t)y$ are SD. $\square$

For positive numbers p and q, a simple calculation shows that

$$\begin{aligned}&\lambda\left(x+\frac{p}{q}y\right)+\left(1-\frac{p}{q}\right)\lambda(y)\\ &=\lambda(x)+\lambda(y)-p\left[\lambda\left(\frac{x}{p}\right)+\lambda\left(\frac{y}{q}\right)-\lambda\left(\frac{x}{p}+\frac{y}{q}\right)\right].\end{aligned} \tag{31}$$

We now interpret Corollary 2 in terms of the numbers p and q.

COROLLARY 3. *Let* $x, y \in \mathbb{H}_n$.
If $0 < p \leqslant q$ *then*

$$\begin{aligned}\lambda(x+y) &\prec \lambda(x)+\lambda(y)-p\left[\lambda\left(\frac{x}{p}\right)+\lambda\left(\frac{y}{q}\right)-\lambda\left(\frac{x}{p}+\frac{y}{q}\right)\right]\\ &\prec \lambda(x)+\lambda(y).\end{aligned} \tag{32}$$

If $0 < q \leqslant p$ *then*

$$\lambda(x+y) \succ \lambda(x)+\lambda(y)-p\left[\lambda\left(\frac{x}{p}\right)+\lambda\left(\frac{y}{q}\right)-\lambda\left(\frac{x}{p}+\frac{y}{q}\right)\right]. \tag{33}$$

Proof. If $0 < p \leqslant q$ then it is a consequence of (30) that

$$\lambda(x+y) \prec \lambda\left(x+\frac{p}{q}y\right)+\left(1-\frac{p}{q}\right)\lambda(y) \prec \lambda(x)+\lambda(y). \tag{34}$$

Combining this with equality (31) yields (32).

If $0 < q \leqslant p$ then it follows from the first inequality of (30) that

$$\lambda\left(x+\frac{p}{q}y\right)+\left(1-\frac{p}{q}\right)\lambda(y) \prec \lambda(x+y). \tag{35}$$

This and (31) lead to (33), as claimed. $\square$

Inequalities (32) and (33) are eigenvalue analogues of (1) and (2), respectively.

Concerning the expression in the square brackets of (32) and (33), it can be viewed from the Ky Fan's inequality that

$$\lambda\left(\frac{x}{p}+\frac{y}{q}\right) \prec \lambda\left(\frac{x}{p}\right)+\lambda\left(\frac{y}{q}\right).$$

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