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RATE OF GROWTH OF DISTRIBUTIONALLY CHAOTIC FUNCTIONS

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Abstract. We investigate the permissible growth rates of functions that are distributionally chaotic with respect to differentiation operators. We improve on the known growth estimates for D-distributionally chaotic entire functions, where growth is in terms of average L^p-norms on spheres of radius $r>0$ as $r\to\infty$, for $1\leqslant p\leqslant\infty$. We compute growth estimates of $\partial/\partial x_k$-distributionally chaotic harmonic functions in terms of the average L^2-norm on spheres of radius $r>0$ as $r\to\infty$. We also calculate sup-norm growth estimates of distributionally chaotic harmonic functions in the case of the partial differentiation operators D^α.

1. Introduction

The term *chaos* first appeared in mathematical literature in an article by Li and Yorke [27], where they studied the dynamical behaviour of interval maps with period three. Schweizer and Smítal [32] subsequently introduced the stronger notion of *distributional chaos* for self-maps of a compact interval.

A continuous map $g\colon Y\to Y$ on a metric space (Y,d) is said to be *Li-Yorke chaotic* if there exists an uncountable set $\Gamma\subset Y$ such that for every pair $(x,y)\in\Gamma\times\Gamma$ of distinct points, we have

$$\liminf_{n\to\infty} d\left(g^n(x),g^n(y)\right)=0 \quad\text{and}\quad \limsup_{n\to\infty} d\left(g^n(x),g^n(y)\right)>0.$$

In this case, Γ is called a *scrambled set* for g and each such pair (x,y) is called a *Li-Yorke pair* for g. This definition captures the behaviour of orbits which are proximal without being asymptotic.

Our setting will be a Fréchet space X endowed with an increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms that define the metric

$$d(x,y):=\sum_{k=1}^{\infty}2^{-k}\frac{\|x-y\|_k}{1+\|x-y\|_k}$$

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under which X is complete, where $x, y \in X$. We let T denote a continuous linear operator on X.

The connection between Li-Yorke chaos and the linear dynamical property of irregularity was identified in [10]. We say that $x \in X$ is an *irregular vector* for T if there exist $m \in \mathbb{N}$ and increasing sequences (j_k) and (n_k) of positive integers such that

$$\lim_{k\to\infty} T^{j_k}x = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T^{n_k}x\|_m = \infty.$$

This notion was introduced by Beauzamy [9] for Banach spaces to describe the local aspects of the dynamics of pairs of vectors and it was generalised to the Fréchet space setting in [14].

We recall if there exists $x \in X$ such that its T-orbit is dense in X, that is

$$\overline{\{T^n x : n \geqslant 0\}} = X,$$

then T is said to be *hypercyclic* and such an $x \in X$ is known as a *hypercyclic vector*. It is well known that hypercyclic vectors are irregular. Comprehensive introductions to the topic of hypercyclicity can be found in the monographs [7] and [24], and a survey of some recent advances in the area can be found in [21].

A natural strengthening of Li-Yorke chaos was introduced by Schweizer and Smítal [32] with the notion of distributional chaos for interval maps. We first recall that the upper and lower densities of a set $A \subset \mathbb{N}$ are defined, respectively, as

$$\overline{\text{dens}}(A) := \limsup_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n},$$
$$\underline{\text{dens}}(A) := \liminf_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.$$

For a continuous map $g\colon Y \to Y$ on a metric space (Y, d), points $x, y \in Y$ and $\delta > 0$, we define

$$F_{x,y}(\delta) := \underline{\text{dens}}(\{n \in \mathbb{N} : d(g^n(x), g^n(y)) < \delta\})$$

and

$$F^*_{x,y}(\delta) := \overline{\text{dens}}(\{n \in \mathbb{N} : d(g^n(x), g^n(y)) < \delta\}).$$

If the pair (x, y) satisfy $F^*_{x,y} \equiv 1$ and $F_{x,y}(\varepsilon) = 0$ for some $\varepsilon > 0$, then (x, y) is called a *distributionally chaotic pair*. The map g is said to be *distributionally chaotic* if there exists an uncountable set $\Gamma \subset Y$ such that every distinct pair $(x, y) \in \Gamma \times \Gamma$ is a distributionally chaotic pair for g.

The study of distributional chaos in the linear dynamical setting was initiated in [29] and it is intrinsically connected to the following properties. The T-orbit of $x \in X$ is said to be *distributionally near to 0* if there exists $A \subset \mathbb{N}$ with $\overline{\text{dens}}(A) = 1$ such that

$$\lim_{n\in A} T^n x = 0.$$

We say $x \in X$ has a *distributionally unbounded orbit* if there exist $m \in \mathbb{N}$ and $B \subset \mathbb{N}$ with $\overline{\text{dens}}(B) = 1$ such that

$$\lim_{n\in B} \|T^n x\|_m = \infty.$$

Combining these properties, $x \in X$ is defined to be a *distributionally irregular vector* for T if its orbit is both distributionally unbounded and distributionally near to 0. This strengthening of irregularity was introduced in [10].

In the Fréchet space setting, it follows from results in [10] and [13] that T admits a distributionally irregular vector if and only if T is distributionally chaotic. Hence, in the sequel our study focuses on distributionally irregular vectors. Distributional chaos has been investigated from many aspects, for instance in [1, 8, 14, 15, 18, 30, 34, 35, 36, 37, 12].

An example of a map that admits a distributionally irregular vector is the differentiation operator $D\colon f \mapsto f'$, acting on the space $H(\mathbb{C})$ of entire functions on $\mathbb{C}$ (this follows from [13, Corollary 17]). Bernal and Bonilla [11] computed growth estimates for D-irregular and D-distributionally irregular entire functions, where growth is in terms of average L^p-norms, for $1 \leqslant p \leqslant \infty$, on spheres of radius $r > 0$ as $r \to \infty$.

We note that permissible growth rates of D-hypercyclic and D-frequently hypercyclic entire functions have previously been investigated in [16, 17, 19, 20, 23, 28, 31, 33]. We recall the notion of frequent hypercyclicity was introduced by Bayart and Grivaux [6], where they defined $T\colon X \to X$ to be *frequently hypercyclic* if there exists $x \in X$ such that for any nonempty open subset $U \subset X$ it holds that

$$\underline{\text{dens}}(\{n : T^n x \in U\}) > 0.$$

Such an $x \in X$ is called a *frequently hypercyclic vector* for T.

In the setting of the space $\mathscr{H}(\mathbb{R}^N)$ of harmonic functions on $\mathbb{R}^N$, for $N \geqslant 2$, Aldred and Armitage [2] considered the linear dynamical properties of partial differentiation operators

$$\frac{\partial}{\partial x_k}\colon \mathscr{H}(\mathbb{R}^N) \to \mathscr{H}(\mathbb{R}^N),$$

where $1 \leqslant k \leqslant N$. They identified sharp L^2-growth rates, on spheres of radius $r > 0$ as $r \to \infty$, of harmonic functions that are universal (and hence hypercyclic) with respect to $\partial/\partial x_k$. Growth estimates in the frequently hypercyclic case were computed by Blasco et al. [16] and sharp growth rates were subsequently identified in [22].

Growth estimates with respect to the sup-norm, on spheres of radius $r > 0$ as $r \to \infty$, were computed by Aldred and Armitage [3] for harmonic functions that are universal (and hence hypercyclic) for general partial differentiation operators

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. The frequently hypercyclic case was subsequently investigated by Blasco et al. [16].

In this article we compute permissible growth rates of irregular and distributionally irregular functions. In Section 3 we improve the growth estimates from [11] for distributionally irregular entire functions and we also provide lower estimates. In Section 4 we investigate average L^2-growth estimates of irregular and distributionally irregular harmonic functions with respect to partial differentiation operators $\partial/\partial x_k$. Then in Section 5 we compute sup-norm growth rates of distributionally irregular harmonic functions in the case of the partial differentiation operators D^α.

2. Preliminaries

In this section we collect some results that are required in the sequel.

We recall that absolutely Cesàro bounded operators cannot be distributionally irregular. For a Banach space X, the continuous linear operator $T: X \to X$ is said to be *absolutely Cesàro bounded* if there exists a constant $C > 0$ such that

$$\sup_{N\in\mathbb{N}} \frac{1}{N} \sum_{j=1}^{N} \|T^j x\| \leqslant C\|x\|$$

for all $x \in X$. If the orbit of $x \in X$ is distributionally unbounded, then it was proven in [15, Proposition 20] that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N} \|T^j x\| = \infty.$$

The following estimate will be needed, it can be found in [16, Lemma 2.2].

LEMMA 2.1. *Let $0 < \alpha \leqslant 2$ and $\beta \in \mathbb{R}$. Then there exists some constant $C > 0$ such that, for all $r > 0$,*

$$\sum_{n=0}^{\infty} \frac{r^{\alpha n}}{(n+1)^{\beta} n!^{\alpha}} \leqslant C \frac{e^{\alpha r}}{r^{(\alpha+2\beta-1)/2}}.$$

We will need a technical result whose proof follows the argument of [16, Theorem 2.4].

LEMMA 2.2. *Let $\alpha, \beta > 0$. Then there exists some $M > 0$ such that, for any non-negative sequence (x_n),*

$$\sup_{m\geqslant 1} \frac{1}{m} \sum_{n=1}^{m} x_n \leqslant M \sup_{R>0} \sum_{n=1}^{\infty} x_n \frac{R^{\alpha n+\beta} e^{-\alpha R}}{n!^{\alpha} n^{\beta-\alpha/2+1/2}}. \tag{2.1}$$

Proof. We consider the functions

$$g_n(R) := \frac{R^{\alpha n+\beta} e^{-\alpha R}}{n!^{\alpha} n^{\beta-\alpha/2+1/2}}, \quad R > 0, \quad n \in \mathbb{N}.$$

The function g_n attains its maximum at $a_n := n + \beta/\alpha$. Moreover by Stirling's formula

$$g_n(a_n) := \frac{(n+\beta/\alpha)^{\alpha n+\beta} e^{-\alpha(n+\beta/\alpha)}}{n!^{\alpha} n^{\beta} n^{-\alpha/2} n^{1/2}} \sim \frac{1}{\sqrt{n}}.$$

The function g_n has an inflection point at $b_n := a_n + \sqrt{n/\alpha+\beta/\alpha^2}$ and, since $\lim_{R\to\infty} g_n(R) = 0$, we have that $g_n(R) \geqslant h_n(R)$ for each $R \in I_n = [a_n, b_n]$, where h_n is the affine map such that $h_n(a_n) = g_n(a_n)$ and $h_n(b_n) = 0$. We fix $m_0 \in \mathbb{N}$ such that

$$m < a_n = n+\beta/\alpha < b_n = a_n + \sqrt{n/\alpha+\beta/\alpha^2} < 3m$$

for $m \geqslant m_0$ and for $n = m,\dots,2m$. We have that

$$\begin{aligned} 2m\left(\sup_{R>0}\sum_{n=1}^{\infty} x_n g_n(R)\right) &\geqslant \int_{[m,3m]}\left(\sum_{n=m}^{2m} x_n g_n(s)\right) ds \geqslant \sum_{n=m}^{2m} x_n \int_{I_n} g_n(s)ds \\ &\geqslant \sum_{n=m}^{2m} x_n \int_{I_n} h_n(s)ds \geqslant C' \sum_{n=m}^{2m} x_n \frac{\sqrt{n/\alpha+\beta/\alpha^2}}{\sqrt{n}} \geqslant \frac{C'}{\sqrt{\alpha}} \sum_{n=m}^{2m} x_n \end{aligned}$$

for $m \geqslant m_0$, where $C' > 0$ is a constant independent of m and (x_n). Therefore we find $M_1 > 0$ with

$$\frac{1}{m}\sum_{n=m}^{2m} x_n \leqslant M_1 \sup_{R>0}\sum_{n=1}^{\infty} x_n \frac{R^{\alpha n+\beta}e^{-\alpha R}}{n!^{\alpha} n^{\beta-\alpha/2+1/2}}, \quad \forall m \geqslant m_0.$$

On the other hand,

$$\sum_{n=m_0}^{m}\left(\frac{1}{n}\sum_{j=n+1}^{2n} x_j\right) \geqslant \sum_{j=m_0+1}^{m} x_j \left(\sum_{\max\{m_0, j/2\}\leqslant n<j} \frac{1}{n}\right) \geqslant \frac{1}{m_0}\sum_{j=m_0+1}^{m} x_j,$$

so we get

$$\begin{aligned} \frac{1}{m}\left(\sum_{j=m_0+1}^{m} x_j\right) &\leqslant \frac{m_0}{m}\sum_{n=m_0}^{m}\left(\frac{1}{n}\sum_{j=n+1}^{2n} x_j\right) \\ &\leqslant m_0 M_1 \sup_{R>0}\sum_{n=1}^{\infty} x_n \frac{R^{\alpha n+\beta}e^{-\alpha R}}{n!^{\alpha} n^{\beta-\alpha/2+1/2}} \end{aligned}$$

for all $m \geqslant m_0$, and we find $M > 0$ independent of (x_n) with

$$\sup_{m\geqslant 1}\frac{1}{m}\sum_{n=1}^{m} x_n \leqslant M \sup_{R>0}\sum_{n=1}^{\infty} x_n \frac{R^{\alpha n+\beta}e^{-\alpha R}}{n!^{\alpha} n^{\beta-\alpha/2+1/2}}. \quad \square$$

The following is a slight improvement of a result that is a consequence of Proposition 7 and Theorems 15 and 19 in [13]. For the convenience of the reader we outline the main steps of the proof.

THEOREM 2.3. (Bernardes et al. [13]) *Let X be a Fréchet space, let $T: X \to X$ be a continuous linear operator, and assume that Y is a separable Fr échet space that is continuously embedded in X. Suppose that:*

(a) *There exists a dense subset* Y_0 *of* Y *with* $T^n(Y_0) \subset Y$ *for each* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} T^n y = 0$ *in* Y *for all* $y \in Y_0$.

(b) *There exist a subset* Y_1 *of* Y, *a map* $S\colon Y_1 \to Y_1$ *with* $TSy = y$ *on* Y_1, *and a vector* $z \in Y_1 \setminus \{0\}$ *such that* $\sum_{n=1}^{\infty} T^n z$ *and* $\sum_{n=1}^{\infty} S^n z$ *converge unconditionally in* X *and* Y, *respectively.*

Then there exists a dense subset of vectors in Y *which are distributionally irregular for* $T\colon X \to X$.

Proof. We let $(\|\cdot\|_k)_{k\in\mathbb{N}}$ denote an increasing fundamental sequence of seminorms on X, and without loss of generality we assume it satisfies

$$\|Tx\|_k \leqslant \|x\|_{k+1} \text{ for all } x \in X \text{ and } k \in \mathbb{N}.$$

We fix another increasing fundamental sequence $\left(\|\cdot\|_k'\right)_{k\in\mathbb{N}}$ of seminorms in Y, with $\|y\|_k \leqslant \|y\|_k'$ for every $y \in Y$.

Arguing as in [13, Theorem 19], we define $w_{k_0} := \sum_{n=1}^{\infty} T^{k_0 n} z + z + \sum_{n=1}^{\infty} S^{k_0 n} z$, where $w_{k_0} \neq 0$ if k_0 is sufficiently large and $T^{k_0} w_{k_0} = w_{k_0}$. Let $y_k := \sum_{n=k}^{\infty} S^{k_0 n} z \in Y$, $k \in \mathbb{N}$. Then $y_k \to 0$ in Y and

$$T^{k_0 j} y_k = \sum_{n=1}^{j-k} T^{k_0 n} z + z + \sum_{n=1}^{\infty} S^{k_0 n} z \to w_{k_0}$$

in X as $j \to \infty$. For $0 \leqslant \ell < k_0$ we have

$$\lim_{j\to\infty} T^{\ell + k_0 j} y_k = T^{\ell} w_{k_0} \text{ in } X$$

and hence $\{T^{\ell} w_{k_0} : 0 \leqslant \ell < k_0\}$ are accumulation points of the orbit of y_k.

Let $m \in \mathbb{N}$ with $\left\|T^{\ell} w_{k_0}\right\|_m \neq 0$, $\ell = 0, \ldots, k_0 - 1$. We define

$$\varepsilon := \frac{1}{2} \min\left\{\left\| T^{\ell} w_{k_0} \right\|_m \ : \ 0 \leqslant \ell < k_0\right\} > 0.$$

Then there exists an increasing sequence (N_k) of positive integers such that

$$\lim_{k\to\infty} \frac{1}{N_k} \left|\left\{1 \leqslant j \leqslant N_k : \left\| T^j y_k \right\|_m > \varepsilon\right\}\right| = 1. \tag{2.2}$$

Without loss of generality we will assume that $m = 1$. We adapt now the proof of [13, Theorem 15] to show that there exists a dense subset of vectors in Y which are distributionally irregular for $T\colon X \to X$. Indeed, by (2.2) and since the sequences (y_k) and $(T^i u)$, $u \in Y_0$, converge to 0 in Y, we can construct inductively a sequence (x_k) of vectors in Y_0 with $\|x_k\|_k' \leqslant 1$, $k \in \mathbb{N}$, and an increasing sequence (n_k) of positive integers such that

$$\left|\left\{1 \leqslant i \leqslant n_k : \|T^i x_k\|_1 > k2^k\right\}\right| > n_k \left(1 - \frac{1}{k^2}\right), \tag{2.3}$$

$$\left| \left\{ 1 \leqslant i \leqslant n_k : \|T^i x_s\|_k' < \frac{1}{k} \right\} \right| > n_k \left(1 - \frac{1}{k^2} \right), \; s = 1, \ldots, k-1. \tag{2.4}$$

We consider an increasing sequence (r_j) of positive integers such that

$$r_{j+1} \geqslant 1 + r_j + n_{r_j+1} \quad \text{for all } j \in \mathbb{N}. \tag{2.5}$$

We fix $\alpha \in \{0,1\}^{\mathbb{N}}$ defined by $\alpha_n = 1$ if and only if $n = r_j$ for some $j \in \mathbb{N}$, and we define the vector

$$u := \sum_i \frac{\alpha_i}{2^i} x_i = \sum_j \frac{\alpha_{r_j}}{2^{r_j}} x_{r_j}$$

with $u \in Y$ by the fact that $\|x_k\|_k' \leqslant 1$, $k \in \mathbb{N}$. The argument of the proof of [13, Theorem 15] yields that $u \in Y$ is a distributionally irregular vector for $T : X \to X$ since $\|y\|_k \leqslant \|y\|_k'$ for every $y \in Y$.

Finally, the set $u + Y_0$ forms a dense subset of vectors in Y which are distributionally irregular for $T : X \to X$, as desired. $\square$

3. Growth of distributionally irregular entire functions

In this section we consider growth of entire functions that are distributionally irregular with respect to the differentiation operator D acting on the space $H(\mathbb{C})$ of entire functions. We first introduce the pertinent definitions that allow us to precisely specify how growth is measured.

For an entire function $f \in H(\mathbb{C})$ and $1 \leqslant p < \infty$, the average L^p-norm is defined as

$$M_p(f,r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}$$

where $r > 0$ and we denote the sup-norm of f by

$$M_\infty(f,r) = \sup_{|z|=r} |f(z)|, \quad r > 0.$$

Optimal growth rates of D-hypercyclic entire functions were identified by Grosse-Erdmann [23] and Shkarin [33]. They proved if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\varphi(r) \to \infty$ as $r \to \infty$, then there exists a D-hypercyclic entire function $f \in H(\mathbb{C})$ such that

$$M_\infty(f,r) \leqslant \varphi(r) \frac{e^r}{\sqrt{r}} \tag{3.1}$$

for r sufficiently large. This growth is optimal, since for the critical rate of $e^r/\sqrt{r}$, there does not exist a D-hypercyclic entire function $f \in H(\mathbb{C})$ such that

$$M_\infty(f,r) \leqslant c \frac{e^r}{\sqrt{r}}, \quad \text{for } r > 0, \tag{3.2}$$

where $c > 0$ is a constant. As noted by Blasco et al. [16, Theorem 2.1], the above growth results extend to M_p-averages for all $1 \leqslant p \leqslant \infty$.

As expected, D-frequently hypercyclic entire functions must grow faster than in the hypercyclic case. Permissible growth of D-frequently hypercyclic entire functions was investigated in [16], and optimal growth was identified by Drasin and Saksman [19]. In [19] they proved for any constant $C>0$, that there exists a D-frequently hypercyclic function $f \in H(\mathbb{C})$ such that

$$M_\infty(f,r) \leqslant C\frac{e^r}{r^{1/4}}, \qquad \text{for all } r>0.$$

The above growth result naturally applies to M_p-averages for $1<p\leqslant\infty$. However, we note that in the case $p=1$, Bonet and Bonilla [17] had previously identified that there exists a D-frequently hypercyclic entire function $f\in H(\mathbb{C})$ with

$$M_1(f,r) \leqslant \varphi(r)\frac{e^r}{r^{1/2}}$$

for all $r>0$ and where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is any function such that $\varphi(r)\to\infty$ as $r\to\infty$.

Bernal and Bonilla [11] subsequently identified optimal growth estimates for D-irregular entire functions. They proved that the critical rate of growth for D-irregular entire functions is the same as given by (3.1) and (3.2) in the D-hypercyclic case.

Initial growth estimates for D-distributionally irregular functions were also obtained in [11]. They proved for $1\leqslant p\leqslant\infty$ and $a=(2\max\{2,p\})^{-1}$, that for every $\varepsilon>0$ there exists a D-distributionally irregular function $f\in H(\mathbb{C})$ such that

$$M_p(f,r) \leqslant C\frac{e^r}{r^{a-\varepsilon}}$$

for some constant $C>0$.

Here we improve the permissible growth estimates for D-distributionally irregular entire functions, while also computing estimates for lower rates of growth.

THEOREM 3.1. *Let* $1\leqslant p\leqslant\infty$.

(i) Let $a=1/(2\max\{2,p\})$. *For any* $\varphi\colon\mathbb{R}_+\to\mathbb{R}_+$ *with* $\varphi(r)\to\infty$ *as* $r\to\infty$, *there exists a* D*-distributionally irregular entire function* f *with*

$$M_p(f,r) \leqslant \varphi(r)\frac{e^r}{r^a}$$

for $r>0$ *sufficiently large.*

(ii) Let $a=1/(2\min\{2,p\})$. *There does not exist a* D*-distributionally irregular entire function* f *that satisfies*

$$M_p(f,r) \leqslant c\frac{e^r}{r^a} \tag{3.3}$$

where $c>0$ *is a constant and for* $r>0$ *sufficiently large.*

Proof. (i) Since

$$M_p(f,r) \leqslant M_2(f,r)$$

for $1 \leqslant p < 2$ we need only prove the result for $p \geqslant 2$. Let $2 \leqslant p \leqslant \infty$ and we assume without loss of generality that $\inf_{r>0} \varphi(r) > 0$. Our strategy is to apply Theorem 2.3, so we consider

$$Y := \left\{ f \in H(\mathbb{C}) \; : \; \lim_{r\to\infty} \frac{M_p(f,r)r^{1/2p}}{\varphi(r)e^r} = 0 \right\}$$

endowed with the natural sup-norm which we denote by $\|\cdot\|_Y$. We note that $(Y, \|\cdot\|_Y)$ is Banach space which is continuously embedded in $H(\mathbb{C})$. Further note that the functions $f \in Y$ satisfy the desired growth condition.

We let $Y_0 = Y_1$ be the space of polynomials and we note that Y_0 is dense in Y. It follows immediately that Theorem 2.3 (a) is satisfied.

Next we check Theorem 2.3 (b). We define the mapping $S \colon Y_1 \to Y_1$ for $g \in Y_1$ as

$$Sg(z) = \int_0^z g(\xi)\,\mathrm{d}\xi.$$

For all $g \in Y_1$ we have that $DSg = g$, and since $\sum_{n=1}^{\infty} D^n g$ is a finite series it converges unconditionally.

It remains to show that $\sum_{n=1}^{\infty} S^n g$ converges unconditionally in Y for any polynomial g. (This calculation appears in [16, Theorem 2.3], but for the convenience of the reader we recall the argument here.)

It suffices to consider monomials $g(z) = z^k$, $k \in \mathbb{N}$, which gives

$$\sum_{n=1}^{\infty} S^n g(z) = \sum_{n=1}^{\infty} \frac{k!}{(k+n)!} z^{k+n}$$

and thus we need to prove that $\sum_{n=1}^{\infty} z^n/n!$ converges unconditionally in Y.

To this end, let $\varepsilon > 0$ and $N \in \mathbb{N}$. By the Hausdorff-Young Inequality (cf. [25]), we obtain for any finite subset $F \subset \mathbb{N}$ that

$$M_p\left(\sum_{n\in F} \frac{z^n}{n!}, r\right) \leqslant \left(\sum_{n\in F} \frac{r^{qn}}{n!^q}\right)^{1/q},$$

where q is the conjugate exponent of p, i.e. $1/p + 1/q = 1$. Hence, if $F \cap \{0,1,\ldots,N\} = \varnothing$, then

$$\left\|\sum_{n\in F} \frac{z^n}{n!}\right\|_Y \leqslant \left(\sup_{r>0} \frac{r^{q/2p}}{\varphi(r)^q e^{qr}} \sum_{n>N} \frac{r^{qn}}{n!^q}\right)^{1/q}.$$

We choose $R > 0$ such that $\varphi(r)^q \geqslant 1/\varepsilon$ for $r \geqslant R$. Then it follows that

$$\sup_{r\leqslant R} \frac{r^{q/2p}}{\varphi(r)^q e^{qr}} \sum_{n>N} \frac{r^{qn}}{n!^q} \leqslant \frac{R^{q/2p}}{\inf_{r>0} \varphi(r)^q} \sum_{n>N} \frac{R^{qn}}{n!^q} \to 0$$

as $N \to \infty$. Moreover, Lemma 2.1 gives that

$$\sup_{r \geqslant R} \frac{1}{\varphi(r)^q} \frac{r^{q/2p}}{e^{qr}} \sum_{n>N} \frac{r^{qn}}{n!^q} \leqslant C\varepsilon$$

for any $N \in \mathbb{N}$, where C is a constant depending only on q. Thus,

$$\left\| \sum_{n \in F} \frac{z^n}{n!} \right\|_Y^q \leqslant (1+C)\varepsilon$$

if $\min F > N$ and N is sufficiently large, so that $\sum_{n=1}^{\infty} z^n/n!$ converges unconditionally in Y.

(ii) For $p = 1$ the result follows from [11, Theorem 7]. Moreover since

$$M_2(f,r) \leqslant M_p(f,r)$$

for $2 < p \leqslant \infty$, it suffices to prove the result for $p \leqslant 2$. So we assume $1 < p \leqslant 2$ and that $f \in H(\mathbb{C})$ satisfies (3.3).

Let $B(r)$ denote the open ball of radius r which is centred at the origin. We define the translation $f_a(z) := f(z+a)$, for $a \in \mathbb{C}$, which is an entire function with $f_a(0) = f(a)$.

We let $R > r$ and it follows from the Hausdorff-Young inequality (cf. [25]) and (3.3) that for $a \in B(r)$

$$\begin{aligned} \left(\sum_{n=0}^{\infty} \left(\frac{|D^n f(a)|}{n!} R^n \right)^q \right)^{1/q} &= \left(\sum_{n=0}^{\infty} \left(\frac{|D^n f_a(0)|}{n!} R^n \right)^q \right)^{1/q} \\ &\leqslant M_p(f_a, R) \leqslant M_p(f, R+r) \leqslant c \frac{e^{R+r}}{(R+r)^{1/2p}} \end{aligned}$$

where q is the conjugate exponent of p with $1/p + 1/q = 1$.

So it follows that

$$\sum_{n=0}^{\infty} |D^n f(a)|^q \frac{R^{qn+q/2p}(1+r/R)^{q/2p} e^{-qR}}{c^q n!^q e^{qr}} \leqslant 1. \tag{3.4}$$

We now apply Lemma 2.2 with $\alpha = q$ and $\beta = q/2p$ to obtain that there exists $C > 0$ depending only on r such that

$$\frac{1}{m} \sum_{n=0}^{m} |D^n f(a)|^q \leqslant C.$$

Next let $S(r) = \{z \in \mathbb{C} : |z| = r\}$ denote the sphere of radius r centered at the origin, $k \in \mathbb{N}$ arbitrary, and let $a_1, \ldots, a_k \in S(r)$. We take the averages

$$\frac{1}{k} \sum_{j=1}^{k} \frac{1}{m} \sum_{n=0}^{m} |D^n f(a_j)|^q \leqslant C$$

and hence

$$\frac{1}{m}\sum_{n=0}^{m}\left(\frac{1}{k}\sum_{j=1}^{k}\left|D^n f(a_j)\right|^q\right) \leqslant C.$$

Note that, if the $a_1,\dots,a_k \in S(r)$ are uniformly distributed,

$$\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k}\left|D^n f(a_j)\right|^q = c_r M_q^q(D^n f,r)$$

where c_r is a constant depending only on r. So it follows that the average

$$\frac{1}{m}\sum_{n=0}^{m} M_q^q(D^n f,r)$$

is bounded and thus f cannot be a distributionally unbounded function for the differentiation operator. □

Although Theorem 3.1 is an improvement on the previously known growth estimates, it naturally leads to the following question (originally posed in [11, Problem 12]).

QUESTION 1. What is the critical order of growth of D-distributionally irregular entire functions?

Since D-hypercyclic and D-irregular entire functions share the same critical growth, one might expect that the optimal growth of D-distributionally irregular entire functions is similar to the frequently hypercyclic case.

4. Growth of irregular and distributionally irregular harmonic functions

We now turn our attention to the permissible growth of harmonic functions that are irregular and distributionally irregular with respect to partial differentiation operators. We begin by recalling some notation and the background results relevant to this topic. Then we collect some auxiliary results (Subsection 4.1) that we require in order to calculate the growth rates (Subsection 4.2).

The space $\mathscr{H}(\mathbb{R}^N)$ of harmonic functions on $\mathbb{R}^N$, for $N \geqslant 2$, is a Fréchet space when equipped with the complete metric

$$d(g,h) := \sum_{n=1}^{\infty} 2^{-n}\frac{|g-h|_{S(n)}}{1+|g-h|_{S(n)}}$$

for $g,h \in \mathscr{H}(\mathbb{R}^N)$, and it corresponds to the topology of local uniform convergence. Above we set $|f|_{S(n)} = \sup_{|x|=n}|f(x)|$ for $f \in \mathscr{H}(\mathbb{R}^N)$.

We denote by $B(x,r)$ and $S(x,r)$, respectively, the open ball and the sphere of radius r (in the euclidean metric) with centre at $x \in \mathbb{R}^N$. When they are centred at the origin of $\mathbb{R}^N$ we simply write $B(r)$ and $S(r)$.

The sup-norm of $h \in \mathscr{H}(\mathbb{R}^N)$ on $S(r)$ is defined as

$$M_\infty(h,r) = \sup_{\|x\|=r} |h(x)|.$$

Let σ_r denote the normalised $(N-1)$-dimensional measure on $S(r)$, so that $\sigma_r(S(r)) = 1$. The L^2-average of $h \in \mathscr{H}(\mathbb{R}^N)$ on $S(r)$ is given by

$$M_2(h,r) = \left(\int_{S(r)} |h|^2 \, \mathrm{d}\sigma_r \right)^{1/2}$$

where $r > 0$ and the corresponding inner product is defined by

$$\langle g,h \rangle_r = \int_{S(r)} gh \, \mathrm{d}\sigma_r, \quad g,h \in \mathscr{H}(\mathbb{R}^N).$$

The following sharp growth rates of harmonic functions that are hypercyclic, with respect to partial differentiation operators, follow from the work of Aldred and Armitage [2].

(I) If $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\varphi(r) \to \infty$ as $r \to \infty$, then there exists a $\partial/\partial x_k$-hypercyclic function $h \in \mathscr{H}(\mathbb{R}^N)$ with

$$M_2(h,r) \leqslant \varphi(r) \frac{e^r}{r^{(N-1)/2}}$$

for $r > 0$ sufficiently large.

(II) Let $\alpha \in \mathbb{N}^N$. There does not exist a D^α-hypercyclic harmonic function $h \in \mathscr{H}(\mathbb{R}^N)$ that satisfies

$$M_2(h,r) \leqslant C \frac{e^r}{r^{(N-1)/2}}$$

for $r > 0$ and any constant $C > 0$.

Permissible growth of $\partial/\partial x_k$-frequently hypercyclic harmonic functions was considered by Blasco et al. [16, Theorem 4.2], and subsequently the following optimal growth rates were identified in [22]: for any constant $C > 0$ there exists a $\partial/\partial x_k$-frequently hypercyclic harmonic function $h \in \mathscr{H}(\mathbb{R}^N)$ such that

$$M_2(h,r) \leqslant C \frac{e^r}{r^{N/2-3/4}}, \qquad \text{for all } r > 0.$$

4.1. Auxiliary results

We gather the notation and background from [2, 4, 5] that is required for our investigation.

We denote by $\mathscr{H}_m(\mathbb{R}^N)$ the space of homogeneous harmonic polynomials on $\mathbb{R}^N$ of homogeneity degree $m \geqslant 0$. The harmonic analogue of the standard power series representation of holomorphic functions states that any $h \in \mathscr{H}(\mathbb{R}^N)$ has a unique expansion of the form

$$h = \sum_{m=0}^{\infty} H_m \tag{4.1}$$

where $H_m \in \mathscr{H}_m(\mathbb{R}^N)$ for each $m \geqslant 0$ and the expansion converges in the metric d, cf. [5, Corollary 5.34]. Moreover, $\langle H_j, H_k \rangle_r = 0$ when $j \neq k$, so by orthogonality one has for any $r > 0$ that the L^2-average of (4.1) is

$$M_2^2(h,r) = \sum_{m=0}^{\infty} M_2^2(H_m, r).$$

We denote the dimension of $\mathscr{H}_m(\mathbb{R}^N)$ by $d_m = d_m(N)$ and it can be shown [5, Proposition 5.8] that $d_0 = 1$ for $N = 2$ and

$$d_m = \frac{N+2m-2}{N+m-2}\binom{N+m-2}{m} \tag{4.2}$$

for $N+m \geqslant 3$. It follows easily from (4.2) that

$$d_m = O(m^{N-2}) \tag{4.3}$$

as $m \to \infty$ (cf. [5, p. 107]). Moreover, for $\alpha \in \mathbb{N}^N$ with $|\alpha| = m$ and $H \in \mathscr{H}_m(\mathbb{R}^N)$, $D^\alpha H$ is constant and it follows from [2, Lemma 1] that

$$|D^\alpha H| \leqslant m!\sqrt{d_m}\, r^{-m} M_2(H,r) \tag{4.4}$$

for $r > 0$. Further details on the spaces $\mathscr{H}_m(\mathbb{R}^N)$ can be found in [5, Chapter 5] and [4, Chapter 2].

We also require an antiderivative for the partial differentiation operators $\partial/\partial x_k$ on $\mathscr{H}(\mathbb{R}^N)$, where $1 \leqslant k \leqslant N$. Suitable linear maps were defined by Aldred and Armitage [2] by using a specific orthogonal representation of harmonic polynomials constructed by Kuran [26]. We denote the n^{th} antiderivative, with respect to the coordinate x_k, by the linear map

$$P_{n,k} \colon \mathscr{H}_m(\mathbb{R}^N) \to \mathscr{H}_{m+n}(\mathbb{R}^N) \tag{4.5}$$

for $m, n \geqslant 0$. For our purposes we do not require to explicitly define the maps $P_{n,k}$, however we will utilise the pertinent properties which are contained in the following fundamental lemma taken from [2, Lemma 4].

LEMMA 4.1. *Let $m, n \geqslant 0$, $N \geqslant 2$ and $1 \leqslant k \leqslant N$. If $H \in \mathscr{H}_m(\mathbb{R}^N)$ then $P_{n,k}(H) \in \mathscr{H}_{m+n}(\mathbb{R}^N)$,*

$$\frac{\partial^n}{\partial x_k^n} P_{n,k}(H) = H$$

and

$$M_2^2(P_{n,k}(H), 1) \leqslant c_{n,m,N} M_2^2(H,1) \tag{4.6}$$

where

$$c_{n,m,N} = \frac{(N+2m-2)!}{n!(N+2m+n-3)!(N+2m+2n-2)}.$$

Similar to line (4.2) in [16], for fixed m we will use the simpler estimate

$$c_{n,m,N} \leqslant \frac{c_m}{(n+m)!^2(n+m+1)^{N-2}} \tag{4.7}$$

for $n \in \mathbb{N}$, where

$$c_m = c_m(N) = (N+2m-2)! \tag{4.8}$$

We note that the different maps $P_{n,k}$ are mutually compatible since for $H \in \mathscr{H}_m(\mathbb{R}^N)$ and $\ell, n \geqslant 0$ we have that

$$P_{\ell+n,k}(H) = P_{\ell,k}\left(P_{n,k}(H)\right).$$

A proof of this fact can be found in [22, Lemma 3.3]. In particular it holds that $\frac{\partial^n}{\partial x_k^n} P_{\ell,k}(H) = P_{\ell-n,k}(H)$ for $\ell > n$.

We also need the following lemma on inequalities between L^2-norms of harmonic functions with respect to N-spheres with different centres. This is essentially known, but for completeness we include a proof. The proof requires the Poisson integral and Harnack's inequality, which we recall below (full details can be found in [4, Sections 1.3 and 1.4]).

We let $\sigma_N = \sigma(S(1))$ be the surface area of the N-sphere $S(1)$, where σ denotes the (unnormalised) surface area measure. The *Poisson kernel* of the ball $B(x_0, r)$ is given by the function

$$K_{x_0,r}(x,y) := \frac{1}{\sigma_N r} \frac{r^2 - \|x - x_0\|^2}{\|x-y\|^N}$$

for $y \in S(x_0, r)$ and $x \in \mathbb{R}^N \setminus \{y\}$.

For a function h continuous on $S(x_0, r)$, the *Poisson integral* is defined as

$$I_{h,x_0,r}(x) := \int_{S(x_0,r)} K_{x_0,r}(x,y)\, h(y)\, \mathrm{d}\sigma(y)$$

for $x \in B(x_0, r)$. It is a fundamental result of potential theory that $I_{h,x_0,r}$ defines a harmonic function on the ball $B(x_0, r)$ with boundary values on $S(x_0, r)$ given by h.

We recall that Harnack's inequality states if h is a positive harmonic function on $B(x_0, r)$, then

$$h(x) \leqslant \frac{(r + \|x - x_0\|)\, r^{N-2}}{(r - \|x - x_0\|)^{N-1}} h(x_0)$$

for each $x \in B(x_0, r)$.

LEMMA 4.2. *Let $N \geqslant 2$. Given $h \in \mathscr{H}(\mathbb{R}^N)$, $r > 0$, $R > r$, and $a \in \mathbb{R}^N$ with $\|a\| \leqslant r$, we consider the translated harmonic function h_a defined by $h_a(x) = h(a+x)$. We then have*

$$M_2(h_a, R) \leqslant C_N M_2(h, r+R),$$

where $C_N > 0$ is a constant that only depends on N.

Proof. For brevity we denote the Poisson integrals of the subharmonic function h^2 on the spheres $S(a,R)$ and $S(r+R)$, respectively, by $I_{a,R}$ and $I_{0,r+R}$.

First observe that

$$M_2^2(h,r+R) = \int_{S(r+R)} |h(y)|^2 \, \mathrm{d}\sigma_{(r+R)}(y) = \frac{1}{\sigma_N (r+R)^{N-1}} \int_{S(r+R)} h(y)^2 \, \mathrm{d}\sigma(y) = I_{0,r+R}(0)$$

and similarly

$$M_2^2(h_a,R) = \int_{S(R)} |h_a(y)|^2 \, \mathrm{d}\sigma_R(y) = \int_{S(a,R)} |h(y)|^2 \, \mathrm{d}\sigma_R(y) = I_{a,R}(a).$$

Next notice that $I_{a,R} = h^2$ on the sphere $S(a,R)$, and that the subharmonicity of h^2 gives that $h^2 \leqslant I_{0,r+R}$ in the ball $B(r+R)$. So it holds that $I_{a,R} = h^2 \leqslant I_{0,r+R}$ on $S(a,R)$. By the maximum principle it follows that $I_{a,R} \leqslant I_{0,r+R}$ on $B(a,R)$ and moreover

$$I_{a,R}(a) \leqslant I_{0,r+R}(a).$$

Next, since $I_{0,r+R}$ is a positive harmonic function on $B(r+R)$, we have by Harnack's inequality and the facts that $\|a\| \leqslant r$ and $r < R$

$$\begin{aligned} I_{0,r+R}(a) &\leqslant \frac{(r+R+\|a\|)\,(r+R)^{N-2}}{(r+R-\|a\|)^{N-1}} I_{0,r+R}(0) \\ &\leqslant \frac{(2r+R)\,(r+R)^{N-2}}{R^{N-1}} I_{0,r+R}(0) \\ &= \left(\frac{2r}{R}+1\right)\left(\frac{r}{R}+1\right)^{N-2} I_{0,r+R}(0) = C_N I_{0,r+R}(0) \end{aligned}$$

where the constant C_N depends only on N and the result follows. □

4.2. Growth rates

We are now ready to identify the optimal M_2-average growth of harmonic functions that are irregular with respect to partial differentiation. We note that the growth rates are the same as in the hypercyclic case.

THEOREM 4.3. *Let* $1 \leqslant k \leqslant N$.

(i) Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be any function with* $\varphi(r) \to \infty$ *as* $r \to \infty$. *Then there exists a* $\partial/\partial x_k$*-irregular harmonic function* $h \in \mathscr{H}(\mathbb{R}^N)$ *with*

$$M_2(h,r) \leqslant \varphi(r) \frac{e^r}{r^{(N-1)/2}}$$

for $r > 0$ *sufficiently large.*

(ii) Let $\alpha \in \mathbb{N}^N$. There does not exist a D^α-irregular harmonic function $h \in \mathscr{H}(\mathbb{R}^N)$ that satisfies

$$M_2(h,r) \leqslant C\frac{e^r}{r^{(N-1)/2}} \tag{4.9}$$

for $r > 0$ and any constant $C > 0$.

Proof. (i) This follows from [2, Theorem 1] since hypercyclic vectors are irregular.
(ii) Let $h \in \mathscr{H}(\mathbb{R}^N)$. We recall that the translation $h_a(x) := h(x+a)$ preserves harmonicity and we further note that $h_a(0) = h(a)$, where $a \in \mathbb{R}^N$. Furthermore it follows from (4.1) that h_a has a unique representation of the form

$$h_a = \sum_{j=0}^{\infty} H_{a,j}$$

where $H_{a,j} \in \mathscr{H}_j(\mathbb{R}^N)$.

For $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^N$ we may differentiate under the summation sign to obtain

$$D^{n\alpha}h(a) = D^{n\alpha}h_a(0) = \sum_{j=0}^{\infty}(D^{n\alpha}H_{a,j})(0) = \left(D^{n\alpha}H_{a,n|\alpha|}\right)(0) \tag{4.10}$$

where we use the convention that $D^{n\alpha} = (D^\alpha)^n$.

Fix $r > 0$ and let $a \in B(r)$. For $R > r$ it follows from (4.10) and (4.4) that

$$\begin{aligned}|D^{n\alpha}h(a)| &\leqslant (n|\alpha|)!\sqrt{d_{n|\alpha|}}R^{-n|\alpha|}M_2(H_{a,n|\alpha|},R)\\ &\leqslant (n|\alpha|)!\sqrt{d_{n|\alpha|}}R^{-n|\alpha|}M_2(h_a,R).\end{aligned}$$

Applying Lemma 4.2 we get that

$$M_\infty(D^{n\alpha}h,r) \leqslant c_N(n|\alpha|)!\sqrt{d_{n|\alpha|}}R^{-n|\alpha|}M_2(h,r+R).$$

Next suppose that (4.9) holds. By (4.3) we know that

$$d_{n|\alpha|} = O((n|\alpha|)^{N-2})$$

as $n \to \infty$ and hence there exists a constant C, independent of n and r, such that

$$M_\infty(D^{n\alpha}h,r) \leqslant C\frac{(n|\alpha|)!(n|\alpha|)^{(N-2)/2}e^{r+R}}{R^{n|\alpha|}(r+R)^{(N-1)/2}}.$$

Applying Stirling's formula and choosing $R = n|\alpha| + (N-1)/2$ we obtain that

$$\begin{aligned}M_\infty(D^{n\alpha}h,r) &\leqslant C\frac{(n|\alpha|)^{n|\alpha|+(N-1)/2}e^{r+n|\alpha|+(N-1)/2}}{(n|\alpha|+(N-1)/2)^{n|\alpha|+(N-1)/2}e^{n|\alpha|}\left(1+\frac{r}{n|\alpha|+(N-1)/2}\right)^{(N-1)/2}}\\ &\leqslant Ce^{r+(N-1)/2}\left(1+\frac{N-1}{2n|\alpha|}\right)^{-n|\alpha|} \leqslant Ce^{r+(N-1)/2}.\end{aligned}$$

So we get that the sequence $\{M_\infty(D^{n\alpha}h,r)\}_n$ is bounded and since h does not have an unbounded orbit it cannot be irregular. □

Next we compute growth rates for $\partial/\partial x_k$-distributionally irregular harmonic functions and we note that the below growth is optimal.

THEOREM 4.4. *Let* $1 \leqslant k \leqslant N$.

(i) Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be any function with* $\varphi(r) \to \infty$ *as* $r \to \infty$*. Then there exists a harmonic function* $h \in \mathscr{H}(\mathbb{R}^N)$ *which is distributionally irregular with respect to the partial differentiation operator* $\partial/\partial x_k$*, such that*

$$M_2(h,r) \leqslant \varphi(r)\frac{e^r}{r^{N/2-3/4}} \tag{4.11}$$

for $r > 0$ *sufficiently large.*

(ii) There does not exist a $\partial/\partial x_k$*-distributionally irregular* $h \in \mathscr{H}(\mathbb{R}^N)$ *satisfying*

$$M_2(h,r) \leqslant c\frac{e^r}{r^{N/2-3/4}} \tag{4.12}$$

where $c > 0$ *is constant and for* $r > 0$ *sufficiently large.*

Proof. (i) Fix $1 \leqslant k \leqslant N$ and we assume without loss of generality that $\inf_{r>0}\varphi(r) > 0$. We consider the space

$$Y := \left\{ h \in \mathscr{H}(\mathbb{R}^N) \ : \ \lim_{r\to\infty} \frac{M_2(h,r)r^{N/2-3/4}}{\varphi(r)e^r} = 0 \right\}$$

endowed with the corresponding sup-norm $\|\cdot\|_Y$. Note that $(Y,\|\cdot\|_Y)$ is a Banach space that is continuously embedded in $\mathscr{H}(\mathbb{R}^N)$, endowed with the topology of local uniform convergence, and that every $h \in Y$ satisfies the desired growth condition.

We will apply Theorem 2.3 and to this end we let $Y_0 = Y_1$ be the space of harmonic polynomials on $\mathbb{R}^N$. The space Y_0 is dense in Y and it follows immediately that part (a) of Theorem 2.3 is satisfied.

For part (b), we use the antiderivative given in (4.5) to define the map $S\colon Y_1 \to Y_1$

$$S\colon \sum_{j=0}^{m} H_j \mapsto \sum_{j=0}^{m} P_{1,k}(H_j)$$

where $H_j \in \mathscr{H}_j(\mathbb{R}^N)$. For all $H \in Y_1$ we have that $\partial/\partial x_k SH = H$ and since $\sum_{n=1}^{\infty} \partial^n/\partial x_k^n H$ is a finite sum it converges unconditionally.

To complete the proof it suffices to show that the series $\sum_{n=1}^{\infty} S^n H = \sum_{n=1}^{\infty} P_{n,k}(H)$ converges unconditionally in Y for any polynomial $H \in \mathscr{H}_j(\mathbb{R}^N)$, $j \geqslant 0$. (This was shown in [16, Theorem 4.2(a)], but for the convenience of the reader we outline the argument here.)

For a finite subset $F \subset \mathbb{N}$, we have by orthogonality, homogeneity, (4.6) and (4.7) that

$$\begin{aligned}\left\|\sum_{n\in F} P_{n,k}(H)\right\|_Y &= \sup_{r>0} \frac{r^{N/2-3/4}}{\varphi(r)e^r} M_2\left(\sum_{n\in F} P_{n,k}(H), r\right)\\ &= \sup_{r>0} \frac{r^{N/2-3/4}}{\varphi(r)e^r}\left(\sum_{n\in F} M_2^2(P_{n,k}(H), r)\right)^{1/2}\\ &\leqslant C\sup_{r>0} \frac{r^{N/2-3/4}}{\varphi(r)e^r}\left(\sum_{n\in F} \frac{r^{2(n+j)}}{(n+j)!^2(n+j+1)^{N-2}}\right)^{1/2}.\end{aligned}$$

By an application of Lemma 2.1, it then follows that $\sum_{n=0}^{\infty} P_{n,k}(H)$ converges unconditionally in Y.

(ii) Fix $1 \leqslant k \leqslant N$. We consider the translated harmonic function $h_a(x) := h(x+a)$ for $a \in \mathbb{R}^N$. It follows from (4.1) that h_a has a unique representation of the form

$$h_a = \sum_{j=0}^{\infty} H_{a,j}$$

where $H_{a,j} \in \mathscr{H}_j(\mathbb{R}^N)$.

For $a \in S(r)$, it follows from (4.4) that

$$\left|\frac{\partial^n}{\partial x_k^n} h(a)\right| = \left|\frac{\partial^n}{\partial x_k^n} h_a(0)\right| \leqslant n!\sqrt{d_n} R^{-n} M_2(H_{a,n}, R)$$

for any $R > r$. It then follows from Lemma 4.2 that

$$\sum_{n=0}^{\infty} \frac{R^{2n}}{n!^2 d_n}\left|\frac{\partial^n}{\partial x_k^n} h(a)\right|^2 \leqslant \sum_{n=0}^{\infty} M_2^2(H_{a,n}, R) = M_2^2(h_a, R) \leqslant C_1 M_2^2(h, r+R)$$

where C_1 is a constant that depends only on N.

We recall that the sequence $(d_n)_n$ given by (4.2) is increasing and satisfies $d_n = O(n^{N-2})$ as $n \to \infty$. So applying our assumption we get that

$$\sum_{n=0}^{\infty} \frac{R^{2n}}{n!^2 d_n}\left|\frac{\partial^n}{\partial x_k^n} h(a)\right|^2 \leqslant C_2 \frac{e^{2(r+R)}}{(r+R)^{N-3/2}}.$$

That is,

$$\sum_{n=0}^{\infty} \frac{R^{2n+N-3/2}}{n!^2 n^{N-2} e^{2R}}\left|\frac{\partial^n}{\partial x_k^n} h(a)\right|^2 \leqslant C_2$$

for some constant C_2 that only depends on r and N. By applying Lemma 2.2 for $\alpha = 2$ and $\beta = N - 3/2$, we obtain

$$\frac{1}{l}\sum_{n=0}^{l}\left|\frac{\partial^n}{\partial x_k^n}h(a)\right|^2 \leqslant C$$

for every $l \in \mathbb{N}$ and $a \in B(r)$, and for some constant C that only depends on r and N. In particular, if we select a finite family $\{a_i \in S(r) : i = 0, \ldots, m\}$ uniformly distributed on $S(r)$ we get

$$\frac{1}{m}\sum_{i=0}^{m}\left(\frac{1}{l}\sum_{n=0}^{l}\left|\frac{\partial^n}{\partial x_k^n}h(a_i)\right|^2\right) = \frac{1}{l}\sum_{n=0}^{l}\left(\frac{1}{m}\sum_{i=0}^{m}\left|\frac{\partial^n}{\partial x_k^n}h(a_i)\right|^2\right) \leqslant C.$$

Taking limits as $m \to \infty$, we conclude that

$$\frac{1}{l}\sum_{n=0}^{l} M_2^2\left(\frac{\partial^n}{\partial x_k^n}h, r\right) \leqslant C$$

for each $l \in \mathbb{N}$, and it follows that h cannot be a distributionally unbounded harmonic function with respect to the operator $\partial/\partial x_k$. □

A natural further question arising from Theorem 4.4 is the following.

QUESTION 2. What are the precise M_p-average growth rates of $\partial/\partial x_k$-distributionally irregular harmonic functions for $p \neq 2$?

5. Growth of D^α-distributionally irregular harmonic functions

In this section we study permissible growth rates, in terms of the sup-norm on spheres, of harmonic functions that are distributionally irregular with respect to general partial differentiation operators D^α, for $\alpha \in \mathbb{N}^N$. The hypercyclic case was considered by the investigation of Aldred and Armitage [3] and the frequently hypercyclic case was studied by Blasco et al. [16].

We begin by recalling some notation and results from [2, 3] which are required in the sequel. Set $c_2 = 1$ and for $N \geqslant 3$ we define the constants

$$c_N = N\left(\prod_{j=1}^{N-1}\frac{(2j)^{2j}}{(2j+1)^{2j+1}}\right)^{1/2N}. \tag{5.1}$$

It was shown in [3, Section 3.2] that for $N \geqslant 3$

$$c_N > \sqrt{\frac{N}{2}} \quad \text{and} \quad c_N = \sqrt{\frac{N}{2}} + o(1), \text{ as } N \to \infty.$$

We also require suitable antiderivatives associated with the partial differentiation operators D^α. Using an inductive construction on the maps from (4.5), in [3, Lemma 2] they identified linear maps

$$P_{n,\alpha} \colon \mathscr{H}_m(\mathbb{R}^N) \to \mathscr{H}_{m+n|\alpha|}(\mathbb{R}^N) \tag{5.2}$$

with the property that $D^{n\alpha}P_{n,\alpha}(H) = H$, for $H \in \mathscr{H}_m(\mathbb{R}^N)$, where $m,n \geqslant 0$. Here we again use the convention that $D^{n\alpha} = (D^\alpha)^n$.

For our purposes, we do not need to explicitly define the maps $P_{n,\alpha}$, however we note that the different maps $P_{n,\alpha}$ are mutually compatible since for $H \in \mathscr{H}_m(\mathbb{R}^N)$ and $\ell,n \geqslant 0$ we have that

$$P_{\ell+n,\alpha}(H) = P_{\ell,\alpha}(P_{n,\alpha}(H))$$

and in particular it holds that $D^{n\alpha}P_{\ell,\alpha}(H) = P_{\ell-n,\alpha}(H)$ for $\ell > n$.

We will utilise the following estimate, which follows from [3, Lemma 4]. For $m,n \in \mathbb{N}$, $\alpha \in \mathbb{N}^N$ and $H \in \mathscr{H}_m(\mathbb{R}^N)$,

$$M_\infty(P_{n,\alpha}(H),r) \leqslant C\frac{n^A|\alpha|^A(m+1)^{(N-1)/2}(c_N r)^{n|\alpha|}}{(n|\alpha|)!}M_\infty(H,r) \tag{5.3}$$

for $r > 0$, where c_N is as defined in (5.1), $A,C > 0$ are constants depending only on N, and we may assume that $A \in \mathbb{N}$.

It follows from results in [3] that for nonzero $\alpha \in \mathbb{R}^N$, there exists a D^α-hypercyclic harmonic function $h \in \mathscr{H}(\mathbb{R}^N)$ such that for any $\varepsilon > 0$, there is some $C_\varepsilon > 0$ with

$$M_\infty(h,r) \leqslant C_\varepsilon e^{(c_N+\varepsilon)r}. \tag{5.4}$$

It also follows from [3] that for $\alpha = (1,\ldots,1)$, there does not exist a D^α-hypercyclic $h \in \mathscr{H}(\mathbb{R}^N)$ that satisfies

$$M_\infty(h,r) \leqslant Ce^{cr},$$

for any $c < \sqrt{N/2}$ and where $C > 0$ is a constant.

In [16] they strengthened growth condition (5.4) and extended it to the frequently hypercyclic case. In particular, they showed if $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\varphi(r)/r^p \to \infty$, as $r \to \infty$, for any $p \geqslant 0$, then there exists a D^α-frequently hypercyclic harmonic function $h \in \mathscr{H}(\mathbb{R}^N)$ such that

$$M_\infty(h,r) \leqslant \varphi(r)e^{c_N r}$$

for $r > 0$ sufficiently large.

Since hypercyclic vectors are irregular, we may immediately infer growth estimates for D^α-irregular harmonic functions from the above results. It is also necessary to mention that Bayart and Ruzsa [8] showed that there are frequently hypercyclic operators that are not distributionally chaotic, answering negatively Problem 36 in [13]. This shows that, in general, we cannot deduce directly the growth estimates for distributional chaos from the corresponding estimates in [16] for frequent hypercyclicity. We give initial growth estimates in the distributionally irregular case in the following theorem.

THEOREM 5.1. *Let $N \geqslant 2$ and $\alpha \in \mathbb{N}^N$ with $\alpha \neq 0$. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a function such that $\varphi(r)/r^p \to \infty$, as $r \to \infty$, for any $p \geqslant 0$. Then there exists a harmonic function $h \in \mathscr{H}(\mathbb{R}^N)$, distributionally irregular with respect to the differentiation operator D^α, such that*

$$M_\infty(h,r) \leqslant \varphi(r)e^{c_N r}$$

for $r > 0$ sufficiently large and where c_N is as given in (5.1).

Proof. The proof is similar to that of Theorem 4.4 (i). We assume without loss of generality that $\inf_{r>0}\varphi(r)>0$ and we consider the space

$$Y := \left\{ h \in \mathscr{H}(\mathbb{R}^N) \ : \ \lim_{r\to\infty} \frac{M_\infty(h,r)}{\varphi(r)e^{c_N r}} = 0 \right\}$$

endowed with the sup-norm $\|\cdot\|_Y$. Note that $(Y, \|\cdot\|_Y)$ is a Banach space which is continuously embedded in $\mathscr{H}(\mathbb{R}^N)$ and that every $h\in Y$ satisfies the desired growth condition.

We will apply Theorem 2.3 and to this end we let $Y_0 = Y_1$ be the space of harmonic polynomials on $\mathbb{R}^N$. Then the space Y_0 is dense in Y and it follows immediately that part (a) of Theorem 2.3 is satisfied with respect to the operator D^α.

For part (b), we define the map $S\colon Y_1 \to Y_1$ by

$$S\colon \sum_{j=0}^{m} H_j \mapsto \sum_{j=0}^{m} P_{1,\alpha}(H_j)$$

where $H_j \in \mathscr{H}_j(\mathbb{R}^N)$ and the antiderivative $P_{1,\alpha}$ is as given in (5.2). For all $H\in Y_1$, note that $D^\alpha S H = H$, and since $\sum_{n=1}^\infty D^{n\alpha}H$ is a finite sum it converges unconditionally.

To complete the proof it suffices to prove that the series $\sum_{n=1}^\infty S^n H$ converges unconditionally in Y for any polynomial $H\in\mathscr{H}_m(\mathbb{R}^N)$. (This calculation appears in [16, Theorem 4.3], but for the convenience of the reader we outline the steps here.)

Let $F\subset\mathbb{N}$ be finite. If $F\cap\{0,1,\ldots,L\}=\varnothing$, then by (5.3), the homogeneity of H, Lemma 2.1 it follows that

$$\begin{aligned}
\left\| \sum_{n\in F} P_{n,\alpha}(H) \right\|_Y &\leqslant \sup_{r>0} \frac{1}{\varphi(r)e^{c_N r}} \sum_{n=L+1}^{\infty} C \frac{n^A |\alpha|^A (m+1)^{(N-1)/2} (c_N r)^{n|\alpha|}}{(n|\alpha|)!} M_\infty(H,r) \\
&\leqslant C \sup_{r>0} \frac{1}{\varphi(r)e^{c_N r}} \sum_{n=L+1}^{\infty} \frac{n^A (c_N r)^{n|\alpha|}}{(n|\alpha|)!} r^m \\
&\leqslant C \sup_{r>0} \frac{r^{m+A}}{\varphi(r)e^{c_N r}} \sum_{n=L+1}^{\infty} \frac{(c_N r)^{n|\alpha|-A}}{(n|\alpha|-A)!} \leqslant C \sup_{r>0} \frac{r^{m+A}}{\varphi(r)}
\end{aligned}$$

where the constants $C>0$ above take different values. Using the assumption that $\varphi(r)/r^{m+A}\to\infty$ as $r\to\infty$, it follows that the series $\sum_{n=1}^\infty S^n H$ converges unconditionally in Y. □

The preceding discussion and theorem naturally give rise to the following question.

QUESTION 3. What are the optimal rates of growth of harmonic functions that are irregular and distributionally irregular with respect to D^α?

The analogous questions remains open for frequent hypercyclicity and even for hypercyclic harmonic functions (cf. [16, Section 6]). Aldred and Armitage [3] conjecture in the hypercyclic case that the sup-norm is of exponential type $\sqrt{N/2}$.

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