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APPROXIMATION BY LINEAR COMBINATIONS OF TRANSLATES OF A SINGLE FUNCTION

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Abstract. We study approximation of periodic functions by arbitrary linear combinations of n translates of a single function. We construct some linear methods of this approximation for univariate functions in the class induced by the convolution with a single function, and prove upper bounds of the L^p-approximation convergence rate by these methods, when $n \to \infty$, for $1 \leqslant p \leqslant \infty$. We also generalize these results to classes of multivariate functions defined as the convolution with the tensor product of a single function. In the case $p = 2$, for this class, we also prove a lower bound of the quantity characterizing best approximation of by arbitrary linear combinations of n translates of arbitrary function.

1. Introduction

The present paper continues investigating the problem of function approximation by arbitrary linear combinations of n translates of a single function which has been studied in [1, 3]. In the last papers, some linear methods were constructed for approximation of periodic functions in a class induced by the convolution with a given function, and prove upper bounds of the L^p-approximation convergence rate by these methods, when $n \to \infty$, for the case $1 < p < \infty$. The main technique of the proofs of the results is based on Fourier analysis, in particular, the multiplier theory. However, this technique cannot be extended to the two important cases $p = 1$ and $p = \infty$. In the present paper, we aim at this approximation problem for the cases $p = 1$ and $p = \infty$ by using a different technique. For convenience of presentation we will do this for $1 \leqslant p \leqslant \infty$.

We shall begin our discussion here by introducing notation used throughout the paper. In this regard, we merely follow closely the presentation in [1, 3]. The d-dimensional torus denoted by $\mathbb{T}^d$ is the cross product of d copies of the interval $[0, 2\pi]$ with the identification of the end points. When $d = 1$, we merely denote the d-torus by $\mathbb{T}$. Functions on $\mathbb{T}^d$ are identified with functions on $\mathbb{R}^d$ which are 2π periodic in each variable. Denote by $L^p(\mathbb{T}^d)$, $1 \leqslant p \leqslant \infty$, the space of integrable functions on $\mathbb{T}^d$ equipped with the norm

$$\|f\|_p := \begin{cases} (2\pi)^{-d/p} \left(\int_{\mathbb{T}^d} |f(\mathbf{x})|^p d\mathbf{x} \right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname{ess\,sup}_{\mathbf{x} \in \mathbb{T}^d} |f(\mathbf{x})|, & p = \infty. \end{cases}$$

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We will consider only real valued functions on $\mathbb{T}^d$. However, all the results in this paper are true for the complex setting. Also, we will use Fourier series of a real valued function in complex form.

Here, we use the notation $\mathbb{N}_m$ for the set $\{1,2,\dots,m\}$. For vectors $\mathbf{x} := (x_l : l \in \mathbb{N}_d)$ and $\mathbf{y} := (y_l : l \in \mathbb{N}_d)$ in $\mathbb{T}^d$ we use $(\mathbf{x},\mathbf{y}) := \sum_{l\in\mathbb{N}_d} x_l y_l$ for the inner product of $\mathbf{x}$ with $\mathbf{y}$. Also, for notational convenience we allow $\mathbb{N}_0$ and $\mathbb{Z}_0$ to stand for the empty set. Given any integrable function f on $\mathbb{T}^d$ and any lattice vector $\mathbf{j} = (j_l : l \in \mathbb{N}_d) \in \mathbb{Z}^d$, we let $\widehat{f}(\mathbf{j})$ denote the $\mathbf{j}$-th Fourier coefficient of f defined by the equation

$$\widehat{f}(\mathbf{j}) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{x})\, e^{-i(\mathbf{j},\mathbf{x})}\, d\mathbf{x}.$$

Frequently, we use the superscript notation $\mathbb{B}^d$ to denote the cross product of d copies of a given set $\mathbb{B}$ in $\mathbb{R}^d$.

Let $S'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Every $f \in S'(\mathbb{T}^d)$ can be identified with the formal Fourier series

$$f = \sum_{\mathbf{j}\in\mathbb{Z}^d} \widehat{f}(\mathbf{j}) e^{i(\mathbf{j},\cdot)},$$

where the sequence $(\widehat{f}(\mathbf{j}) : \quad \mathbf{j} \in \mathbb{Z}^d)$ forms a tempered sequence.

Let $\lambda : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be a bounded function. With the univariate λ we associate the multivariate tensor product function λ_d given by

$$\lambda_d(\mathbf{x}) := \prod_{l=1}^{d} \lambda(x_l), \quad \mathbf{x} = (x_l : l \in \mathbb{N}_d),$$

and introduce the function $\varphi_{\lambda,d}$, defined on $\mathbb{T}^d$ by the equation

$$\varphi_{\lambda,d}(\mathbf{x}) := \sum_{\mathbf{j}\in\mathbb{Z}^d} \lambda_d(\mathbf{j})\, e^{i(\mathbf{j},\mathbf{x})}. \tag{1}$$

Moreover, in the case that $d = 1$ we merely write φ_λ for the univariate function $\varphi_{\lambda,1}$. We introduce a subspace of $L^p(\mathbb{T}^d)$ defined as

$$\mathscr{H}_{\lambda,p}(\mathbb{T}^d) := \left\{ f : f = \varphi_{\lambda,d} * g,\ g \in L^p(\mathbb{T}^d) \right\},$$

with norm

$$\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)} := \|g\|_p,$$

where $f_1 * f_2$ is the convolution of two functions f_1 and f_2 on $\mathbb{T}^d$.

As in [1, 3], we are concerned with the following concept. Let $\mathbb{W}$ be a prescribed subset of $L^p(\mathbb{T}^d)$ and $\psi \in L^p(\mathbb{T}^d)$ be a given function. We are interested in the approximation in $L^p(\mathbb{T}^d)$-norm of all functions $f \in \mathbb{W}$ by arbitrary linear combinations of n translates of the function ψ, that is, by the functions in the set $\{\psi(\cdot - \mathbf{y}_l) : \mathbf{y}_l \in \mathbb{T}^d, l \in \mathbb{N}_n\}$ and measure the error in terms of the quantity

$$M_n(\mathbb{W},\psi)_p := \sup_{f\in\mathbb{W}} \inf \left\{ \left\| f - \sum_{l\in\mathbb{N}_n} c_l \psi(\cdot - \mathbf{y}_l) \right\|_p : c_l \in \mathbb{R}, \mathbf{y}_l \in \mathbb{T}^d \right\}.$$

The aim of the present paper is to investigate the convergence rate, when $n \to \infty$, of $M_n(U_{\lambda,p}(\mathbb{T}^d),\psi)_p$ for $1 \leqslant p \leqslant \infty$, where

$$U_{\lambda,p}(\mathbb{T}^d) := \left\{ f \in \mathscr{H}_{\lambda,p}(\mathbb{T}^d) : \quad \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)} \leqslant 1 \right\}$$

is the unit ball in $\mathscr{H}_{\lambda,p}(\mathbb{T}^d)$. We shall also obtain a lower bound for the convergence rate as $n \to \infty$ of the quantity

$$M_n(U_{\lambda,2}(\mathbb{T}^d))_2 := \inf\left\{ M_n(U_{\lambda,2}(\mathbb{T}^d),\psi)_2 : \psi \in L^2(\mathbb{T}^d) \right\},$$

which gives information about the best choice of ψ.

This paper is organized in the following manner. In Section 2, we give the necessary background from Fourier analysis and construct a method for approximation of functions in the univariate case. In Section 3, we extend the method of approximation developed in Section 2 to the multivariate case, in particular, prove upper bounds for the approximation error and convergence rate, we also prove a lower bound of $M_n(U_{\lambda,2}(\mathbb{T}^d))_2$.

2. Univariate approximation

In this section, we construct a linear method in the form of a linear combination of translates of a function φ_β defined as in (1) for approximation of univariate functions in $\mathscr{H}_{\lambda,p}(\mathbb{T})$. We give upper bounds of the approximation error for various λ and β.

Let $\lambda, \beta, \vartheta : \mathbb{R} \to \mathbb{R}$ be given 2-times continuously differentiable functions and ϑ be such that

$$\vartheta(x) := \begin{cases} 1, & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}], \\ 0, & \text{if } x \notin (-1,1). \end{cases}$$

Corresponding to these functions we define the functions $\mathscr{G}$ and H_m as

$$\mathscr{G}(x) := \frac{\lambda(x)}{\beta(x)}, \quad H_m(x) := \sum_{k\in\mathbb{Z}} \vartheta(k/m)\mathscr{G}(k)\, e^{ikx}. \tag{2}$$

For a function $f \in \mathscr{H}_{\lambda,p}(\mathbb{T})$ represented as $f = \varphi_\lambda * g$, $g \in L^p(\mathbb{T})$, we define the operator

$$Q_{m,\beta}(f) := \frac{1}{2m+1}\sum_{k=0}^{2m} V_m(g)\left(\frac{2\pi k}{2m+1}\right)\varphi_\beta\left(\cdot - \frac{2\pi k}{2m+1}\right), \tag{3}$$

where $V_m(g) := H_m * g$. Finally, we define for a function $h : \mathbb{R} \to \mathbb{R}$,

$$\sigma_m(h;f)(x) := \sum_{k\in\mathbb{Z}} h(k/m)\,\widehat{f}_k e^{ikx}.$$

Let us obtain upper estimates for the error of approximating a function $f \in \mathscr{H}_{\lambda,p}(\mathbb{T})$ by the trigonometric polynomial $Q_{m,\beta}(f)$ a linear combination of $2m+1$ translates of the function φ_β.

DEFINITION 1. A 2-times continuously differentiable function $\psi:\mathbb{R}\to\mathbb{R}$ is called a function of monotone type if there exists a positive constant c_0 such that

$$|\psi(x)|\geqslant c_0|\psi(y)|,\quad |\psi''(x)|\geqslant c_0|\psi''(y)|\quad \text{for all } 2|y|\geqslant|x|\geqslant|y|/2.$$

We put

$$\varepsilon_m := J_m(\lambda)+\sup_{|x|\in[-m,m]}\left(|\mathscr{G}(x)|+m^2\sup_{|x|\in[-m,m]}|\mathscr{G}''(x)|\right)J_m(\beta),$$

where for a 2-times continuously differentiable function ψ,

$$J_m(\psi):=\int_{|x|\geqslant m}\left(\left|\frac{\psi(x)}{m}\right|+\left|x\psi''(x)\right|\right)dx.$$

THEOREM 1. *Let $1\leqslant p\leqslant\infty$. Assume that the functions λ,β are of monotone type. Then there exists a positive constant c such that for all $f\in\mathscr{H}_{\lambda,p}(\mathbb{T})$ and $m\in\mathbb{N}$,*

$$\|f-Q_{m,\beta}(f)\|_p\leqslant c\varepsilon_m\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T})}.$$

Before we give the proof of the above theorem, we recall a lemma proved in [6], [7].

LEMMA 1. *Let $1\leqslant p\leqslant\infty$, $f\in L^p(\mathbb{T})$ and $h:\mathbb{R}\to\mathbb{R}$ be 2-times continuously differentiable function, supported on $[-1,1]$. Then there exists a constant c_1 independent of f,h,m such that*

$$\|\sigma_m(h;f)\|_p\leqslant c_1\|h''\|_\infty\|f\|_p.$$

We also need a Landau's inequality for derivatives [4].

LEMMA 2. *Let $f\in L^\infty(\mathbb{R})$ be 2-times continuously differentiable function. Then*

$$\|f'\|_\infty^2\leqslant 4\|f\|_\infty\|f''\|_\infty.$$

In particular,

$$\|f'\|_\infty\leqslant\|f\|_\infty+\|f''\|_\infty.$$

Proof of Theorem 1. Let $f\in\mathscr{H}_{\lambda,p}(\mathbb{T})$ be represented as $\varphi_{\lambda,d}*g$ for some $g\in L^p(\mathbb{T})$. We define the kernel $P_m(x,t)$ for $x,t\in\mathbb{T}$ as

$$P_m(x,t):=\frac{1}{2m+1}\sum_{k=0}^{2m}\varphi_\beta\left(x-\frac{2\pi k}{2m+1}\right)H_m\left(\frac{2\pi k}{2m+1}-t\right).$$

It is easy to obtain from the definition (3) that

$$Q_{m,\beta}(f)(x)=\frac{1}{2\pi}\int_{\mathbb{T}}P_m(x,t)g(t)\,dt.$$

We now use equation (1), the definition of the trigonometric polynomial H_m given in equation (2) and the easily verified fact, for $k,s\in\mathbb{Z}, s\in[-m,m]$, that

$$\frac{1}{2m+1}\sum_{\ell=0}^{2m} e^{ik(t-(2\pi\ell/(2m+1)))}e^{is((2\pi\ell/(2m+1))-t)} = \begin{cases} 0, & \text{if } \frac{k-s}{2m+1}\notin\mathbb{Z},\\ e^{i(k-k_m)t}, & \text{if } \frac{k-s}{2m+1}\in\mathbb{Z},\end{cases}$$

to conclude that

$$P_m(x,t)=\sum_{k\in\mathbb{Z}}\gamma(k)e^{ikx}e^{-ik_m t},$$

where $\gamma(k)=\vartheta(k_m/m)\mathscr{G}(k_m)\beta(k)$ and $k_m\in[-m,m]$ satisfy $(k-k_m)/(2m+1)\in\mathbb{Z}$. Hence,

$$\begin{aligned} Q_{m,\beta}(f)(x) &= \sum_{k>m}\gamma(k)e^{ikx}\widehat{g}(k_m)+\sum_{k<-m}\gamma(k)e^{ikx}\widehat{g}(k_m)+\sum_{k=-m}^{m}\gamma(k)e^{ikx}\widehat{g}(k_m)\\ &=: \mathscr{A}_m(x)+\mathscr{B}_m(x)+\mathscr{C}_m(x).\end{aligned}$$

Consequently,

$$\|f-Q_{m,\beta}(f)\|_p\leqslant\|\mathscr{A}_m\|_p+\|\mathscr{B}_m\|_p+\|f-\mathscr{C}_m\|_p. \tag{4}$$

For each $j\in\mathbb{N}$, we define the functions $\Lambda_{j,m}(x)$, $\mathscr{J}_m(x)$, $\mathscr{K}_{j,m}(x)$, $\mathscr{D}_{j,m}(x)$ and the set $I_{j,m}$ as follows

$$\begin{gathered} \Lambda_{j,m}(x):=\beta(mx+j(2m+1)),\qquad \mathscr{J}_m(x):=\mathscr{G}(mx),\\ \mathscr{K}_{j,m}(x):=\Lambda_{j,m}(x)\vartheta(x)\mathscr{J}_m(x),\qquad \mathscr{D}_{j,m}(x):=\sum_{k\in I_{j,m}}\gamma(k)e^{ikx}\widehat{g}(k_m),\\ I_{j,m}:=\{k\in\mathbb{Z}:\quad (2m+1)j-m\leqslant k\leqslant(2m+1)j+m\}.\end{gathered}$$

Then we have

$$\mathscr{A}_m(x)=\sum_{j\in\mathbb{N}}\sum_{k\in I_{j,m}}\gamma(k)e^{ikx}\widehat{g}(k_m)=\sum_{j\in\mathbb{N}}\mathscr{D}_{j,m}(x), \tag{5}$$

and for all $k\in I_{j,m}$,

$$\begin{aligned}\gamma(k)&=\beta(k)\vartheta(k_m/m)\mathscr{G}(k_m)=\beta(j(2m+1)+k_m)\vartheta(k_m/m)\mathscr{G}(k_m)\\ &=\Lambda_{j,m}(k_m/m)\vartheta(k_m/m)\mathscr{G}(k_m)=\Lambda_{j,m}(k_m/m)\vartheta(k_m/m)\mathscr{J}_m(k_m/m)\\ &=\mathscr{K}_{j,m}(k_m/m).\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{D}_{j,m}(x)&=\sum_{k\in I_{j,m}}\gamma(k)e^{ikx}\widehat{g}(k_m)=\sum_{k_m\in[-m,m]}\mathscr{K}_{j,m}(k_m/m)e^{i(j(2m+1)+k_m)x}\widehat{g}(k_m)\\ &=e^{ij(2m+1)x}\sum_{k_m\in[-m,m]}\mathscr{K}_{j,m}(k_m/m)e^{ik_m x}\widehat{g}(k_m)=e^{ij(2m+1)x}\sigma_m(\mathscr{K}_{j,m};g).\end{aligned}$$

Therefore, by Lemma 1, there exists a constant c_1 such that

$$\|\mathscr{D}_{j,m}\|_p \leqslant c_1 \|(\mathscr{K}_{j,m})^{''}\|_\infty \|g\|_p.$$

Then it follows from (5) that

$$\|\mathscr{A}_m\|_p \leqslant \sum_{j\in\mathbb{N}} \|\mathscr{D}_{j,m}\|_p \leqslant c_1 \sum_{j\in\mathbb{N}} \|(\mathscr{K}_{j,m})^{''}\|_\infty \|g\|_p. \tag{6}$$

From the definition of $\mathscr{K}_{j,m}$, $\operatorname{supp}\vartheta \subset [-1,1]$, and $\|\vartheta\|_\infty \leqslant 2\|\vartheta^{'}\|_\infty \leqslant 4\|\vartheta^{''}\|_\infty$, we deduce that

$$\begin{aligned}
\|(\mathscr{K}_{j,m})^{''}\|_\infty &\leqslant 4\|\vartheta^{''}\|_\infty \sup_{x\in[-1,1]} \Big(|\Lambda_{j,m}(x)\mathscr{J}_m(x)| + |(\Lambda_{j,m}\mathscr{J}_m)^{'}(x)| + |(\Lambda_{j,m}\mathscr{J}_m)^{''}(x)|\Big) \\
&\leqslant 4\|\vartheta^{''}\|_\infty \Big[\sup_{x\in I_{j,m}} \Big(|\beta(x)| + m|\beta^{'}(x)| + m^2|\beta^{''}(x)|\Big) \sup_{x\in[-m,m]} |\mathscr{G}(x)| \\
&\quad + m \sup_{x\in I_{j,m}} \Big(|\beta(x)| + m|\beta^{'}(x)|\Big) \sup_{x\in[-m,m]} |\mathscr{G}^{'}(x)| \\
&\quad + m^2 \sup_{x\in I_{j,m}} |\beta(x)| \sup_{x\in[-m,m]} |\mathscr{G}^{''}(x)|\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|(\mathscr{K}_{j,m})^{''}\|_\infty \leqslant 4\|\vartheta^{''}\|_\infty &\sup_{x\in I_{j,m}} \Big(|\beta(x)| + m|\beta^{'}(x)| + m^2|\beta^{''}(x)|\Big) \times \\
&\times \sup_{x\in[-m,m]} \Big(|\mathscr{G}(x)| + m|\mathscr{G}^{'}(x)| + m^2|\mathscr{G}^{''}(x)|\Big)
\end{aligned}$$

for all $j \in \mathbb{N}$. Therefore, it follows from (6) that

$$\begin{aligned}
\|\mathscr{A}_m\|_p \leqslant 4c_1\|\vartheta^{''}\|_\infty &\sum_{j\in\mathbb{N}} \sup_{x\in I_{j,m}} \Big(|\beta(x)| + m|\beta^{'}(x)| + m^2|\beta^{''}(x)|\Big) \times \\
&\times \sup_{x\in[-m,m]} \Big(|\mathscr{G}(x)| + m|\mathscr{G}^{'}(x)| + m^2|\mathscr{G}^{''}(x)|\Big) \|g\|_p.
\end{aligned}$$

So, by Lemma 2, we have

$$\begin{aligned}
\|\mathscr{A}_m\|_p \leqslant 16c_1\|\vartheta^{''}\|_\infty &\sum_{j\in\mathbb{N}} \sup_{x\in I_{j,m}} \Big(|\beta(x)| + m^2|\beta^{''}(x)|\Big) \times \\
&\times \sup_{x\in[-m,m]} \Big(|\mathscr{G}(x)| + m^2|\mathscr{G}^{''}(x)|\Big) \|g\|_p. \qquad (7)
\end{aligned}$$

Since the function α, β is of monotone type, there exists a positive constant c_0 such that

$$|\alpha(x)| \geqslant c_0|\alpha(y)|, |\alpha^{''}(x)| \geqslant c_0|\alpha^{''}(y)|, |\beta(x)| \geqslant c_0|\beta(y)|, |\beta^{''}(x)| \geqslant c_0|\beta^{''}(y)| \tag{8}$$

for all $4|y| \geqslant |x| \geqslant |y|/4$. Hence,

$$\sup_{|x|\in I_{j,m}} |\beta(x)| \leqslant \frac{c_0}{m} \int_{|x|\in I_{j,m}} |\beta(x)| dx,$$

$$\sup_{|x|\in I_{j,m}} |m^2\beta''(x)| \leqslant c_0 m \int_{|x|\in I_{j,m}} |\beta''(x)| dx.$$

So,

$$\sum_{j\in\mathbb{N}} \sup_{|x|\in I_{j,m}} \left(|\beta(x)| + |m^2\beta''(x)|\right) \leqslant c_0 \int_{|x|\geqslant m} \left(\frac{|\beta(x)|}{m} + \left|m\beta''(x)\right|\right) dx \leqslant c_0 J_m(\beta).$$

Combining this with (7), we obtain that

$$\|\mathscr{A}_m\|_p \leqslant 16 c_0 c_1 \|\vartheta''\|_\infty \varepsilon_m \|g\|_p. \tag{9}$$

Similarly,

$$\|\mathscr{B}_m\|_p \leqslant 16 c_0 c_1 \|\vartheta''\|_\infty \varepsilon_m \|g\|_p. \tag{10}$$

Next, we will estimate $\|f - \mathscr{C}_m\|_p$. Notice that $\gamma(k) = \vartheta(k/m)\mathscr{G}(k)\beta(k) = \vartheta(k/m)\lambda(k)$ for $k \in [-m,m]$, and then

$$\begin{aligned} \sigma_m(\vartheta; f)(x) &= \sum_{k\in\mathbb{Z}} \vartheta(k/m)\widehat{f}(k)e^{ikx} = \sum_{k=-m}^{m} \vartheta(k/m)\lambda(k)\widehat{g}(k)e^{ikx} \\ &= \sum_{k=-m}^{m} \gamma(k)\widehat{g}(k)e^{ikx} = \mathscr{C}_m(x), \end{aligned}$$

and therefore,

$$\|f - \mathscr{C}_m\|_p = \|f - \sigma_m(\vartheta; f)\|_p. \tag{11}$$

We define the functions $S(x)$, $\Phi_{j,m}(x)$ and $\Psi_{j,m}(x)$ as

$$S(x) := \vartheta(x) - \vartheta(x/2), \quad \Phi_{j,m}(x) := \lambda(2^j mx), \quad \Psi_{j,m}(x) := S(x)\Phi_{j,m}(x).$$

Clearly, we have that

$$(\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m)))\lambda(k) = S(k/(2^j m))\Phi_{j,m}(k/(2^j m)) = \Psi_{j,m}(k/(2^j m)),$$

which together with

$$\begin{aligned} \sigma_{2^{j+1}m}(\vartheta; f) - \sigma_{2^j m}(\vartheta; f) &= \sum_{k\in\mathbb{Z}} (\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m)))\widehat{f}(k)e^{ikx} \\ &= \sum_{k\in\mathbb{Z}} (\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m)))\lambda(k)\widehat{g}(k)e^{ikx} \end{aligned}$$

implies that

$$\sigma_{2^{j+1}m}(\vartheta; f) - \sigma_{2^j m}(\vartheta; f) = \sum_{k\in\mathbb{Z}} \Psi_{j,m}(k/(2^j m))\widehat{g}(k)e^{ikx} = \sigma_{2^j m}(\Psi_{j,m}; g).$$

Then by Lemma 1, we obtain

$$\|\sigma_{2^{j+1}m}(\vartheta;f)-\sigma_{2^jm}(\vartheta;f)\|_p \leqslant c_1\|\Psi''_{j,m}\|_\infty\|g\|_p. \tag{12}$$

Moreover, from the definition of $\Psi_{j,m}$, $\operatorname{supp}S\subset[-2,-1/2]\cup[1/2,2]$, and $\|S\|_\infty\leqslant 2\|S'\|_\infty\leqslant 4\|S''\|_\infty\leqslant 8\|\vartheta''\|_\infty$, we have that

$$\begin{aligned}
|\Psi''_{j,m}(x)| &= |S''(x)\Phi_{j,m}(x)+2S'(x)\Phi'_{j,m}(x)+S(x)\Phi''_{j,m}(x)|\\
&\leqslant 8\|\vartheta''\|_\infty \sup_{|x|\in[1/2,2]}\Big(|\Phi_{j,m}(x)|+\Phi'_{j,m}(x)|+|\Phi''_{j,m}(x)|\Big)\\
&\leqslant 16\|\vartheta''\|_\infty \sup_{|x|\in[1/2,2]}\Big(|\Phi_{j,m}(x)|+|\Phi''_{j,m}(x)|\Big)\\
&= 16\|\vartheta''\|_\infty \sup_{|x|\in[2^{j-1}m,2^{j+1}m]}\Big(|\lambda(x)|+(2^jm)^2|\lambda''(x)|\Big)\\
&\leqslant 64\|\vartheta''\|_\infty \sup_{|x|\in[2^{j-1}m,2^{j+1}m]}\Big(|\lambda(x)|+|x^2\lambda''(x)|\Big).
\end{aligned}$$

Combining this and (12), we deduce

$$\|\sigma_{2^{j+1}m}(\vartheta;f)-\sigma_{2^jm}(\vartheta;f)\|_p \leqslant 64c_1\|\vartheta''\|_\infty \sup_{|x|\in[2^{j-1}m,2^{j+1}m]}\Big(|\lambda(x)|+|x^2\lambda''(x)|\Big)\|g\|_p.$$

Therefore, by (11) and $\lim_{m\to\infty}\|f-\sigma_{2^jm}(\vartheta;f)\|_p=0$, we have that

$$\begin{aligned}
\|f-\mathscr{C}_m\|_p &\leqslant \sum_{j=0}^{\infty}\|\sigma_{2^{j+1}m}(\vartheta;f)-\sigma_{2^jm}(\vartheta;f)\|_p\\
&\leqslant 64c_1\|\vartheta''\|_\infty\sum_{j=0}^{\infty}\sup_{|x|\in[2^{j-1}m,2^{j+1}m]}\Big(|\lambda(x)|+|x^2\lambda''(x)|\Big)\|g\|_p.
\end{aligned} \tag{13}$$

Since (8),

$$\sup_{|x|\in[2^{j-1}m,2^{j+1}m]}|\lambda(x)|\leqslant\frac{c_0}{2^jm}\int_{|x|\in[2^jm,2^{j+1}m]}|\lambda(x)|dx\leqslant\frac{c_0}{m}\int_{|x|\in[2^jm,2^{j+1}m]}|\lambda(x)|dx,$$

and

$$\sup_{|x|\in[2^{j-1}m,2^{j+1}m]}|x^2\lambda''(x)|\leqslant 2c_0\int_{|x|\in[2^jm,2^{j+1}m]}|x\lambda''(x)|dx.$$

So,

$$\begin{aligned}
\sum_{j=0}^{\infty}\sup_{|x|\in[2^{j-1}m,2^{j+1}m]}\Big(|\lambda(x)|+|x^2\lambda''(x)|\Big) &\leqslant 2c_0\int_{|x|\geqslant m}\Big(\frac{|\lambda(x)|}{m}+|x\lambda''(x)|\Big)dx\\
&= 2c_0J_m(\lambda).
\end{aligned}$$

Hence, by (13), we deduce

$$\|f-\mathscr{C}_m\|_p \leqslant 128c_0c_1\|\vartheta''\|_\infty \varepsilon_m\|g\|_p. \tag{14}$$

Combining (9), (10) and (14) we have

$$\|f-Q_{m,\beta}(f)\|_p \leqslant c\varepsilon_m\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T})}. \quad \square$$

From the above theorem, by letting $\lambda=\beta$, we obtain the following corollary.

COROLLARY 1. *Let $1\leqslant p\leqslant\infty$ and λ be of monotone type. Then there exists a positive constant c such that for all $f\in\mathscr{H}_{\lambda,p}(\mathbb{T})$ and $m\in\mathbb{N}$,*

$$\|f-Q_{m,\lambda}(f)\|_p \leqslant cJ_m(\lambda)\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T})}.$$

DEFINITION 2. Let $r,\kappa\in\mathbb{R}$. A function $f:\mathbb{R}\to\mathbb{R}$ will be called a mask of type (r,κ) if f is an even, 2 times continuously differentiable such that for $t\geqslant 1$, $f(t)=|t|^{-r}(\log(|t|+1))^{-\kappa}F(\log|t|)$ for some $F:\mathbb{R}\to\mathbb{R}$ such that $|F^{(k)}(t)|\leqslant a_1$ for all $t\geqslant 1, k=0,1,2$.

THEOREM 2. *Let $1\leqslant p\leqslant\infty$, $1<r<\infty$, $\kappa\in\mathbb{R}$ and the function λ be a mask of type (r,κ). Then there exists a positive constant c such that for all $f\in\mathscr{H}_{\lambda,p}(\mathbb{T})$ and $m\in\mathbb{N}$,*

$$\|f-Q_{m,\lambda}(f)\|_p \leqslant cm^{-r}(\log m)^{-\kappa}\ \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T})}.$$

Proof. Since the function λ be a mask of type (r,κ) and $r>1$,

$$\int_{|x|\geqslant m}\left|\frac{\lambda(x)}{m}\right|dx \leqslant a_1\int_{|x|\geqslant m}\frac{|x|^{-r}(\log(|x|+1))^{-\kappa}}{m}dx \leqslant a_2m^{-r}(\log(m+1))^{-\kappa} \quad \forall m\in\mathbb{N}. \tag{15}$$

On the other hand,

$$\begin{aligned}
&\int_{|x|\geqslant m}|x\lambda''(x)|dx\\
&\leqslant \int_{|x|\geqslant m}|x|\left(\left(|x|^{-r}(\log(|x|+1))^{-\kappa}\right)''|F(\log|x|)| + \left(|x|^{-r}(\log(|x|+1))^{-\kappa}\right)'\right.\\
&\quad\left.\times|F'(\log|x|)|/|x| + \left(|x|^{-r}(\log(|x|+1))^{-\kappa}\right)|F''(\log|x|)-F'(\log|x|)|/x^2\right)dx\\
&\leqslant a_1\int_{|x|\geqslant m}|x|\Big(\left(|x|^{-r}(\log(|x|+1))^{-\kappa}\right)'' + 2\left(|x|^{-r}(\log(|x|+1))^{-\kappa}\right)'/|x|\\
&\quad + 2\left(|x|^{-r}(\log(|x|+1))^{-\kappa}\right)/x^2\Big)dx \leqslant a_3m^{-r}(\log(m+1))^{-\kappa}.
\end{aligned}$$

Hence, by (15), we deduce

$$J_m(\lambda)\leqslant a_4m^{-r}(\log(m+1))^{-\kappa}.$$

From this and Corollary 1, we complete the proof. $\square$

COROLLARY 2. *For* $1 \leqslant p \leqslant \infty$, $1 < r < \infty$ *and* $\lambda(x) = \beta(x) = x^{-r}$ *for* $x \neq 0$, $\mathscr{H}_{\lambda,p}(\mathbb{T})$ *becomes the Korobov space* $K_p^r(\mathbb{T})$. *Then we have the estimate as in* [1]*:*

$$M_n(U_{\lambda,p}(\mathbb{T}), \kappa_r)_p \leqslant cm^{-r}$$

where κ_r *is the Korobov function.*

DEFINITION 3. A function $f : \mathbb{R} \to \mathbb{R}$ is called a function of exponent type if f is 2 times continuously differentiable and there exists a positive constant s such that $f(t) = e^{-s|t|}F(|t|)$ for some decreasing function $F : [0, +\infty) \to (0, +\infty)$.

THEOREM 3. *Let* $1 \leqslant p \leqslant \infty$, $1 < r < \infty$, $\kappa \in \mathbb{Z}$, *the function* λ *be a mash of type* (r, κ), *the function* β *of exponent type. Then there exists a positive constant* c *such that for all* $f \in \mathscr{H}_{\lambda,p}(\mathbb{T})$ *and* $m \in \mathbb{N}$, *we have*

$$\|f - Q_{m,\beta}(f)\|_p \leqslant cm^{-r}(\log(m+1))^{-\kappa}\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T})}.$$

Proof. We will use the notation in the proof of Theorem 1. For $k \in I_{j,m}$ we have $k_m = k - j(2m+1)$ and then

$$|\gamma(k)| = \left|\beta(k_m + j(2m+1))\vartheta(k_m/m)\frac{\lambda(k_m)}{\beta(k_m)}\right|$$

$$= e^{-sj(2m+1))}\frac{|\lambda(k_m)F(k_m + j(2m+1))|}{|F(k_m)|} \leqslant b_1 e^{-sj(2m+1))}.$$

Hence,

$$\left\|\sum_{k\in I_{j,m}} \gamma(k)e^{ikx}\widehat{g}(k_m)\right\|_p \leqslant 3b_1 m e^{-sj(2m+1))}\|g\|_p.$$

This implies that

$$\begin{aligned}\|\mathscr{A}_m\|_p &= \left\|\sum_{j\in\mathbb{N}}\sum_{k\in I_{j,m}} \gamma(k)e^{ikx}\widehat{g}(k_m)\right\|_p \\ &\leqslant 3b_1\sum_{j\in\mathbb{N}} m e^{-sj(2m+1))}\|g\|_p \leqslant b_2 m^{-r}(\log(m+1))^{-\kappa}\|g\|_p.\end{aligned} \tag{16}$$

Similarly,

$$\|\mathscr{B}_m\|_p \leqslant b_2 m^{-r}(\log(m+1))^{-\kappa}\|g\|_p. \tag{17}$$

We also known that in the proof of Theorem 1 that

$$\|f - \mathscr{C}_m\|_p \leqslant b_3 \sum_{j=0}^{\infty} \sup_{|x|\in[2^{j-1}m, 2^{j+1}m]} \left(|\lambda(x)| + |x^2\lambda''(x)|\right)\|g\|_p. \tag{18}$$

We see that

$$\sup_{|x|\in[2^{j-1}m,2^{j+1}m]} |\lambda(x)| \leqslant b_4 \int_{|x|\in[2^j m, 2^{j+1}m]} \frac{|\lambda(x)|}{|x|}dx$$

$$\sup_{|x|\in[2^{j-1}m,2^{j+1}m]} |x^2\lambda''(x)| \leqslant b_4 \int_{|x|\in[2^j m,2^{j+1}m]} |x\lambda''(x)|dx.$$

So,

$$\sum_{j=0}^{\infty} \sup_{|x|\in[2^{j-1}m,2^{j+1}m]} \left(|\lambda(x)| + |x^2\lambda''(x)|\right) \leqslant b_4 \int_{|x|\geqslant m} \left(\frac{|\lambda(x)|}{|x|} + \left|x\lambda''(x)\right|\right) dx.$$

Hence, by (18), we deduce that

$$\|f - \mathscr{C}_m\|_p \leqslant b_3 b_4 \|g\|_p \int_{|x|\geqslant m} \left(\frac{|\lambda(x)|}{|x|} + \left|x\lambda''(x)\right|\right) dx \leqslant b_5 m^{-r}(\log(m+1))^{-\kappa}\|g\|_p.$$

Combining this, (16), (17) and (4), we complete the proof. $\square$

3. Multivariate approximation

In this section, we make use of the univariate operators $Q_{m,\lambda}$ to construct multivariate operators on sparse Smolyak grids for approximation of functions from $\mathscr{H}_{\lambda,p}(\mathbb{T}^d)$. Based on this approxiation with certain restriction on the function λ we prove an upper bound of $M_n(U_{\lambda,p}(\mathbb{T}^d),\varphi_{\lambda,d})_p$ for $1 \leqslant p \leqslant \infty$ as well as a lower bound of $M_n(U_{\lambda,2}(\mathbb{T}^d))_2$. The results obtained in this section generalize some results in [1, 2].

3.1. Error estimates for functions in the space $\mathscr{H}_{\lambda,p}(\mathbb{T}^d)$

For $\mathbf{m} \in \mathbb{N}^d$, let the multivariate operator $Q_{\mathbf{m}}$ in $\mathscr{H}_{\lambda,p}(\mathbb{T}^d)$ be defined by

$$Q_{\mathbf{m}} := \prod_{j=1}^{d} Q_{m_j,\lambda}, \tag{19}$$

where the univariate operator $Q_{m_j,\lambda}$ is applied to the univariate function f by considering f as a function of variable x_j with the other variables held fixed, $\mathbb{Z}^d_+ := \{\mathbf{k} \in \mathbb{Z}^d : k_j \geqslant 0,\ j \in \mathbb{N}_d\}$ and k_j denotes the jth coordinate of $\mathbf{k}$.

Set $\mathbb{Z}^d_{-1} := \{\mathbf{k} \in \mathbb{Z}^d : k_j \geqslant -1,\ j \in \mathbb{N}_d\}$. For $k \in \mathbb{Z}_{-1}$, we define the univariate operator T_k in $\mathscr{H}_{\lambda,p}(\mathbb{T})$ by

$$T_k := \mathrm{I} - Q_{2^k,\lambda},\ k \geqslant 0, \quad T_{-1} := \mathrm{I},$$

where I is the identity operator. If $\mathbf{k} \in \mathbb{Z}^d_{-1}$, we define the mixed operator $T_{\mathbf{k}}$ in $\mathscr{H}_{\lambda,p}(\mathbb{T}^d)$ in the manner of the definition of (19) as

$$T_{\mathbf{k}} := \prod_{i=1}^{d} T_{k_i}.$$

Set $|\mathbf{k}| := \sum_{j\in\mathbb{N}_d} |k_j|$ for $\mathbf{k} \in \mathbb{Z}^d_{-1}$ and $\mathbf{k}^{-\kappa}_{(2)} = \prod_{j=1}^{d}(k_j+2)^{-\kappa}$.

LEMMA 3. *Let $1 \leqslant p \leqslant \infty$, $1 < r < \infty$, $0 \leqslant \kappa < \infty$ and the function λ be a mask of type (r,κ). Then we have for any $f \in \mathscr{H}_{\lambda,p}(\mathbb{T}^d)$ and $\mathbf{k} \in \mathbb{Z}^d_{-1}$,*

$$\|T_{\mathbf{k}}(f)\|_p \leqslant C\mathbf{k}_{(2)}^{-\kappa} 2^{-r|\mathbf{k}|}\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)}$$

with some constant C independent of f and $\mathbf{k}$.

Proof. We prove the lemma by induction on d. For $d=1$ it follows from Theorems 2. Assume the lemma is true for $d-1$. Set $\mathbf{x}' := \{x_j : j \in \mathbb{N}_{d-1}\}$ and $\mathbf{x} = (\mathbf{x}', x_d)$ for $\mathbf{x} \in \mathbb{R}^d$. We temporarily denote by $\|f\|_{p,\mathbf{x}'}$ and $\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^{d-1}),\mathbf{x}'}$ or $\|f\|_{p,x_d}$ and $\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}),x_d}$ the norms applied to the function f by considering f as a function of variable $\mathbf{x}'$ or x_d with the other variable held fixed, respectively. For $\mathbf{k} = (\mathbf{k}', k_d) \in \mathbb{Z}^d_{-1}$, we get by Theorems 2 and the induction assumption

$$\begin{aligned}
\|T_{\mathbf{k}}(f)\|_p &= \|\|T_{\mathbf{k}'}T_{k_d}(f)\|_{p,\mathbf{x}'}\|_{p,x_d} \ll \|2^{-r|\mathbf{k}'|}\mathbf{k}'^{-\kappa}_{(2)}\|T_{k_d}(f)\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^{d-1}),\mathbf{x}'}\|_{p,x_d} \\
&= 2^{-r|\mathbf{k}'|}\mathbf{k}'^{-\kappa}_{(2)}\|\|T_{k_d}(f)\|_{p,x_d}\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^{d-1}),\mathbf{x}'} \\
&\ll 2^{-r|\mathbf{k}'|}\mathbf{k}'^{-\kappa}_{(2)}\|2^{-rk_d}(k_d+2)^{-\kappa}\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}),x_d}\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^{d-1}),\mathbf{x}'} \\
&= 2^{-r|\mathbf{k}|}\prod_{j=1}^{d}(k_j+2)^{-\kappa}\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)}. \quad \square
\end{aligned}$$

Let the univariate operator q_k be defined for $k \in \mathbb{Z}_+$, by

$$q_k := Q_{2^k,\lambda} - Q_{2^{k-1},\lambda}, \ k > 0, \ \ q_0 := Q_{1,\lambda},$$

and in the manner of the definition of (19), the multivariate operator $q_{\mathbf{k}}$ for $\mathbf{k} \in \mathbb{Z}^d_+$, by

$$q_{\mathbf{k}} := \prod_{j=1}^{d} q_{k_j}.$$

For $\mathbf{k} \in \mathbb{Z}^d_+$, we write $\mathbf{k} \to \infty$ if $k_j \to \infty$ for each $j \in \mathbb{N}_d$.

THEOREM 4. *Let $1 \leqslant p \leqslant \infty$, $1 < r < \infty$, $0 \leqslant \kappa < \infty$ and the function λ be a mask of type (r,κ). Then every $f \in \mathscr{H}_{\lambda,p}(\mathbb{T}^d)$ can be represented as the series*

$$f = \sum_{\mathbf{k} \in \mathbb{Z}^d_+} q_{\mathbf{k}}(f) \tag{20}$$

converging in L^p-norm, and we have for $\mathbf{k} \in \mathbb{Z}^d_+$,

$$\|q_{\mathbf{k}}(f)\|_p \leqslant C 2^{-r|\mathbf{k}|}\mathbf{k}_{(2)}^{-\kappa}\|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)} \tag{21}$$

with some constant C independent of f and $\mathbf{k}$.

Proof. Let $f \in \mathscr{H}_{\lambda,p}(\mathbb{T}^d)$. In a way similar to the proof of Lemma 3, we can show that

$$\|f - Q_{2^{\mathbf{k}}}(f)\|_p \ll \max_{j \in \mathbb{N}_d} 2^{-rk_j} k_j^{-\kappa} \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)},$$

and therefore,

$$\|f - Q_{2^{\mathbf{k}}}(f)\|_p \to 0, \ \mathbf{k} \to \infty,$$

where $2^{\mathbf{k}} = (2^{k_j} : \ j \in \mathbb{N}_d)$. On the other hand,

$$Q_{2^{\mathbf{k}}} = \sum_{s_j \leqslant k_j, \ j \in \mathbb{N}_d} q_{\mathbf{s}}(f).$$

This proves (20). To prove (21) we notice that from the definition it follows that

$$q_{\mathbf{k}} \ = \sum_{e \subset \mathbb{N}_d} (-1)^{|e|} T_{\mathbf{k}^e},$$

where $\mathbf{k}^e$ is defined by $k_j^e = k_j$ if $j \in e$, and $k_j^e = k_j - 1$ if $j \notin e$. Hence, by Lemma 3

$$\begin{aligned}\|q_{\mathbf{k}}(f)\|_p \ &\leqslant \sum_{e \subset \mathbb{N}_d} \|T_{\mathbf{k}^e}(f)\|_p \ll \sum_{e \subset \mathbb{N}_d} 2^{-r|\mathbf{k}^e|} (\mathbf{k}^e_{(2)})^{-\kappa} \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)} \\ &\ll 2^{-r|\mathbf{k}|} \mathbf{k}_{(2)}^{-\kappa} \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)}. \quad \square\end{aligned}$$

For approximation of $f \in \mathscr{H}_{\lambda,p}(\mathbb{T}^d)$, we introduce the linear operator $P_m, m \in \mathbb{N}$, by

$$P_m(f) := \sum_{|\mathbf{k}| \leqslant m} q_{\mathbf{k}}(f). \tag{22}$$

We give an upper bound for the error of the approximation of functions $f \in \mathscr{H}_{\lambda,p}(\mathbb{T}^d)$ by the operator P_m in the following theorem.

THEOREM 5. *Let $1 \leqslant p \leqslant \infty$, $1 < r < \infty$, $0 \leqslant \kappa < \infty$ and the function λ be a mask of type (r,κ). Then, we have for every $m \in \mathbb{N}$ and $f \in \mathscr{H}_{\lambda,p}(\mathbb{T}^d)$,*

$$\|f - P_m(f)\|_p \leqslant C\, 2^{-rm} m^{d-1-\kappa} \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)}$$

with some constant C independent of f and m.

Proof. From Theorem 4 we deduce that

$$\begin{aligned}\|f - P_m(f)\|_p &= \Big\| \sum_{|\mathbf{k}| > m} q_{\mathbf{k}}(f) \Big\|_p \leqslant \sum_{|\mathbf{k}| > m} \|q_{\mathbf{k}}(f)\|_p \\ &\ll \sum_{|\mathbf{k}| > m} 2^{-r|\mathbf{k}|} \mathbf{k}_{(2)}^{-\kappa} \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)} \ll \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)} \sum_{|\mathbf{k}| > m} 2^{-r|\mathbf{k}|} \mathbf{k}_{(2)}^{-\kappa} \\ &\ll 2^{-rm} m^{d-1-\kappa} \|f\|_{\mathscr{H}_{\lambda,p}(\mathbb{T}^d)}. \quad \square\end{aligned}$$

3.2. Convergence rate

We choose a positive integer $m \in \mathbb{N}$, a lattice vector $\mathbf{k} \in \mathbb{Z}_+^d$ with $|\mathbf{k}| \leqslant m$ and another lattice vector $\mathbf{s} = (s_j : j \in \mathbb{N}_d) \in \prod_{j\in\mathbb{N}_d} Z[2^{k_j+1}+1]$ to define the vector $\mathbf{y}_{\mathbf{k},\mathbf{s}} = \left(\frac{2\pi s_j}{2^{k_j+1}+1} : j \in \mathbb{N}_d\right)$. The Smolyak grid on $\mathbb{T}^d$ consists of all such vectors and is given as

$$G^d(m) := \left\{\mathbf{y}_{\mathbf{k},\mathbf{s}} : |\mathbf{k}| \leqslant m, \mathbf{s} \in \otimes_{j\in\mathbb{N}_d} Z[2^{k_j+1}+1]\right\}.$$

A simple computation confirms, for $m \to \infty$ that

$$|G^d(m)| = \sum_{|\mathbf{k}|\leqslant m} \prod_{j\in\mathbb{N}_d} (2^{k_j+1}+1) \asymp 2^d m^{d-1},$$

so, $G^d(m)$ is a sparse subset of a full grid of cardinality 2^{dm}. Moreover, by the definition of the linear operator P_m given in equation (22) we see that the range of P_m is contained in the subspace

$$\operatorname{span}\{\varphi_{\lambda,d}(\cdot - \mathbf{y}) : \mathbf{y} \in G^d(m)\}.$$

Other words, P_m defines a multivariate method of approximation by translates of the function $\varphi_{\lambda,d}$ on the sparse Smolyak grid $G^d(m)$. An upper bound for the error of this approximation of functions from $\mathscr{H}_{\lambda,p}(\mathbb{T}^d)$ is given in Theorem 5.

Now, we are ready to prove the next theorem, thereby establishing an upper bound of $M_n(U_{\lambda,p}, \varphi_{\lambda,d})_p$.

THEOREM 6. *If $1 \leqslant p \leqslant \infty$, $1 < r < \infty$, $0 \leqslant \kappa < \infty$ and the function λ be a mask of type (r,κ), then*

$$M_n(U_{\lambda,p}(\mathbb{T}^d), \varphi_{\lambda,d})_p \ll n^{-r}(\log n)^{(r+1)(d-1)-\kappa}.$$

Proof. If $n \in \mathbb{N}$ and m is the largest positive integer such that $|G^d(m)| \leqslant n$, then $n \asymp 2^m m^{d-1}$ and by Theorem 5 we have that

$$\begin{aligned} M_n(U_{\lambda,p}(\mathbb{T}^d), \varphi_{\lambda,d})_p &\leqslant \sup_{f\in U_{\lambda,p}(\mathbb{T}^d)} \|f - P_m(f)\|_p \\ &\ll 2^{-rm} m^{d-1-\kappa} \asymp n^{-r}(\log n)^{(r+1)(d-1)-\kappa}. \quad \square \end{aligned}$$

For $p = 2$, we are able to establish a lower bound for $M_n(U_{\lambda,2}(\mathbb{T}^d), \varphi_{\lambda,d})_2$. We prepare some auxiliary results. Let $\mathbb{P}_q(\mathbb{R}^l)$ be the set of algebraic polynomials on $\mathbb{R}^l$ of total degree at most q, and

$$\mathbb{E}^m := \{\mathbf{t} = (t_j : j \in \mathbb{N}_m) : |t_j| = 1, j \in \mathbb{N}_m\}.$$

We define the polynomial maifold

$$\mathbb{M}_{m,l,q} := \left\{(p_j(\mathbf{u}) : j \in \mathbb{N}_m) : p_j \in \mathbb{P}_q(\mathbb{R}^l), j \in \mathbb{N}_m, \mathbf{u} \in \mathbb{R}^l\right\}.$$

Denote by $\|\mathbf{x}\|_2$ the Euclidean norm of a vector $\mathbf{x}$ in $\mathbb{R}^m$. The following lemma was proven in [5].

LEMMA 4. *Let $m,l,q \in \mathbb{N}$ satisfy the inequality $l\log(\frac{4emq}{l}) \leqslant \frac{m}{4}$. Then there is a vector $\mathbf{t} \in \mathbb{E}^m$ and a positive constant c such that*

$$\inf\left\{\|\mathbf{t}-\mathbf{x}\|_2 : \mathbf{x} \in \mathbb{M}_{m,l,q}\right\} \geqslant c m^{1/2}.$$

THEOREM 7. *If $1 < r < \infty$, $0 \leqslant \kappa < \infty$ and the function λ be a mask of type (r,κ), then we have that*

$$n^{-r}(\log n)^{r(d-2)-d\kappa} \ll M_n(U_{\lambda,2})_2 \ll n^{-r}(\log n)^{(r+1)(d-1)-\kappa}. \tag{23}$$

Proof. The upper bound of (23) is in Theorem 6. Let us prove the lower bound by developing a technique used in the proofs of [5, Theorem 1.1] and [1, Theorem 4.4]. For a positive number a we define a subset $\mathbb{H}(a)$ of lattice vectors by

$$\mathbb{H}(a) := \left\{\mathbf{k} = (k_j : j \in \mathbb{N}_d) \in \mathbb{Z}^d : \prod_{j\in\mathbb{N}_d} |k_j| \leqslant a\right\}.$$

Notice that $|\mathbb{H}(a)| \asymp a(\log a)^{d-1}$ when $a \to \infty$. To apply Lemma 4, for any $n \in \mathbb{N}$, we take $q = \lfloor n(\log n)^{-d+2}\rfloor + 1$, $m = 5(2d+1)\lfloor n\log n\rfloor$ and $l = (2d+1)n$. With these choices we obtain

$$|\mathbb{H}(q)| \asymp m \tag{24}$$

and

$$q \asymp m(\log m)^{-d+1} \tag{25}$$

as $n \to \infty$. Moreover, we have that

$$\lim_{n\to\infty} \frac{l}{m}\log\left(\frac{4emq}{l}\right) = \frac{1}{5},$$

and therefore, the assumption of Lemma 4 is satisfied for $n \to \infty$.

Now, let us specify the polynomial manifold $\mathbb{M}_{m,l,q}$. To this end, we put $\zeta := q^{-r}m^{-1/2}(\log q)^{-d\kappa}$ and let $\mathbb{Y}$ be the set of trigonometric polynomials on $\mathbb{T}^d$, defined by

$$\mathbb{Y} := \left\{f = \zeta \sum_{\mathbf{k}\in\mathbb{H}(q)} a_{\mathbf{k}} t_{\mathbf{k}} : \mathbf{t} = (t_{\mathbf{k}} : \mathbf{k} \in \mathbb{H}(q)) \in \mathbb{E}^{|\mathbb{H}(q)|}\right\}.$$

If $f \in \mathbb{Y}$ and

$$f = \zeta \sum_{\mathbf{k}\in\mathbb{H}(q)} a_{\mathbf{k}} t_{\mathbf{k}},$$

then $f = \varphi_{\lambda,d} * g$ for some trigonometric polynomial g such that

$$\|g\|^2_{L^2(\mathbb{T}^d)} \leqslant \zeta^2 \sum_{\mathbf{k}\in\mathbb{H}(q)} |\lambda(\mathbf{k})|^{-2}.$$

Since

$$\zeta^2 \sum_{\mathbf{k}\in\mathbb{H}(q)} |\lambda(\mathbf{k})|^{-2} \leqslant \zeta^2 q^{2r} \sum_{\mathbf{k}\in\mathbb{H}(q)} \left|\log\prod_{j=1}^{d} k_j\right|^{2\kappa}$$

$$\leqslant \zeta^2 q^{2r} \sum_{\mathbf{k}\in\mathbb{H}(q)} \left|\sum_{j=1}^{n} \log k_j\right|^{2d\kappa}$$

$$\leqslant \zeta^2 q^{2r} (\log q)^{2d\kappa} |\mathbb{H}(q)| = m^{-1}|\mathbb{H}(q)|,$$

by (24) that there is a positive constant c such that $\|g\|_{L^2(\mathbb{T}^d)} \leqslant c$ for all $n \in \mathbb{N}$. Therefore, we can either adjust functions in $\mathbb{Y}$ by dividing them by c, or we can assume without loss of generality that $c = 1$, and obtain $\mathbb{Y} \subseteq U_{\lambda,2}(\mathbb{T}^d)$.

We are now ready to prove the lower bound for $M_n(U_{\lambda,2}(\mathbb{T}^d))_2$. We choose any $\varphi \in L^2(\mathbb{T}^d)$ and let v be any function formed as a linear combination of n translates of the function φ:

$$v = \sum_{j\in\mathbb{N}_n} c_j \varphi(\cdot - \mathbf{y}_j).$$

By the well-known Bessel inequality we have for a function

$$f = \zeta \sum_{\mathbf{k}\in\mathbb{H}(q)} a_{\mathbf{k}} t_{\mathbf{k}} \in \mathbb{Y},$$

that

$$\|f - v\|^2_{L^2(\mathbb{T}^d)} \geqslant \zeta^2 \sum_{\mathbf{k}\in\mathbb{H}(q)} \left| t_{\mathbf{k}} - \frac{\widehat{\varphi}(\mathbf{k})}{\zeta} \sum_{j\in\mathbb{N}_n} c_j e^{i(\mathbf{y}_j,\mathbf{k})} \right|^2. \tag{26}$$

We introduce a polynomial manifold so that we can use Lemma 4 to get a lower bound for the expressions on the left hand side of inequality (26). To this end, we define the vector $\mathbf{c} = (c_j : j \in \mathbb{N}_n) \in \mathbb{R}^n$ and for each $j \in \mathbb{N}_n$, let $\mathbf{z}_j = (z_{j,l} : l \in \mathbb{N}_d)$ be a vector in $\mathbb{C}^d$ and then concatenate these vectors to form the vector $\mathbf{z} = (\mathbf{z}_j : j \in \mathbb{N}_n) \in \mathbb{C}^{nd}$. We employ the standard multivariate notation

$$\mathbf{z}_j^{\mathbf{k}} = \prod_{l\in\mathbb{N}_d} z_{j,l}^{k_l}$$

and require vectors $\mathbf{w} = (\mathbf{c},\mathbf{z}) \in \mathbb{R}^n \times \mathbb{C}^{nd}$ and $\mathbf{u} = (\mathbf{c}, \Re\mathbf{z}, \Im\mathbf{z}) \in \mathbb{R}^l$ to be written in concatenate form. Now, we introduce for each $\mathbf{k} \in \mathbb{H}(q)$ the polynomial $\mathbf{q}_{\mathbf{k}}$ defined at $\mathbf{w}$ as

$$\mathbf{q}_{\mathbf{k}}(\mathbf{w}) := \frac{\widehat{\varphi}(\mathbf{k})}{\zeta} \sum_{\mathbf{j}\in\mathbb{H}(q)} c_{\mathbf{j}} \mathbf{z}^{\mathbf{j}}.$$

We only need to consider the real part of $\mathbf{q}_{\mathbf{k}}$, namely, $\mathbf{p}_{\mathbf{k}} = \Re\mathbf{q}_{\mathbf{k}}$ since we have that

$$\inf\left\{ \sum_{\mathbf{k}\in\mathbb{H}(q)} \left| t_{\mathbf{k}} - \frac{\widehat{\varphi}(\mathbf{k})}{\zeta} \sum_{j\in\mathbb{N}_n} c_j e^{i(\mathbf{y}_j,\mathbf{k})} \right|^2 : c_j \in \mathbb{R}, \mathbf{y}_j \in \mathbb{T}^d \right\}$$

$$\geqslant \inf\left\{ \sum_{\mathbf{k}\in\mathbb{H}(q)} |t_{\mathbf{k}} - p_{\mathbf{k}}(\mathbf{u})|^2 : \mathbf{u} \in \mathbb{R}^l \right\}.$$

Therefore, by Lemma 4 and (25) we conclude there is a vector $\mathbf{t}^0 = (t^0_{\mathbf{k}} : \mathbf{k} \in \mathbb{H}(q)) \in \mathbb{E}^{h_q}$ and the corresponding function

$$f^0 = \zeta \sum_{\mathbf{k} \in \mathbb{H}(q)} t^0_{\mathbf{k}} \chi_{\mathbf{k}} \in \mathbb{Y}$$

for which there is a positive constant c such that for every v of the form

$$v = \sum_{j \in \mathbb{N}_n} c_j \varphi(\cdot - \mathbf{y}_j),$$

we have that

$$\| f^0 - v\|_{L^2(\mathbb{T}^d)} \geqslant c\zeta m^{\frac{1}{2}} = q^{-r}(\log q)^{-d\kappa} \asymp n^{-r}(\log n)^{r(d-2)-d\kappa}$$

which proves the lower bound of (23). □

Similar to the proof of the above theorem, we can prove the following theorem for the case $-\infty < \kappa < 0$.

THEOREM 8. *If* $1 < r < \infty, -\infty < \kappa < 0$ *and the function* λ *be a mask of type* (r,κ)*, then we have that*

$$n^{-r}(\log n)^{r(d-2)-\kappa} \ll M_n(U_{\lambda,2}(\mathbb{T}^d))_2 \ll n^{-r}(\log n)^{(r+1)(d-1)-d\kappa}.$$

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