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PRE–GRÜSS AND GRÜSS–OSTROWSKI LIKE INEQUALITIES IN BANACH SPACES

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Abstract. For a given Banach space and its dual space we investigate a Chebyshev type functional. We derive a pre-Grüss inequality for the functional. We discuss various variants of assumptions leading to this inequality. To do so, we employ some superquadratic as well as convex control functions in order to weaken the classical Dragomir's condition. Next, we establish a corresponding Grüss-Ostrowski like inequalities for the space $L^p_{[a,b]}$.

1. Introduction

The classical Grüss' inequality [16] asserts that if $f,g:[a,b]\to\mathbb{R}$ are two integrable functions on $[a,b]$ such that

$$\alpha_0 \leqslant f(t) \leqslant \beta_0 \quad \text{and} \quad \gamma_0 \leqslant g(t) \leqslant \delta_0 \quad \text{for all } t\in[a,b]$$

with $\alpha_0,\beta_0,\gamma_0,\delta_0\in\mathbb{R}$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\right|\leqslant\frac{1}{4}(\beta_0-\alpha_0)(\delta_0-\gamma_0).$$

As usual, the symbol $L^p_{[a,b]}$ for $1\leqslant p<\infty$ denotes the space of p-power integrable functions on interval $[a,b]$ equipped with the norm $\|f\|_p=\left(\int_a^b|f(t)|^p\,dt\right)^{1/p}$, and $L^\infty_{[a,b]}$ denotes the space of all essentially bounded functions on $[a,b]$ with the norm $\|f\|_\infty=\operatorname{ess\,sup}_{x\in[a,b]}|f(x)|$.

It is known by Ostrowski's inequality [25, p. 468] that if $f:[a,b]\to\mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leqslant\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \quad \text{for } x\in[a,b].$$

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A Grüss-Ostrowski type inequality due to Dragomir and Wang [14] says that if $f:[a,b]\to\mathbb{R}$ is a differentiable function with bounded derivative and

$$\alpha_0 \leqslant f'(t) \leqslant \beta_0 \quad \text{for } t\in[a,b],$$

where $\alpha_0,\beta_0\in\mathbb{R}$, then for $x\in[a,b]$,

$$\left| f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right| \leqslant \frac{1}{4}(b-a)(\beta_0-\alpha_0).$$

Over the years Grüss and Ostrowski type inequalities and their applications have been studied by many mathematicians (see e.g., [13, 14, 15, 19, 20, 22, 23, 24, 30, 32, 33, 34, 35]). Dragomir [8] proved a generalization of Grüss inequality in inner product spaces. Niezgoda [28] developed this idea by introducing some class of bounding support functions in place of bounding constants $\alpha_0,\beta_0,\gamma_0,\delta_0$.

In the present paper we deal with pre-Grüss and Grüss-Ostrowski like inequalities in Banach spaces. In Section 2, we derive a pre-Grüss type inequality and discuss numerous variants of assumptions ensuring the validity of this inequality. For this end, we use superquadratic and/or convex control functions and the notion of G-majorization on a Banach space. As applications, in Section 3 we employ the obtained results to establish some corresponding inequalities on the $L^p_{[a,b]}$-space of p-power integrable functions.

2. Pre-Grüss type inequalities on a Banach space

Throughout $(X,\|\cdot\|)$ is a real Banach space and X^* stands for the dual space of X, i.e., the space of all real bounded linear functionals on X. For an $x^*\in X^*$ and $x\in X$, we write $\langle x,x^*\rangle$ instead of the value $x^*(x)$ of x^* at x. The norm on X^* is given by $\|x^*\|_* = \sup_{\|x\|=1}|\langle x,x^*\rangle|$ for $x^*\in X^*$ and $x\in X$.

By $\mathbb{B}(X)$ we denote the set of all bounded linear operators on the Banach space X. For $L\in\mathbb{B}(X)$, the operator norm of L is defined by

$$\|L\| = \sup_{\|y\|=1}\|Ly\|.$$

Thus one has

$$\|Lx\| \leqslant \|L\|\,\|x\| \quad \text{for any } x\in X. \tag{1}$$

Throughout $e\in X$ and $e^*\in X^*$ are two elements such that $\langle e,e^*\rangle=1$.

The *Chebyshev functional* $T_{e,e^*}: X\times X^*\to\mathbb{R}$ is defined by

$$T_{e,e^*}(x,x^*) = \langle x,x^*\rangle - \langle x,e^*\rangle\langle e,x^*\rangle \quad \text{for any } x\in X \text{ and } x^*\in X^*. \tag{2}$$

By standard algebra we obtain

$$T_{e,e^*}(x,x^*) = \langle x-\langle x,e^*\rangle e,\, x^*-\langle e,x^*\rangle e^*\rangle \quad \text{for any } x\in X \text{ and } x^*\in X^*. \tag{3}$$

We introduce linear operators $Q : X \to X$ and $S : X^* \to X^*$ by

$$Qx = x - \langle x, e^* \rangle e \quad \text{for } x \in X, \quad \text{and} \quad Sx^* = x^* - \langle e, x^* \rangle e^* \quad \text{for } x^* \in X^*. \tag{4}$$

It is not hard to check that $\ker Q = \operatorname{span} e$ and $\ker S = \operatorname{span} e^*$.

By using (4) one can verify that the operators Q and S are idempotent, i.e.,

$$Q^2 = Q \quad \text{and} \quad S^2 = S.$$

Our interest lies in establishing Grüss-Ostrowski type inequalities. To do so, we shall use the following result presenting pre-Grüss like inequality (6).

LEMMA 1. *Let $x \in X$ and $x^* \in X^*$. Under the above notation, if*

$$\|x - x_0\| \leqslant r \quad \textit{for some } x_0 \in \operatorname{span} e \textit{ and } r > 0, \tag{5}$$

then

$$|\langle x, x^* \rangle - \langle x, e^* \rangle \langle e, x^* \rangle| \leqslant r \|S(x^* - x_0^*)\|_* \tag{6}$$

for any $x_0^* \in \operatorname{span} e^*$.

Proof. In light of (3) and (4), we see that

$$\langle x, x^* \rangle - \langle x, e^* \rangle \langle e, x^* \rangle = \langle x - \langle x, e^* \rangle e, x^* - \langle e, x^* \rangle e^* \rangle$$
$$= \langle Qx, Sx^* \rangle = \langle Q(x - x_0), S(x^* - x_0^*) \rangle,$$

since $x_0 \in \operatorname{span} e = \ker Q$ and $x_0^* \in \operatorname{span} e^* = \ker S$ (cf. [28, p. 120]).

Likewise, we have

$$\langle x, x^* \rangle - \langle x, e^* \rangle \langle e, x^* \rangle = \langle x - \mu e, x^* - \langle e, x^* \rangle e^* \rangle = \langle x - \mu e, S(x^* - x_0^*) \rangle$$

for any $\mu \in \mathbb{R}$, because $\langle e, x^* - \langle e, x^* \rangle e^* \rangle = 0$ (cf. [26, p. 233]).

By putting $\mu = \mu_0$, where $x_0 = \mu_0 e \in \operatorname{span} e$, and using Hölder type inequality (1) and (5), we get

$$|\langle x, x^* \rangle - \langle x, e \rangle \langle e, x^* \rangle| = |\langle x - x_0, S(x^* - x_0^*) \rangle|$$
$$\leqslant \|x - x_0\| \|S(x^* - x_0^*)\|_* \leqslant r \|S(x^* - x_0^*)\|_*,$$

which proves (6). $\square$

REMARK 1. Concerning (6), we do not estimate the norm $\|S(x^* - x_0^*)\|$, because it will be calculated explicitly in future applications of (6).

A function $\varphi : X \to \mathbb{R}$ is said to be *Gateaux differentiable* if for each $x, h \in X$ there exists the directional derivative

$$\nabla_h \varphi(x) = \lim_{t \to 0} \frac{\varphi(x + th) - \varphi(x)}{t},$$

and for each $x \in X$ the functional $h \mapsto \nabla_h \varphi(x)$ from X to $\mathbb{R}$ is linear and continuous. This functional is denoted by $\nabla\varphi(x)$ and is called the *gradient* of φ at x. So, it holds that $\nabla\varphi(x) \in X^*$ and

$$\nabla_h \varphi(x) = \langle h, \nabla\varphi(x)\rangle \quad \text{for } x, h \in X.$$

A Gateaux differentiable function $\varphi : X \to \mathbb{R}$ is called *superquadratic* on a nonempty set $U \subset X$, if

$$\varphi(z+h) \geqslant \varphi(z) + \langle h, \nabla\varphi(z)\rangle + \varphi(h) \quad \text{for all } z, z+h \in U \tag{7}$$

(cf. [1, 2, 3]).

We now employ superquadratic functions to establish statements satisfying condition (5) (see (9)).

The following result is in line of [9, Lemma 2.1] and [28, Lemma 4.1].

LEMMA 2. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a strictly increasing function such that the function $\varphi = \psi(\|\cdot\|) : X \to \mathbb{R}$ is Gateaux differentiable. Assume that φ is superquadratic on a nonempty set $U \subset X$.*

If $\beta, x, x_0 \in X$ are such that $x - x_0 \in U$, $\beta - x_0 \in U$ and

$$\langle \beta - x, \nabla\varphi\,(x - x_0)\rangle + \varphi(\beta - x) \geqslant 0, \tag{8}$$

then

$$\|x - x_0\| \leqslant \|\beta - x_0\|, \tag{9}$$

that is, x belongs to the $\|\cdot\|$-ball of radius $r = \|\beta - x_0\|$ centered at the point x_0.

In particular, if $x_0 = \frac{\alpha+\beta}{2}$ for some $\alpha \in X$ and

$$\left\langle \beta - x, \nabla\varphi\left(x - \frac{\alpha+\beta}{2}\right)\right\rangle + \varphi(\beta - x) \geqslant 0, \tag{10}$$

then

$$\left\| x - \frac{1}{2}(\alpha+\beta)\right\| \leqslant \left\| \frac{1}{2}(\beta - \alpha)\right\|. \tag{11}$$

Proof. By setting $z = x - x_0$ and $h = \beta - x$, we get $z + h = \beta - x_0$. Next, by (7) we obtain

$$\varphi\,(\beta - x_0) \geqslant \varphi\,(x - x_0) + \langle \beta - x, \nabla\varphi\,(x - x_0)\rangle + \varphi(\beta - x).$$

Therefore (8) ensures that

$$\varphi\,(\beta - x_0) \geqslant \varphi\,(x - x_0).$$

That is,

$$\psi\,(\|\beta - x_0\|) \geqslant \psi\,(\|x - x_0\|). \tag{12}$$

Since ψ is strictly increasing on $[0,\infty)$, ψ is invertible on $[0,\infty)$ and ψ^{-1} is strictly increasing on $\psi([0,\infty))$. For this reason (12) implies that

$$\|\beta - x_0\| \geqslant \|x - x_0\|,$$

as desired.

To see the implication (10) $\Rightarrow$ (11), use (8) $\Rightarrow$ (9) with the substitution $x_0 = \frac{1}{2}(\alpha+\beta)$ and $\beta - x_0 = \frac{1}{2}(\beta-\alpha)$. $\square$

We now discuss a simplification of condition (8).

LEMMA 3. *Under the assumptions of Lemma 2, let $0 < c \in \mathbb{R}$ be such that the function $\varphi = \psi(\|\cdot\|)$ has the property*

$$\varphi(h) = c\,\langle h, \nabla\varphi(h)\rangle \quad \textit{for } h \in X. \tag{13}$$

Then condition (8) takes the form

$$\langle \beta - x, \nabla\varphi\,(x - x_0) + c\,\nabla\varphi(\beta - x)\rangle \geqslant 0. \tag{14}$$

Consequently, if conditions (13) and (14) are satisfied, then inequality (9) holds.

Proof. Under the validity of condition (13), we have

$$\varphi(\beta - x) = c\,\langle \beta - x, \nabla\varphi(\beta - x)\rangle.$$

From this condition (8) can be restated as

$$\langle \beta - x, \nabla\varphi\,(x - x_0)\rangle + c\,\langle \beta - x, \nabla\varphi(\beta - x)\rangle \geqslant 0,$$

which gives

$$\langle \beta - x, \nabla\varphi\,(x - x_0) + c\,\nabla\varphi(\beta - x)\rangle \geqslant 0,$$

completing the proof of (14).

To see the last assertion of Lemma 3, use Lemma 2. $\square$

We now interpret the crucial conditions (8), (13) and (14).

EXAMPLE 1. Let X be a real linear space endowed with an inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{1/2}$. Obviously, $X^* = X$, $\|\cdot\|_* = \|\cdot\|$ and functionals in X^* are induced by vectors in X via the inner product $\langle\cdot,\cdot\rangle$. Let $\beta, x, x_0 \in X$.

Consider the function $\psi(t) = t^2$ for $t \in [0,\infty)$. Then $\varphi(z) = \|z\|^2 = \langle z, z\rangle$ and $\nabla\varphi(z) = 2z$ for $z \in X$. So, $\nabla\varphi\,(x - x_0) = 2\,(x - x_0)$ and $\nabla\varphi(\beta - x) = 2(\beta - x)$. In addition, φ is superquadratic on X in the sense of (7).

In consequence, condition (8) takes the form

$$\langle \beta - x, 2\,(x - x_0)\rangle + \|\beta - x\|^2 \geqslant 0.$$

Evidently,

$$\varphi(h) = \|h\|^2 = \frac{1}{2}\langle h, 2h\rangle = \frac{1}{2}\langle h, \nabla\varphi(h)\rangle \quad \text{for } h \in X,$$

which guarantees that (13) is satisfied with $c = \frac{1}{2}$.

Therefore condition (8) can be replaced by (14). Here (14) can be rewritten as

$$\langle \beta - x, 2(x - x_0) + (\beta - x)\rangle \geqslant 0,$$

which reduces to

$$\langle \beta - x, x + \beta - 2x_0\rangle \geqslant 0.$$

Let $\alpha \in X$. With the substitution $x_0 = \frac{1}{2}(\alpha + \beta)$, the last inequality becomes

$$\langle \beta - x, x - \alpha\rangle \geqslant 0. \tag{15}$$

This is the classical condition due to Dragomir [7, 8, 9, 10, 11] intended to prove Grüss type inequalities in the context of inner product spaces.

As noted in [28, Lemma 4.1], statement (15) amounts to the condition

$$\alpha \leqslant_C x \leqslant_{\mathrm{dual}C} \beta \tag{16}$$

for some cone preorder $\leqslant_C$ on X generated by a convex cone $C \subset X$, where $\mathrm{dual}C = \{z \in X : \langle z, v\rangle \geqslant 0 \text{ for all } v \in C\}$. In the special case when C is self-dual, i.e., $C = \mathrm{dual}C$, then (16) simplies to

$$\alpha \leqslant_C x \leqslant_C \beta. \tag{17}$$

For example, if $X = \mathbb{R}^n$ and $x = (x_1, \dots, x_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, and $C = \mathbb{R}^n_+$, then (17) means

$$\alpha_i \leqslant x_i \leqslant \beta_i \quad \text{for } i = 1\dots, n.$$

The next result is an extension of [28, Theorem 4.2] from inner product spaces to Banach spaces.

THEOREM 1. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a strictly increasing function such that $\varphi = \psi(\|\cdot\|) : X \to \mathbb{R}$ is a Gateaux differentiable function. Assume φ is superquadratic on a nonempty set $U \subset X$. Let $\beta, x, x_0 \in X$ and $x^* \in X^*$. Assume that*

(i) $x_0 \in \mathrm{span}\, e$,

(ii) $x - x_0 \in U$ *and* $\beta - x_0 \in U$,

(iii)

$$\langle \beta - x, \nabla\varphi(x - x_0)\rangle + \varphi(\beta - x) \geqslant 0. \tag{18}$$

Then

$$|\langle x, x^*\rangle - \langle x, e^*\rangle\langle e, x^*\rangle| \leqslant \|\beta - x_0\| \|S(x^* - x_0^*)\|_* \tag{19}$$

for any $x_0^* \in \mathrm{span}\, e^*$.

Proof. By (ii) and Lemma 2 we deduce that

$$\|x-x_0\| \leqslant \|\beta - x_0\|. \tag{20}$$

By (i) and (20) we see that condition (5) is satisfied with $r = \|\beta - x_0\|$.

It is now enough to utilize inequality (6) from Lemma 1. □

COROLLARY 1. *Under the assumptions of Theorem 1, let* $x_0 = \frac{\alpha+\beta}{2}$ *for some* $\alpha \in X$.

If

$$\left\langle \beta - x, \nabla\varphi\left(x - \frac{\alpha+\beta}{2}\right)\right\rangle + \varphi(\beta - x) \geqslant 0, \tag{21}$$

then

$$|\langle x, x^*\rangle - \langle x, e^*\rangle\langle e, x^*\rangle| \leqslant \frac{1}{2}\|\beta-\alpha\|\|S(x^*-x_0^*)\|_* \tag{22}$$

for any $x_0^* \in \operatorname{span} e^*$.

Proof. With $x_0 = \frac{\alpha+\beta}{2}$ we have $\|\beta - x_0\| = \left\|\frac{\beta-\alpha}{2}\right\|$. By making use inequality (19) in Theorem 1 we obtain (22), as wanted. □

THEOREM 2. *Let* $\psi : [0,\infty) \to \mathbb{R}$ *be a strictly increasing function such that* $\varphi = \psi(\|\cdot\|) : X \to \mathbb{R}$ *is a Gateaux differentiable convex function. Let* $\beta, x, x_0 \in X$ *and* $x^* \in X^*$. *Assume that*

(i) $x_0 \in \operatorname{span} e$,

(ii)

$$\langle \beta - x, \nabla\varphi(x - x_0)\rangle \geqslant 0. \tag{23}$$

Then

$$|\langle x, x^*\rangle - \langle x, e^*\rangle\langle e, x^*\rangle| \leqslant \|\beta - x_0\|\|S(x^*-x_0^*)\|_* \tag{24}$$

for any $x_0^* \in \operatorname{span} e^*$.

Proof. By virtue of the gradient inequality for φ we can write

$$\varphi(z+h) \geqslant \varphi(z) + \langle h, \nabla\varphi(z)\rangle \quad \text{for all } z, h \in X.$$

By using the substitutions $z = x - x_0$ and $h = \beta - x$ we derive

$$\varphi(\beta - x_0) \geqslant \varphi(x - x_0) + \langle \beta - x, \nabla\varphi(x - x_0)\rangle.$$

With the help of (23) we infer that

$$\varphi(\beta - x_0) \geqslant \varphi(x - x_0).$$

In other words,
$$\psi(\|\beta - x_0\|) \geqslant \psi(\|x - x_0\|). \tag{25}$$
Since the function ψ is strictly increasing on $[0,\infty)$, it is invertible and the inverse ψ^{-1} is strictly increasing on $\psi([0,\infty))$. Therefore we deduce from (25) that
$$\|x - x_0\| \leqslant \|\beta - x_0\|. \tag{26}$$
So, condition (5) with $r = \|\beta - x_0\|$ is fulfilled by (26) and (i).

By employing inequality (6) from Lemma 1, we obtain the desired assertion (24). □

COROLLARY 2. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a strictly increasing function such that $\varphi = \psi(\|\cdot\|) : X \to \mathbb{R}$ is a Gateaux differentiable convex function. Let $\alpha, \beta, x \in X$ and $x^* \in X^*$. Assume that*

(i) $\frac{\alpha+\beta}{2} \in \operatorname{span} e$,

(ii)
$$\left\langle \beta - x, \nabla\varphi\left(x - \frac{\alpha+\beta}{2}\right)\right\rangle \geqslant 0. \tag{27}$$

Then
$$|\langle x, x^*\rangle - \langle x, e^*\rangle\langle e, x^*\rangle| \leqslant \frac{1}{2}\|\beta - \alpha\|\|S(x^* - x_0^*)\|_* \tag{28}$$
for any $x_0^* \in \operatorname{span} e^*$.

Proof. By putting $x_0 = \frac{\alpha+\beta}{2}$, one has $\|\beta - x_0\| = \left\|\frac{\beta-\alpha}{2}\right\|$. Now, it is enough apply inequality (24) from Theorem 2 to see (28). □

In Corollary 2, in the situation when X is an inner product space and $\psi(t) = t^2$ for $t \in [0,\infty)$, condition (27) means that
$$\left\langle \beta - x, x - \frac{\alpha+\beta}{2}\right\rangle \geqslant 0,$$
which corresponds to Dragomir's condition (15) (see Example 1).

Throughout, unless stated otherwise, it is assumed that $G \subset \mathbb{B}(X)$ is a semigroup of continuous linear operators from X into X.

Given $x, y \in X$ we say that y is G-*majorized* by x, written as $y \prec_G x$, if y belongs to the convex hull of the set Gx, i.e.,
$$y = \sum_{i=1}^{m} t_i g_i x$$
for some positive integer m, operators $g_i \in G$ and real numbers $t_i \in [0,1]$ for $i = 1,\ldots,m$ such that $\sum_{i=1}^{m} t_i = 1$.

It is readily seen that the relation $\prec_G$ is a preorder on X, i.e., $\prec_G$ is reflexive and transitive on X. Furthermore, for any $x \in X$ it holds that $\{y \in X : y \prec_G x\} = \operatorname{conv} Gx$, where $\operatorname{conv} Gx$ is the convex hull of the set $Gx = \{gx : g \in G\}$.

A function $\Phi : X \to \mathbb{R}$ is called G*-increasing*, if for all $x, y \in X$,

$$y \prec_G x \quad \text{implies} \quad \Phi(y) \leqslant \Phi(x).$$

A function $\Phi : X \to \mathbb{R}$ is called G*-invariant*, if

$$\Phi(gx) = \Phi(x) \quad \text{for all } x \in X \text{ and } g \in G.$$

A function $\Phi : X \to \mathbb{R}$ is called G*-subinvariant*, if

$$\Phi(gx) \leqslant \Phi(x) \quad \text{for all } x \in X \text{ and } g \in G.$$

A G-increasing function on X must be necessarily G-subinvariant on X, because $gx \prec_G x$ for all $x \in X$ and $g \in G$.

It follows that if a function $\Phi : X \to \mathbb{R}$ is convex and G-subinvariant on X, then Φ is G-increasing on X. In fact, taking any $x, y \in X$ such that $y \prec_G x$, we get $y = \sum_{i=1}^{m} t_i g_i x$ for some $g_i \in G$ and $t_i \in [0,1]$, $i = 1, \ldots, m$, with $\sum_{i=1}^{m} t_i = 1$. Hence

$$\Phi(y) = \Phi\left(\sum_{i=1}^{m} t_i g_i x\right) \leqslant \sum_{i=1}^{m} t_i \Phi(g_i x) \leqslant \sum_{i=1}^{m} t_i \Phi(x) = \Phi(x),$$

as claimed.

So, if a function $\Phi : X \to \mathbb{R}$ is convex and G-invariant on X, then Φ is G-increasing on X. In consequence, if a norm on X is G-subinvariant then it is G-increasing. In particular, if a norm on X is G-invariant then it is G-increasing.

We introduce the subspace

$$M_G(X) = \{x \in X : gx = x \text{ for all } g \in G\}.$$

This subspace consists of all minimal points for the preorder $\prec_G$ on X.

LEMMA 4. *Let* $x_0 \in \operatorname{span} e$ *with* $e \in M_G(X)$ *and* $\|\cdot\|$ *be* G*-subinvariant. Let* $x, \beta \in X$.

Then

$$x \prec_G \beta \quad \textit{implies} \quad \|x - x_0\| \leqslant \|\beta - x_0\|.$$

Proof. Let $x \prec_G \beta$. Then $x = \sum_{i=1}^{m} t_i g_i \beta$ for some $g_i \in G$ and $t_i \in [0,1]$, $i = 1, \ldots, m$, with $\sum_{i=1}^{m} t_i = 1$. Since $e \in M_G(X)$, we have $g_i e = e$ for $i = 1, \ldots, m$. Hence $g_i x_0 = x_0$ for $i = 1, \ldots, m$, because $x_0 = c_0 e$ for some $c_0 \in \mathbb{R}$. Therefore $x_0 = \sum_{i=1}^{m} t_i g_i x_0$.

So, we obtain $x - x_0 = \sum_{i=1}^{m} t_i g_i(\beta - x_0)$, whence, $x - x_0 \prec_G \beta - x_0$. Moreover, $\|\cdot\|$ is G-subinvariant and convex. Consequently, $\|\cdot\|$ is G-increasing. For this reason we get $\|x - x_0\| \leqslant \|\beta - x_0\|$, as required. $\square$

THEOREM 3. *Let $G \subset \mathbb{B}(X)$ be a semigroup and $\|\cdot\|$ be G-subinvariant on X. Let $\beta, x, x_0 \in X$ and $x^* \in X^*$. Assume that*

(i) $x_0 \in \operatorname{span} e$ *with* $e \in M_G(X)$,

(ii) $x \prec_G \beta$.

Then

$$|\langle x, x^* \rangle - \langle x, e^* \rangle \langle e, x^* \rangle| \leqslant \|\beta - x_0\| \|S(x^* - x_0^*)\|_* \tag{29}$$

for any $x_0^* \in \operatorname{span} e^*$.

Proof. In light of Lemma 4, by (i)–(ii), we obtain $\|x - x_0\| \leqslant \|\beta - x_0\|$. By taking $r = \|\beta - x_0\|$ and applying inequality (6) from Lemma 1 we deduce that (29) holds valid. $\square$

COROLLARY 3. *Let $G \subset \mathbb{B}(X)$ be a semigroup and $\|\cdot\|$ be G-subinvariant on X. Let $\alpha, \beta, x \in X$ and $x^* \in X^*$. Assume that*

(i) $\frac{\alpha+\beta}{2} \in \operatorname{span} e$ *with* $e \in M_G(X)$,

(ii) $x \prec_G \beta$.

Then

$$|\langle x, x^* \rangle - \langle x, e^* \rangle \langle e, x^* \rangle| \leqslant \frac{1}{2} \|\beta - \alpha\| \|S(x^* - x_0^*)\|_* \tag{30}$$

for any $x_0^* \in \operatorname{span} e^*$.

Proof. We set $x_0 = \frac{\alpha+\beta}{2}$. Then $\|\beta - x_0\| = \left\| \frac{\beta-\alpha}{2} \right\|$. By making use inequality (29) from Theorem 3 we obtain (30), as wanted. $\square$

3. Applications for L^p functions

In this section we are concerned with interpretations and applications of the results obtained in Section 2. We show some integral pre-Grüss and Grüss-Ostrowski type inequalities for L^p-functions with restrictions.

We consider the spaces $X = L^p_{[a,b]}$ and $X^* = L^q_{[a,b]}$ with $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. For $x = f \in L^p_{[a,b]}$ and $x^* = g \in L^q_{[a,b]}$, we have

$$\|x\| = \|f\|_p = \left(\int_a^b |f(t)|^p \, dt \right)^{1/p} \text{ and } \|x^*\|_* = \|g\|_q = \left(\int_a^b |g(t)|^q \, dt \right)^{1/q}, \tag{31}$$

$$\langle x, x^* \rangle = \langle f, g \rangle = \int_a^b f(t)g(t)\,dt. \tag{32}$$

By setting $\mathbf{1}(t) = 1$ for $t \in [a,b]$, we put

$$e = \frac{1}{(b-a)^{1/2}}\mathbf{1} \quad \text{and} \quad e^* = \frac{1}{(b-a)^{1/2}}\mathbf{1}.$$

It is easily seen that $e \in L^p_{[a,b]}$, $e^* \in L^q_{[a,b]}$ and $\langle e, e^* \rangle = 1$.

Here Chebyshev functional (2) is given by

$$T_{e,e^*}(f,g) = T(f,g) = \int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \int_a^b g(t)\,dt. \tag{33}$$

It is not hard to verify that

$$Qf = f - \langle f, e^* \rangle e = f - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \mathbf{1}, \tag{34}$$

$$Sg = g - \langle e, g \rangle e^* = g - \frac{1}{b-a}\int_a^b g(t)\,dt \cdot \mathbf{1}. \tag{35}$$

The next result can be compared to [6, Theorem 2], [12, p. 2], [21, Lemma 1].

COROLLARY 4. *Let* $f \in L^p_{[a,b]}$ *and* $g \in L^q_{[a,b]}$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$. *If*

$$\left(\int_a^b |f(t) - c_0|^p\,dt \right)^{1/p} \leqslant r \tag{36}$$

for some $c_0 \in \mathbb{R}$ *and* $r > 0$, *then*

$$\left| \int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \int_a^b g(t)\,dt \right| \leqslant r \left(\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^q dt \right)^{1/q}. \tag{37}$$

Proof. With the notation $x = f$ and $x^* = g$ and $x_0 = c_0\mathbf{1}$, by (33), (34) and (35) we obtain

$$\langle x, x^* \rangle - \langle x, e^* \rangle \langle e, x^* \rangle = \int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \int_a^b g(t)\,dt, \tag{38}$$

$$\|x-x_0\| = \|f-c_0\mathbf{1}\|_p = \left(\int_a^b |f(t)-c_0|^p\,dt\right)^{1/p}, \tag{39}$$

$$\|Qx\| = \|f-\langle f,e^*\rangle e\|_p = \left(\int_a^b \left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}, \tag{40}$$

$$\|Sx^*\|_* = \|g-\langle e,g\rangle e^*\|_q = \left(\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^q dt\right)^{1/q}. \tag{41}$$

By making use of (36) we see that the condition

$$\|x-x_0\| \leqslant r$$

is satisfied. In conclusion, by Lemma 1 applied for $x_0^*=0$, we establish the inequality

$$|\langle x,x^*\rangle - \langle x,e^*\rangle\langle e,x^*\rangle| \leqslant r\|Sx^*\|_*,$$

which proves (37) via (38)–(41). □

COROLLARY 5. *Let* $f,\beta \in L^p_{[a,b]}$ *and* $g \in L^q_{[a,b]}$ *with* $\frac{1}{p}+\frac{1}{q}=1$, $2 \leqslant p < \infty$. *Assume that for some* $c_0 \in \mathbb{R}$,

(i) $x_0 = c_0\mathbf{1}$ *is a constant function,*

(ii) $f \geqslant c_0\mathbf{1}$ *and* $\beta \geqslant c_0\mathbf{1}$,

(iii) $p\,(f(t)-c_0)^{p-1}(\beta(t)-f(t)) + |\beta(t)-f(t)|^p \geqslant 0$ *a.e. on* $[a,b]$, *or, more generally,*

$$\int_a^b \left[p\,(f(t)-c_0)^{p-1}(\beta(t)-f(t)) + |\beta(t)-f(t)|^p\right] dt \geqslant 0. \tag{42}$$

Then we have the inequality

$$\left|\int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \int_a^b g(t)\,dt\right|$$

$$\leqslant \|\beta-c_0\mathbf{1}\|_p \cdot \left(\int_a^b \left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^q dt\right)^{1/q}. \tag{43}$$

Proof. We consider the functions $\psi(u) = u^p$ for $u \in [0,\infty)$, and

$$\varphi(z) = \|z\|^p = \int_a^b |z(t)|^p \, dt \quad \text{for } z \in L^p_{[a,b]}$$

(see (31)).

Then for $z,h \in L^p_{[a,b]}$,

$$\nabla_h \varphi(z) = \int_a^b p|z(t)|^{p-2} z(t)h(t)\, dt = \langle h, p|z|^{p-2} z\rangle \tag{44}$$

(see [18, pp. 350–351]), where $|z|^{p-2}z \in L^q_{[a,b]}$, because $z \in L^p_{[a,b]}$ and $(p-1)q = p$. Therefore,

$$\nabla\varphi(z) = p|z|^{p-2} z. \tag{45}$$

We shall show that the function φ is superquadratic for $z \geqslant 0$ and $z+h \geqslant 0$ in the sense of (7).

For $p \geqslant 2$ the function ψ is superquadratic on $[0,\infty)$ in the sense of [1, 2, 3], that is,

$$(u+s)^p \geqslant u^p + pu^{p-1}s + |s|^p \quad \text{for } u, u+s \in [0,\infty).$$

By substituting $u = z(t)$ and $s = h(t)$ for $t \in [a,b]$, we obtain

$$(z(t)+h(t))^p \geqslant (z(t))^p + p(z(t))^{p-1}h(t) + |h(t)|^p \quad \text{for } t \in [a,b],$$

because $z \geqslant 0$ and $z+h \geqslant 0$. Hence,

$$\int_a^b (z(t)+h(t))^p \, dt \geqslant \int_a^b \left[(z(t))^p + p(z(t))^{p-1}h(t) + |h(t)|^p\right] dt,$$

and further,

$$\int_a^b |z(t)+h(t)|^p \, dt \geqslant \int_a^b \left[|z(t)|^p + p|z(t)|^{p-2}z(t)h(t) + |h(t)|^p\right] dt,$$

which means

$$\|z+h\|^p \geqslant \|z\|^p + \langle h, \nabla\|z\|^p\rangle + \|h\|^p \quad \text{for } z,h \in L^p_{[a,b]} \text{ such that } z, z+h \geqslant 0.$$

In conclusion, (7) is met for the set $U = \{v \in L^p_{[a,b]} : v \geqslant 0\}$.

On the other hand, by using the substitutions

$$x = f, \;\; x_0 = c_0\mathbf{1}, \;\; x^* = g, \;\; x_0^* = 0,$$

$$z = x - x_0 = f - c_0\mathbf{1},$$

$$h = \beta - x = \beta - f,$$

we see that conditions (42), (44) and (45) via (32) imply that

$$\langle \beta - x, \nabla\varphi(x - x_0)\rangle + \varphi(\beta - x) \geqslant 0.$$

Thus all assumptions of Theorem 1 are verified to hold. So, we are allowed to apply inequality (19) in Theorem 1, which easily leads to (43). $\square$

In the case of $p = 2$ in Corollary 5, condition (iii) can be equivalently restated as

$$\int_a^b [\beta(t) - f(t)]\,[f(t) - 2c_0 + \beta(t)]\; dt \geqslant 0.$$

Specifically, for $c_0 = \frac{\alpha+\beta}{2}$ with $\alpha, \beta \in L^2_{[a,b]}$ the last inequality holds whenever

$$[\beta(t) - f(t)]\,[f(t) - \alpha(t)] \geqslant 0 \;\; \text{a.e. on } [a,b],$$

which is of Dragomir's type (see Example 1). The latter is met, e.g., if $\alpha(t) \leqslant f(t) \leqslant \beta(t)$ a.e. on $[a,b]$.

For $s,t \in [a,b]$, we denote

$$P_s(t) = \begin{cases} t - a & \text{if } t \in [a,s], \\ t - b & \text{if } t \in (s,b]. \end{cases}$$

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function differentiable on $[a,b]$. Suppose $f', \beta \in L^p_{[a,b]}$, where $2 \leqslant p < \infty$.*

Assume that for some $c_0 \in \mathbb{R}$,

(i) $x_0 = c_0\mathbf{1}$ *is a constant function,*

(ii) $f' \geqslant c_0\mathbf{1}$ *and* $\beta \geqslant c_0\mathbf{1}$,

(iii) $p\,(f'(t) - c_0)^{p-1}\,(\beta(t) - f'(t)) + |\beta(t) - f'(t)|^p \geqslant 0$ *a.e. on* $[a,b]$, *or, more generally,*

$$\int_a^b \left[p\left(f'(t) - c_0\right)^{p-1} \left(\beta(t) - f'(t)\right) + |\beta(t) - f'(t)|^p \right] dt \geqslant 0.$$

Then for any $s \in [a,b]$ we have the inequality

$$\left| f(s) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b) - f(a)}{b-a} \cdot \left(s - \frac{a+b}{2}\right) \right| \leqslant \|\beta - c_0\mathbf{1}\|_p \cdot \frac{(b-a)^{1/q}}{2(q+1)^{1/q}}, \tag{46}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The first part of this proof is based on the method shown in the proof of [14, Theorem 2.1].

By taking into account Corollary 5 applied to the functions f' and P_s (in place f and g, respectively), we estimate $|T(f', P_s)|$ as follows

$$\left| \int_a^b f'(t)P_s(t)\,dt - \frac{1}{b-a}\int_a^b f'(t)\,dt \cdot \int_a^b P_s(t)\,dt \right|$$

$$\leqslant \|\beta - c_0\mathbf{1}\|_p \cdot \left\| P_s - \frac{1}{b-a}\int_a^b P_s(t)\,dt \right\|_q . \tag{47}$$

Montgomery identity (see [31]) states that

$$f(s) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b f'(t)P_s(t)\,dt. \tag{48}$$

It is known that

$$\frac{1}{b-a}\int_a^b f'(t)\,dt = \frac{f(b)-f(a)}{b-a}. \tag{49}$$

It is not hard to check that

$$\frac{1}{b-a}\int_a^b P_s(t)\,dt = s - \frac{a+b}{2}. \tag{50}$$

It is calculated in [29] that

$$\left\| P_s - \frac{1}{b-a}\int_a^b P_s(t)\,dt \right\|_q = \left\| P_s - \left(s - \frac{a+b}{2}\right) \right\|_q = \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}} \quad \text{for } 1 < q < \infty. \tag{51}$$

In summary, by combining (47), (48), (49), (50) and (51), we find that

$$\left| (b-a)f(s) - \int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a} \cdot (b-a)\left(s - \frac{a+b}{2}\right) \right|$$

$$\leqslant \|\beta - c_0\mathbf{1}\|_p \cdot \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}. \tag{52}$$

Now, we deduce from (52) that (46) holds valid. □

Finally, we present an interpretation of Theorem 2.

COROLLARY 6. *Let* $f, \beta \in L^p_{[a,b]}$ *and* $g \in L^q_{[a,b]}$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $2 \leqslant p < \infty$. *Assume that for some* $c_0 \in \mathbb{R}$,

(i) $x_0 = c_0\mathbf{1}$ *is a constant function,*

(ii)

$$\int_a^b \left[p\,|f(t) - c_0|^{p-2} \left(f(t) - c_0\right)\left(\beta(t) - f(t)\right)\right] dt \geqslant 0. \tag{53}$$

Then

$$\left| \int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \int_a^b g(t)\,dt \right|$$
$$\leqslant \|\beta - c_0\mathbf{1}\|_p \cdot \left(\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|^q dt \right)^{1/q}. \tag{54}$$

Proof. It follows from Theorem 2 via a similar method as that in the proof of Corollary 5. □

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