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OPTIMAL CONSTANTS OF THE MIXED LITTLEWOOD INEQUALITIES: THE COMPLEX CASE

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Abstract. In this paper, among other results, we obtain an extension of a kind of Khinchine inequality given by R. Blei, namely, the Blei–Khinchine inequality. As an application we obtain the optimal constants of the mixed Littlewood inequalities, for complex scalars.

1. Introduction

The origins of the theory of summability of multilinear forms and absolutely summing multilinear operators are probably associated to Littlewood's (ℓ_1, ℓ_2) mixed inequalities, published in 1930. A very detailed introduction to the theory of absolutely summing operators can be found in [10], while the multilinear theory has been recently explored in different contexts by various authors (see [8, 19, 20, 25] and the references therein) with applications in other fields as Quantum Information Theory and Theoretical Computer Science (see [3, 21, 29] and the references therein).

From now on $\mathbb{K}$ will denote the real scalar field $\mathbb{R}$ or the complex scalar field $\mathbb{C}$ and, for any $s \geqslant 1$, we denote the conjugate index of s by s^*, i.e., $1/s + 1/s^* = 1$ (as usual we consider $1/0 = \infty$ and $1/\infty = 0$). Littlewood's (ℓ_1, ℓ_2)-mixed inequalities ([17], 1930) assert that there are (optimal) constants $\mathscr{L}^{\mathbb{K}}_{(2,1)} \geqslant 1$ and $\mathscr{L}^{\mathbb{K}}_{(1,2)} \geqslant 1$ such that

$$\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \left| A(e_j, e_k) \right|^2 \right)^{\frac{1}{2}} \leqslant \mathscr{L}^{\mathbb{K}}_{(2,1)} \|A\| \tag{1.1}$$

and

$$\left(\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \left| A(e_j, e_k) \right| \right)^2 \right)^{\frac{1}{2}} \leqslant \mathscr{L}^{\mathbb{K}}_{(1,2)} \|A\| \tag{1.2}$$

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for all continuous bilinear forms $A : c_0 \times c_0 \to \mathbb{K}$. Here and henceforth e_n represents the canonical vector with 1 at the n-th entry, and zero otherwise, in a sequence space and

$$\|A\| := \sup\{|A(x,y)| : \|x\| \leqslant 1 \text{ and } \|y\| \leqslant 1\}.$$

The inequality (1.2) was obtained in 1933 by Orlicz, working in a different context (see [6, pages 23–25]).

The exponents of Littlewood's (ℓ_1, ℓ_2)-mixed inequalities are optimal in the sense that, fixing the exponent 1, the exponent 2 cannot be replaced by a smaller exponent (nor the exponent 1 can be replaced by a smaller one). On the other hand, the optimality of the constants $\mathscr{L}_{(1,2)}^{\mathbb{K}}$ and $\mathscr{L}_{(2,1)}^{\mathbb{K}}$ is summarized in the following way (see [6, page 31]):

$$\begin{cases} \mathscr{L}_{(1,2)}^{\mathbb{R}} = \mathscr{L}_{(2,1)}^{\mathbb{R}} = \sqrt{2}, \\ \mathscr{L}_{(1,2)}^{\mathbb{C}} = \mathscr{L}_{(2,1)}^{\mathbb{C}} = 2/\sqrt{\pi}. \end{cases}$$

In 1934 Hardy and Littlewood [14] pushed the subject further, extending the above results to bilinear forms defined on ℓ_p spaces (when $p = \infty$, as usual, we consider c_0 instead of ℓ_∞, and for any function f we shall consider $f(\infty) := \lim_{s\to\infty} f(s)$): for $p, q \geqslant 2$, with $1/p + 1/q < 1$, there is a (optimal) constant $\mathscr{L}_{(p,q,2,\lambda)}^{\mathbb{K}} \geqslant 1$ such that

$$\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\left|A(e_j,e_k)\right|^2\right)^{\frac{\lambda}{2}}\right)^{\frac{1}{\lambda}} \leqslant \mathscr{L}_{(p,q,2,\lambda)}^{\mathbb{K}}\|A\| \tag{1.3}$$

with $\lambda := \frac{pq}{pq-p-q}$, for all continuous bilinear forms $A : \ell_p \times \ell_q \to \mathbb{K}$. Observe that the inequality (1.3) is the extension of the inequality (1.1) to bilinear forms defined on $\ell_p \times \ell_q$. On the other hand, note that when $1/p + 1/q \leqslant 1/2$ we have $\lambda = \frac{pq}{pq-p-q} \leqslant 2$, and by a well-known result sometimes credited to Minkowski (see [12, Corollary 5.4.2]), we obtain

$$\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\left|A(e_j,e_k)\right|^{\lambda}\right)^{\frac{2}{\lambda}}\right)^{\frac{1}{2}} \leqslant \left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\left|A(e_j,e_k)\right|^{2}\right)^{\frac{\lambda}{2}}\right)^{\frac{1}{\lambda}}.$$

Therefore, for $p, q \geqslant 2$, with $1/p + 1/q \leqslant 1/2$, there is a (optimal) constant $\mathscr{L}_{(p,q,\lambda,2)}^{\mathbb{K}} \geqslant 1$ such that

$$\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\left|A(e_j,e_k)\right|^{\lambda}\right)^{\frac{2}{\lambda}}\right)^{\frac{1}{2}} \leqslant \mathscr{L}_{(p,q,\lambda,2)}^{\mathbb{K}}\|A\|, \tag{1.4}$$

for all continuous bilinear forms $A : \ell_p \times \ell_q \to \mathbb{K}$. Observe that this is the extension of the inequality (1.2) to bilinear forms defined on $\ell_p \times \ell_q$. The inequalities (1.3) and (1.4) were obtained in 1934 by Hardy and Littlewood (see [14, Theorems 1 and 4]). The exponents in the inequalities (1.3) and (1.4) are optimal in the sense that λ can not be improved keeping the exponent 2 nor the exponent 2 can be improved keeping the exponent λ. Looking at this result, the natural question is: why does $1/p + 1/q = 1/2$ separate the rank of validity of the two extensions (1.3) and (1.4)? This question is answered in [9, Appendix]. On the other hand, we observe that fixing the parameter $q=\infty$ (or $p=\infty$), the sum $1/p+1/q$ is always less than or equal to $1/2$, whenever $p\geqslant 2$ (or $q\geqslant 2$), and hence the two extensions (1.3) and (1.4) are valid, and look as follows:

THEOREM 1.1. (Littlewood's (ℓ_{p^*},ℓ_2) mixed inequality) *Let $p\geqslant 2$. There is a (optimal) constant $\mathscr{L}^{\mathbb{K}}_{(p,\infty,2,p^*)}$ such that*

$$\left(\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|A(e_j,e_k)|^2\right)^{\frac{p^*}{2}}\right)^{\frac{1}{p^*}}\leqslant \mathscr{L}^{\mathbb{K}}_{(p,\infty,2,p^*)}\|A\|,$$

for all continuous bilinear forms $A:\ell_p\times c_0\to\mathbb{K}$.

THEOREM 1.2. (Littlewood's (ℓ_2,ℓ_{p^*}) mixed inequality) *Let $p\geqslant 2$. There is a (optimal) constant $\mathscr{L}^{\mathbb{K}}_{(p,\infty,p^*,2)}$ such that*

$$\left(\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}|A(e_j,e_k)|^{p^*}\right)^{\frac{2}{p^*}}\right)^{\frac{1}{2}}\leqslant \mathscr{L}^{\mathbb{K}}_{(p,\infty,p^*,2)}\|A\|,$$

for all continuous bilinear forms $A:\ell_p\times c_0\to\mathbb{K}$.

REMARK 1.3. The optimal constants $\mathscr{L}^{\mathbb{K}}_{(p,\infty,p^*,2)}$ and $\mathscr{L}^{\mathbb{K}}_{(p,\infty,2,p^*)}$, for all $p\geqslant 2$, were obtained, in the real case, in the recent papers [22, 23]. In fact, it was proved that for all $p\geqslant 2$ we have

$$\mathscr{L}^{\mathbb{R}}_{(p,\infty,p^*,2)}=\mathscr{L}^{\mathbb{R}}_{(p,\infty,2,p^*)}=A^{-1}_{\frac{p}{p-1}},$$

where $A_{\frac{p}{p-1}}$ denotes the optimal constant in the Khinchine inequality (formally introduced in Section 2). On the other hand, in the complex case, the only known estimates for the optimal constants are

$$1\leqslant \mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)}\leqslant\frac{2}{\sqrt{\pi}},\quad\text{and}\quad 1\leqslant \mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}\leqslant\frac{2}{\sqrt{\pi}},$$

for all $p\geqslant 2$.

Theorems 1.1 and 1.2 are usually called mixed Littlewood inequalities (see [18, 22, 23]).

The first main goal of the present paper is to obtain, for all $p\geqslant 2$, the optimal values of $\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)}$ and $\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}$. We recall that the optimal estimates for $\mathscr{L}^{\mathbb{R}}_{(p,\infty,p^*,2)}$ and $\mathscr{L}^{\mathbb{R}}_{(p,\infty,2,p^*)}$ are presented in [22, 23], as a consequence of the Khinchine inequality, a result from Probability frequently used in Functional Analysis. In fact, many modern proofs of the Hardy–Littlewood and related inequalities depend on this inequality. The second main objective of this work is to extend the Khinchine inequality to an appropriate environment that will allow us to obtain the optimal estimates of $\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)}$ and $\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}$.

This paper is organized as follows: in Section 2, inspired by a result of [6], we obtain an extension of the Khinchine inequality, namely, the Blei–Khinchine inequality. In Section 3, we use the results of Section 2 for the sake of reaching our main goal. The last section sketches consequences of our approach to the multilinear setting; one of them is obtaining optimal estimates for the constants in the multilinear version of the mixed Littlewood inequalities.

2. An extension of the Khinchine inequality

The Khinchine inequality, proved in 1923 by A. Khinchine ([15]), asserts that for any $p>0$ there is a constant $A_p>0$ such that

$$A_p\left(\sum_{j=1}^{N}\left|a_j\right|^2\right)^{\frac{1}{2}} \leqslant \left(\frac{1}{2^N}\sum_{\eta\in\{1,-1\}^N}\left|\sum_{j=1}^{N}\eta_j a_j\right|^p\right)^{\frac{1}{p}} \tag{2.1}$$

for all sequences of scalars $(a_j)_{j=1}^N$ and all positive integers N. This inequality is strongly related to the development of the theory of summing linear and multilinear operators.

Obviously, $A_p=1$ for all $p\geqslant 2$. In 1982 Haagerup ([13]) furnished the optimal values of the constant A_p for all $p>0$.

The counterpart for the average $\frac{1}{2^N}\sum_{\eta\in\{1,-1\}^N}\left|\sum_{j=1}^{N}\eta_j a_j\right|^p$ in the complex framework is

$$\left(\frac{1}{2\pi}\right)^N\int_0^{2\pi}\cdots\int_0^{2\pi}\left|\sum_{j=1}^{N}a_j e^{it_j}\right|^p dt_1\cdots dt_N. \tag{2.2}$$

For the sake of simplicity we shall denote (2.2) by

$$\mathbb{E}\left|\sum_{j=1}^{N}a_j\varepsilon_j\right|^p$$

where ε_j are Steinhaus variables; i.e. variables which are uniformly distributed on the circle S^1. The following version of the Khinchine inequality holds and in this case it is known as the Khinchine inequality for Steinhaus variables:

THEOREM 2.1. (Khinchine's inequality for Steinhaus variables) *For every* $0<p<\infty$*, there is a (optimal) constant* $\widetilde{A_p}$ *such that*

$$\widetilde{A_p}\left(\sum_{n=1}^{N}|a_n|^2\right)^{\frac{1}{2}} \leqslant \left(\mathbb{E}\left|\sum_{n=1}^{N}a_n\varepsilon_n\right|^p\right)^{\frac{1}{p}} \tag{2.3}$$

for every positive integer N *and all scalars* $a_1,\ldots,a_N$*, where* ε_n *are Steinhaus variables.*

Obviously, $\widetilde{A_p} = 1$ for all $p \geqslant 2$. Recently, in 2014 König [16] proved that the optimal constants $\widetilde{A_p}$ are

$$\widetilde{A_p} = \left(\Gamma\left(\frac{p+2}{2}\right)\right)^{\frac{1}{p}}, \quad \text{for } 0.4756 \approx p_1 \leqslant p < 2 \tag{2.4}$$

and

$$\widetilde{A_p} = \sqrt{2}\left(\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p+2}{2}\right)\sqrt{\pi}}\right)^{\frac{1}{p}}, \quad \text{for } 0 < p < p_1 \approx 0.4756. \tag{2.5}$$

Above and henceforth Γ denotes the famous Gamma function. The exact definition of the critical value p_1 is the following: $p_1 \in (0,1)$ is the unique real number satisfying

$$1 = \sqrt{2}\left(\frac{\Gamma\left(\frac{p_1+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{p_1}{2}+1\right)^2}\right)^{\frac{1}{p}}.$$

In [6, chapter II: section 6] it was introduced a kind of Khinchine inequality that extends and unifies the inequalities (2.1) and (2.3). Before stating this Blei–Khinchine inequality, we need to introduce some notation and results.

Let $p_1, p_2 \in [1,\infty]$ and N be a positive integer. We recall that for a continuous bilinear form $A : \ell_{p_1}^N \times \ell_{p_2}^N \to \mathbb{C}$, the sup-norm of A is given by

$$\|A\| = \sup\left\{\left|\sum_{i,j=1}^{N} a_{ij}x_i y_j\right| : \|x\|_{\ell_{p_1}^N} \leqslant 1, \|y\|_{\ell_{p_2}^N} \leqslant 1\right\},$$

where $A(e_i, e_j) = a_{ij}$, for all $i, j \in \{1,\ldots,N\}$, and $\ell_{p_k}^N$ is $\mathbb{C}^N$, endowed with the ℓ_{p_k} norm (we remember that, when $p_k = \infty$, we consider c_0 instead of ℓ_∞).

For each integer $M \geqslant 2$, we consider

$$\begin{cases} T_M := \left\{\exp\left(\frac{2j\pi}{M}i\right) : j = 0,\ldots,M-1\right\}, \\ T_\infty = \{\exp(ti) : t \in [0,2\pi)\}. \end{cases}$$

and

$$D_M := conv(T_M) \text{ and } D_\infty := conv(T_\infty),$$

where *conv* means the convex hull. Observe that D_∞ is the closed unit disk $\mathbb{D}$ and, trivially, $D_M \subseteq D_\infty$. Obviously, D_M is a convex and closed absorbing set in $\mathbb{C}$.

LEMMA 2.2. *Let $M \geqslant 3$ be an integer. If $r_M := \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{M}\right)\right)^{\frac{1}{2}}$, then*

$$B[0, r_M] \subseteq D_M,$$

where $B[0, r_M]$ denotes the closed ball with center in 0 and radius r_M.

Proof. Note that $0 \in D_M$. In fact,

$$0 = \frac{1}{M} + \frac{1}{M}\exp\left(\frac{2\pi}{M}i\right) + \cdots + \frac{1}{M}\exp\left(\frac{2(M-1)\pi}{M}i\right),$$

and $\sum_{i=0}^{M-1}\frac{1}{M}=1$. We also know that D_M is a regular polygon with apothem given by

$$\left|\frac{1}{2}\exp\left(\frac{2j\pi}{M}i\right)+\frac{1}{2}\exp\left(\frac{2(j+1)\pi}{M}i\right)\right|.$$

Computing the apothem, we have

$$\begin{aligned}
&\left|\frac{1}{2}\exp\left(\frac{2j\pi}{M}i\right)+\frac{1}{2}\exp\left(\frac{2(j+1)\pi}{M}i\right)\right|^2\\
&=\frac{1}{4}\left(\left(\cos\left(\frac{2j\pi}{M}\right)+\cos\left(\frac{2(j+1)\pi}{M}\right)\right)^2+\left(\sin\left(\frac{2j\pi}{M}\right)+\sin\left(\frac{2(j+1)\pi}{M}\right)\right)^2\right)\\
&=\frac{1}{4}\left(\left(2+2\cos\left(\frac{2j\pi}{M}\right)\cos\left(\frac{2(j+1)\pi}{M}\right)\right)+2\sin\left(\frac{2j\pi}{M}\right)\sin\left(\frac{2(j+1)\pi}{M}\right)\right)\\
&=\frac{1}{4}\left(2+2\cos\left(\frac{2\pi}{M}\right)\right)\\
&=r_M^2.
\end{aligned}$$

Thus, it is possible to draw a circle inside D_M with radius r_M. □

Let N and M be positive integers, $M\geqslant 3$. For any bilinear form $A:\ell_p^N\times c_0^N\to\mathbb{C}$ we define the norm

$$\|A\|_M:=\sup\left\{|A(x,y)|:\|x\|_{\ell_p}\leqslant 1\text{ and }y\in T_M^N\right\}.$$

The following basic result, whose aim is to get approximations of the $\sup$-norm $\|A\|$, is a simple consequence of Lemma 2.2:

THEOREM 2.3. *Let N,M be positive integers, $M\geqslant 3$, and $p\in[1,\infty]$. Then*

$$\|A\|_M\leqslant\|A\|\leqslant r_M^{-1}\|A\|_M,$$

for all bilinear forms $A:\ell_p^N\times c_0^N\to\mathbb{C}$, where r_M is as in Lemma 2.2.

For each $M\geqslant 2$, let

$$\Omega_M:=\left\{\frac{2j\pi}{M}:j=0,\ldots,M-1\right\}.$$

Let $(a_n)_{n=1}^N$ be an array of scalars, $0<p<\infty$, and $M\geqslant 2$. We define

$$E_{M,p}\left((a_n)_{n=1}^N\right)=\left(\left(\frac{1}{M}\right)^N\sum_{\beta\in\Omega_M^N}\left|\sum_{n=1}^N a_n e^{i\beta_n}\right|^p\right)^{\frac{1}{p}}.$$

Using the Dominated Convergence Theorem it is possible to prove that

$$\lim_{M\to\infty}E_{M,p}\left((a_n)_{n=1}^N\right)=\left(\mathbb{E}\left|\sum_{n=1}^N a_n\varepsilon_n\right|^p\right)^{\frac{1}{p}}.$$

We need the following auxiliary result:

LEMMA 2.4. *Let* $(a_n)_{n=1}^N$ *be an array of scalars,* $0<p<\infty$, *and* $M \geqslant 2$. *Then*

$$E_{M,p}\left((a_n)_{n=1}^N\right) = E_{M,p}\left((a_n e^{is_n})_{n=1}^N\right)$$

for all $s=(s_1,\ldots,s_N)\in\Omega_M^N$.

Proof. If $s=(s_1,\ldots,s_N)\in\Omega_M^N$, then

$$\begin{aligned} E_{M,p}\left((a_n e^{is_n})_{n=1}^N\right) &= \left(\left(\frac{1}{M}\right)^N \sum_{\beta\in\Omega_M^N}\left|\sum_{n=1}^N a_n e^{is_n} e^{i\beta_n}\right|^p\right)^{\frac{1}{p}} \\ &= \left(\left(\frac{1}{M}\right)^N \sum_{\beta\in\Omega_M^N}\left|\sum_{n=1}^N a_n e^{i(s_n+\beta_n)}\right|^p\right)^{\frac{1}{p}} \\ &= \left(\left(\frac{1}{M}\right)^N \sum_{\gamma\in\Omega_M^N}\left|\sum_{n=1}^N a_n e^{i\gamma_n}\right|^p\right)^{\frac{1}{p}} \\ &= E_{M,p}\left((a_n)_{n=1}^N\right). \quad \square \end{aligned}$$

Now, we enunciate and prove the announced extension of the Khinchine inequality, for $1\leqslant p\leqslant 2$, that extends and unifies the inequalities (2.1) and (2.3). We emphasize that the theorem below was introduced by Blei for $p=1$, in [6, Chapter II: Section 6].

THEOREM 2.5. (Blei–Khinchine inequality) *For every* $1\leqslant p\leqslant 2$, *and* $M\geqslant 2$, *there is a (optimal) constant* $\mathscr{B}_{M,p}$ *such that*

$$\left(\sum_{n=1}^N |a_n|^2\right)^{1/2} \leqslant \mathscr{B}_{M,p}\cdot E_{M,p}\left((a_n)_{n=1}^N\right), \tag{2.6}$$

for every positive integer N *and all scalars* $a_1,\ldots,a_N$. *Moreover, for all* $M\geqslant 3$,

$$\mathscr{B}_{M,p} \leqslant \mathscr{L}^{\mathbb{C}}_{(p^*,\infty,p,2)}\cdot r_M^{-1}, \tag{2.7}$$

where

$$r_M = \left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{2\pi}{M}\right)\right)^{\frac{1}{2}}.$$

Observe that the inequality (2.6) in Theorem 2.5 is a direct consequence of Lemma 2.2. However, since the estimate (2.7) will be used in the next section, we give the following proof of the Blei–Khinchine inequality:

Proof. The case $M=2$ is the inequality (2.1) and thus we only need to prove the case $M\geqslant 3$. Let $1\leqslant p\leqslant 2$, and let $(a_n)_{n=1}^N$ be an array of scalars, such that $E_{M,p}\left((a_n)_{n=1}^N\right)=1$. Then, by the previous lemma,

$$E_{M,p}\left((a_n e^{is_n})_{n=1}^N\right) = \left(\left(\frac{1}{M}\right)^N \sum_{\beta\in\Omega_M^N}\left|\sum_{n=1}^N a_n e^{is_n} e^{i\beta_n}\right|^p\right)^{\frac{1}{p}} = 1,$$

for all $s \in \Omega_M^N$. Thus,

$$\left(\left(\frac{1}{M}\right)^N \sum_{r\in T_M^N} \left|\sum_{n=1}^N a_n w_n r_n\right|^p\right)^{\frac{1}{p}} = 1,$$

for all $w \in T_M^N$. Note that

$$\left(\left(\frac{1}{M}\right)^N \sum_{r\in T_M^N} \left|\sum_{n=1}^N a_n w_n r_n\right|^p\right)^{\frac{1}{p}} = \left(\left(\frac{1}{M}\right)^N \sum_{i=1}^{M^N} \left|\sum_{n=1}^N a_n \tau_n^{(i)} w_n\right|^p\right)^{\frac{1}{p}} = 1 \tag{2.8}$$

for all $w \in T_M^N$, where $\tau_n^{(i)} \in T_M$, for all $n \in \{1,\dots,N\}$ and $i \in \{1,\dots,M^N\}$.

Consider the bilinear form $A : \ell_{p^*}^{M^N} \times c_0^N \to \mathbb{C}$ given by

$$A(e_i,e_n) = \frac{a_n \tau_n^{(i)}}{M^{\frac{N}{p}}}, \quad n \in \{1,\dots,N\} \text{ and } i \in \{1,\dots,M^N\}.$$

Note that $\|A\|_M \leqslant 1$. In fact, using Hölder's inequality and the equality (2.8) we get

$$\begin{aligned}\left|\sum_{i=1}^{M^N}\sum_{n=1}^N A(e_i,e_n) w_n z_i\right| &= \left|\sum_{i=1}^{M^N}\sum_{n=1}^N \frac{a_n \tau_n^{(i)}}{M^{\frac{N}{p}}} w_n z_i\right| \\ &\leqslant \left(\sum_{i=1}^{M^N}\left|\sum_{n=1}^N \frac{a_n \tau_n^{(i)}}{M^{\frac{N}{p}}} w_n\right|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{M^N} |z_i|^{p^*}\right)^{\frac{1}{p^*}} \\ &\leqslant \left(\left(\frac{1}{M}\right)^N \sum_{i=1}^{M^N}\left|\sum_{n=1}^N a_n \tau_n^{(i)} w_n\right|^p\right)^{\frac{1}{p}} \\ &\overset{(2.8)}{=} 1\end{aligned}$$

for all $w \in T_M^N$, and $z \in \ell_{p^*}$, with $\|z\|_{\ell_{p^*}} \leqslant 1$. Therefore, $\|A\|_M \leqslant 1$. Moreover, using Theorem 1.2 and Theorem 2.3, and the above norm estimate, we have

$$\begin{aligned}\left(\sum_{n=1}^N |a_n|^2\right)^{1/2} &= \left(\sum_{n=1}^N \left(\sum_{i=1}^{M^N}\left|\left(\frac{1}{M}\right)^{\frac{N}{p}} a_n \tau_n^{(i)}\right|^p\right)^{2/p}\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=1}^N \left(\sum_{i=1}^{M^N} |A(e_i,e_n)|^p\right)^{2/p}\right)^{\frac{1}{2}} \\ &\leqslant \mathscr{L}^{\mathbb{C}}_{(p^*,\infty,p,2)} \|A\| \\ &\leqslant \mathscr{L}^{\mathbb{C}}_{(p^*,\infty,p,2)} \cdot r_M^{-1} \|A\|_M \\ &\leqslant \mathscr{L}^{\mathbb{C}}_{(p^*,\infty,p,2)} \cdot r_M^{-1}.\end{aligned}$$

Thus, the inequality follows and

$$\mathscr{B}_{M,p} \leqslant \mathscr{L}^{\mathbb{C}}_{(p^*,\infty,p,2)} \cdot r_M^{-1},$$

as asserted. □

3. Applications of the Blei–Khinchine inequality

In this section, as an application of the Blei–Khinchine inequality, we obtain the optimal constants $\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)}$ and $\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}$, for all $p \geqslant 2$. We start with the following proposition, showing that $\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)} \leqslant \widetilde{A_{\frac{p}{p-1}}}^{-1}$, for all $p \geqslant 2$. This estimate is somewhat new; for real scalars, in [22, Theorem 2] it was proved that $\mathscr{L}^{\mathbb{R}}_{(p,\infty,p^*,2)} \leqslant A^{-1}_{\frac{p}{p-1}}$ but for complex scalars the only known estimate is $\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)} \leqslant \frac{2}{\sqrt{\pi}}$. The proof is simple and follows the lines of the proof of [22, Theorem 2]. We shall include a short proof for the sake of completeness.

PROPOSITION 3.1. (Littlewood's (ℓ_{p^*},ℓ_2) mixed inequality) *Let* $p \in [2,\infty]$. *We have*

$$\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)} \leqslant \widetilde{A_{\frac{p}{p-1}}}^{-1}.$$

Proof. Let $T : \ell_p^n \times c_0^n \to \mathbb{C}$ be a bilinear form with $p \geqslant 2$. Then, invoking the Khinchine inequality for Steinhaus variables and recalling that the weak p^*-norm of $(e_j)_{j=1}^n$ in ℓ_p^n is 1 and that all continuous linear functionals are absolutely $\left(\frac{p}{p-1},\frac{p}{p-1}\right)$-summing with constant 1, we have

$$\begin{aligned}
\left(\sum_{i=1}^n\left(\sum_{j=1}^n\left|T\left(e_i,e_j\right)\right|^2\right)^{\frac{1}{2}\frac{p}{p-1}}\right)^{\frac{p-1}{p}} &\leqslant \widetilde{A_{\frac{p}{p-1}}}^{-1}\left(\sum_{i=1}^n\int_0^1\left|\sum_{j=1}^n r_j(t)T\left(e_i,e_j\right)\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\\
&= \widetilde{A_{\frac{p}{p-1}}}^{-1}\left(\int_0^1\sum_{i=1}^n\left|T\left(e_i,\sum_{j=1}^n r_j(t)e_j\right)\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\\
&\leqslant \widetilde{A_{\frac{p}{p-1}}}^{-1}\left(\int_0^1\left\|T\left(\cdot,\sum_{j=1}^n r_j(t)e_j\right)\right\|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}}\\
&\leqslant \widetilde{A_{\frac{p}{p-1}}}^{-1}\|T\|,
\end{aligned}$$

for all bilinear forms $T : \ell_p^n \times c_0^n \to \mathbb{C}$. Thus, the inequality follows and

$$\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)} \leqslant \widetilde{A_{\frac{p}{p-1}}}^{-1},$$

as asserted. □

If $p \geqslant 2$, we have $p^* \leqslant 2$, and thus by [12, Corollary 5.4.2] and Proposition 3.1 we obtain

$$\left(\sum_{j=1}^\infty\left(\sum_{i=1}^\infty|T(e_i,e_j)|^{p^*}\right)^{\frac{2}{p^*}}\right)^{\frac{1}{2}} \leqslant \left(\sum_{i=1}^\infty\left(\sum_{j=1}^\infty|T(e_i,e_j)|^2\right)^{\frac{p^*}{2}}\right)^{\frac{1}{p^*}} \leqslant \mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}\|T\|.$$

for all continuous bilinear forms $T : \ell_p \times c_0 \to \mathbb{C}$. Then, Proposition 3.1 implies the next result:

PROPOSITION 3.2. (Littlewood's (ℓ_2,ℓ_{p^*}) mixed inequality) *Let* $p\in[2,\infty]$. *We have*

$$\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)}\leqslant\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}.$$

On the other hand, inequality (2.7) combined with Proposition 3.1 and Proposition 3.2 give us

$$r_M\cdot\mathscr{B}_{M,\frac{p}{p-1}}\leqslant\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)}\leqslant\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}\leqslant\widetilde{A_{\frac{p}{p-1}}}^{-1}$$

for all $M\geqslant 3$ and for all $p\in[2,\infty]$. Thus, making $M\to\infty$, we have that r_M turns 1, and $\mathscr{B}_{M,\frac{p}{p-1}}$ turns $\widetilde{A_{\frac{p}{p-1}}}^{-1}$, and the above inequalities become

$$\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)}=\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}=\widetilde{A_{\frac{p}{p-1}}}^{-1}$$

for all $p\in[2,\infty]$. In short, we have proved the following:

THEOREM 3.3. *For all* $p\in[2,\infty]$, *the optimal constants in the complex mixed Littlewood inequalities are*

$$\mathscr{L}^{\mathbb{C}}_{(p,\infty,2,p^*)}=\mathscr{L}^{\mathbb{C}}_{(p,\infty,p^*,2)}=\widetilde{A_{\frac{p}{p-1}}}^{-1}.$$

REMARK 3.4. We recall that the particular case $p=\infty$ was previously obtained in [6, page 31].

4. Remarks on the multilinear case

Let m be a positive integer and $1\leqslant p_1,\ldots,p_m\leqslant\infty$. From now on, for $\mathbf{p}:=(p_1,\ldots,p_m)\in[1,+\infty]^m$, let

$$\left|\frac{1}{\mathbf{p}}\right|:=\frac{1}{p_1}+\cdots+\frac{1}{p_m}.$$

In the last 40 years, several multilinear variants of the classical Hardy–Littlewood inequalities have appeared. As in the bilinear case, if we want all mixed inequalities to be valid, the condition that must be imposed is $|1/\mathbf{p}|\leqslant 1/2$. In this environment, one of the most general versions of the Hardy–Littlewood inequality for m-linear forms was presented in [1] (following our convention, c_0 is understood as the proper substitute of ℓ_∞ when the parameter $p_j\to\infty$):

THEOREM 4.1. (Hardy–Littlewood inequality, [1]) *Let* $2\leqslant p_1,\ldots,p_m\leqslant\infty$, *with* $|1/\mathbf{p}|\leqslant 1/2$ *and* $\mathbf{q}:=(q_1,\ldots,q_m)\in\left[(1-|1/\mathbf{p}|)^{-1},2\right]^m$. *The following assertions are equivalent:*

(a) *There is a constant* $C^{\mathbb{K}}_{m,\mathbf{p},\mathbf{q}}\geqslant 1$ *such that*

$$\left(\sum_{i_1=1}^{\infty}\left(\cdots\left(\sum_{i_m=1}^{\infty}|A(e_{i_1},\ldots,e_{i_m})|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}\leqslant C^{\mathbb{K}}_{m,\mathbf{p},\mathbf{q}}\|A\|$$

for every continuous m*-linear form* $A:\ell_{p_1}\times\cdots\times\ell_{p_m}\to\mathbb{K}$.

(b) The exponents $q_1,\ldots,q_m \in \left[(1-|1/\mathbf{p}|)^{-1},2\right]$ *satisfy*

$$\frac{1}{q_1}+\cdots+\frac{1}{q_m} \leqslant \frac{m+1}{2}-\left|\frac{1}{\mathbf{p}}\right|.$$

REMARK 4.2. According to [1] and [2], the constants $C^{\mathbb{R}}_{m,\mathbf{p},\mathbf{q}}$ are dominated by $\left(\sqrt{2}\right)^{m-1}$, while $C^{\mathbb{C}}_{m,\mathbf{p},\mathbf{q}}$ are dominated by $\left(2/\sqrt{\pi}\right)^{m-1}$.

Observe that, for $k \in \{1,\ldots,m\}$, if we consider $\mathbf{t}=(t_1,\ldots,t_m)$, with $\frac{1}{t_k}=1-\left|\frac{1}{\mathbf{p}}\right|$, and $t_j=2$ for every $j\neq k$, we have

$$\frac{1}{t_1}+\cdots+\frac{1}{t_m}=\frac{m+1}{2}-\left|\frac{1}{\mathbf{p}}\right|.$$

Thus, the following inequality is a particular case of Theorem 4.1:

THEOREM 4.3. (see [27]) *Let* $p_1,\ldots,p_m \in [2,\infty]$ *be such that* $0<|1/\mathbf{p}|\leqslant 1/2$. *Define*

$$\Lambda := \frac{1}{1-\left|\frac{1}{\mathbf{p}}\right|},$$

and, for $k\in\{1,\ldots,m\}$, *consider* $t_k=\Lambda$, *and* $t_j=2$ *for every* $j\neq k$. *Then, there are constants* $C^{\mathbb{K}}_{m,\mathbf{p},\mathbf{q}}\geqslant 1$ *such that*

$$\left(\sum_{i_1=1}^{n}\left(\cdots\left(\sum_{i_m=1}^{n}\left|A\left(e_{i_1},\ldots,e_{i_m}\right)\right|^{t_m}\right)^{\frac{t_{m-1}}{t_m}}\cdots\right)^{\frac{t_1}{t_2}}\right)^{\frac{1}{t_1}} \leqslant C^{\mathbb{K}}_{m,\mathbf{p},\mathbf{q}}\|A\|, \tag{4.1}$$

for all continuous m*-linear forms* $A:\ell_{p_1}\times\cdots\times\ell_{p_m}\to\mathbb{K}$.

If we look for a common thread in the “different” historical proofs of the Hardy–Littlewood inequalities, we necessarily observe that Theorem 4.3 implies a Hardy–Littlewood inequality (and for this reason it has its own special interest).

The Hardy–Littlewood inequalities appeared for the first time for bilinear forms in 1930 [17, Theorem 1], with $\ell_{p_1}=\ell_{p_2}=c_0$, and then in 1931 [7, Theorem I], 1934 [14, Theorem 1], 1981 [27, Theorem A], 2016 [11, Proposition 3.1], 2016 [1, Lemma 2.1]. The role of Theorem 4.3, in the proofs of the Hardy–Littlewood inequalities in the above references, is essentially the same (this is described in Bayart’s paper [4] in what he calls Abstract Hardy–Littlewood Method). In fact, in these references, Theorem 4.3 was always used as the starting point of the proof of the respective Hardy–Littlewood inequality.

Using Theorem 4.3 we can get the following extension of Theorem 1.1 and Theorem 1.2: for $p\in[2,\infty]$ and $m\in\mathbb{N}$, $m\geqslant 2$, there are positive constants

$$\mathscr{L}^{\mathbb{K}}_{1,m,(p,\infty,\ldots,\infty)},\ \mathscr{L}^{\mathbb{K}}_{2,m,(p,\infty,\ldots,\infty)},\ \ldots,\ \mathscr{L}^{\mathbb{K}}_{m,m,(p,\infty,\ldots,\infty)}$$

such that

$$\begin{cases} \left(\sum_{j_1=1}^{\infty} \left(\sum_{j_2,\cdots,j_m=1}^{\infty} \left| A(e_{j_1},\cdots,e_{j_m}) \right|^2 \right)^{\frac{p^*}{2}} \right)^{\frac{1}{p^*}} \leqslant \mathscr{L}^{\mathbb{K}}_{1,m,(p,\infty,\ldots,\infty)} \|A\|, \\ \vdots \\ \left(\sum_{j_2,\cdots,j_m=1}^{\infty} \left(\sum_{j_1=1}^{\infty} \left| A(e_{j_1},\cdots,e_{j_m}) \right|^{p^*} \right)^{\frac{2}{p^*}} \right)^{\frac{1}{2}} \leqslant \mathscr{L}^{\mathbb{K}}_{m,m,(p,\infty,\ldots,\infty)} \|A\|, \end{cases} \tag{4.2}$$

for all continuous m-linear forms $A\colon \ell_p \times c_0 \times \cdots \times c_0 \to \mathbb{K}$. These are also called *mixed Littlewood inequalities* (see [22, 23]).

The inequalities in (4.2) have their own interest; for instance, in [23] it was proved that the real mixed Littlewood inequalities are equivalent to the Khinchine inequality.

According to Remark 4.2,

$$\mathscr{L}^{\mathbb{R}}_{k,m,(p,\infty,\ldots,\infty)} \leqslant \left(\sqrt{2}\right)^{m-1} \text{ and } \mathscr{L}^{\mathbb{C}}_{k,m,(p,\infty,\ldots,\infty)} \leqslant \left(2/\sqrt{\pi}\right)^{m-1}.$$

In the recent years, several authors ([22, 23, 24, 26]) have worked on estimating the optimal constants of (4.2) and managed to solve the problem for the case $\mathbb{K} = \mathbb{R}$. The following table summarizes the results obtained thus far:

Case	Year	Optimal constant
(i)	2016, [24]	$\mathscr{L}^{\mathbb{R}}_{1,m,(\infty,\ldots,\infty)} = \left(\sqrt{2}\right)^{m-1}$
(ii)	2018, [26]	$\mathscr{L}^{\mathbb{R}}_{k,m,(\infty,\ldots,\infty)} = \left(\sqrt{2}\right)^{m-1}$
(iii)	2019, [22, 23]	$\mathscr{L}^{\mathbb{R}}_{k,m,(p,\infty,\ldots,\infty)} = A_{\frac{p}{p-1}}^{-(m-1)}$; $p \in [2,\infty]$.

In the case of complex scalars, despite the results achieved in the real case, the optimal constants for all values of p are unknown. In this section we will obtain the optimal constants for the cases (i)-(iii) when $\mathbb{K} = \mathbb{C}$.

By [12, Corollary 5.4.2] it is simple to verify that the constants in (4.2) satisfy the following estimate:

$$\mathscr{L}^{\mathbb{C}}_{m,m,(p,\infty,\ldots,\infty)} \leqslant \cdots \leqslant \mathscr{L}^{\mathbb{C}}_{2,m,(p,\infty,\ldots,\infty)} \leqslant \mathscr{L}^{\mathbb{C}}_{1,m,(p,\infty,\ldots,\infty)}. \tag{4.3}$$

The following two well-known theorems are natural and useful extensions of the Khinchine inequalities, for Rademacher functions and Steinhaus variables, to the multilinear setting (see [23, 28]):

THEOREM 4.4. (Multiple Khinchine inequality) *For every $0 < p < \infty$ and $m \in \mathbb{N}$ there is a (optimal) constant $J_{m,p} \geqslant 1$, such that regardless of the array of scalars $(y_{i_1\ldots i_m})_{i_1,\ldots,i_m=1}^{\infty}$ we have*

$$J_{m,p} \left(\sum_{i_1,\ldots,i_m=1}^{N} |y_{i_1\ldots i_m}|^2 \right)^{\frac{1}{2}} \leqslant \left(\int_0^1 \cdots \int_0^1 \left| \sum_{i_1,\ldots,i_m=1}^{N} r_{i_1}(t_1)\ldots r_{i_m}(t_m) y_{i_1\ldots i_m} \right|^p dt_1 \ldots dt_m \right)^{\frac{1}{p}},$$

for all $N \in \mathbb{N}$, where $r_{i_j}(t_j)$ are the Rademacher functions, for all $j \in \{1,\ldots,m\}$ and $i_j \in \{1,\ldots,N\}$.

For the sake of simplicity, we write

$$\begin{aligned}&\mathbb{E}_m\left|\sum_{n_1,\ldots,n_m=1}^{N} a_{n_1\ldots n_m}\varepsilon_{n_1}^{(1)}\cdots\varepsilon_{n_m}^{(m)}\right|^p\\&=\left(\frac{1}{2\pi}\right)^{Nm}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|\sum_{n_1,\ldots,n_m=1}^{N} e^{it_{n_1}^{(1)}}\cdots e^{it_{n_m}^{(m)}}a_{n_1\ldots n_m}\right|^p dt_1^{(1)}\cdots dt_N^{(1)}\cdots dt_1^{(m)}\cdots dt_N^{(m)},\end{aligned}$$

where $\varepsilon_{n_j}^{(j)}$ are Steinhaus variables and then, the multiple Khinchine inequality for Steinhaus variables reads as follows:

THEOREM 4.5. (Multiple Khinchine inequality for Steinhaus variables) *For every $0<p<\infty$ and $m\in\mathbb{N}$ there is a (optimal) constant $S_{m,p}\geqslant 1$, such that regardless of the array of scalars $(a_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^{\infty}$ we have*

$$S_{m,p}\left(\sum_{i_1\ldots i_m=1}^{N}|a_{i_1,\ldots,i_m}|^2\right)^{1/2}\leqslant\left(\mathbb{E}_m\left|\sum_{i_1,\ldots,i_m=1}^{N}\varepsilon_{i_1}^{(1)}\cdots\varepsilon_{i_m}^{(m)}a_{i_1\ldots i_m}\right|^p\right)^{\frac{1}{p}},$$

for all $N\in\mathbb{N}$, where $\varepsilon_{i_j}^{(j)}$ are the Steinhaus variables, for all $j\in\{1,\ldots,m\}$ and $i_j\in\{1,\ldots,N\}$.

The final solution giving the optimal constant $J_{m,p}$ in Theorem 4.4 was obtained in 2019 [23]:

$$J_{m,p}=(A_p)^m$$

for all $m\in\mathbb{N}$ and for all $0<p<\infty$, where A_p is the optimal value of the constant in (2.1). By using the same technique in [23] (in the case of Steinhaus variables, we use [16, Theorem 1] instead of [13, p. 239], according [23, Lemma 3.3]), we can obtain the following optimal estimates for the constants in Theorem 4.5:

$$S_{m,p}=\left(\widetilde{A_p}\right)^m$$

for all $m\in\mathbb{N}$ and for all $0<p<\infty$, where $\widetilde{A_p}$ denotes the best constants in the Khinchine inequality for Steinhaus variables.

The multiple Khinchine inequality for Steinhaus variables plays a crucial role to improve the estimates for the constants in the Hardy–Littlewood inequalities for complex scalars (see [2, 5, 28]). In our case, it will help us to obtain the following estimate:

PROPOSITION 4.6. (Multilinear mixed Littlewood inequality) *Let $p\in[2,\infty]$ and $m\geqslant 2$. We have*

$$\mathscr{L}_{1,m,(p,\infty,\ldots,\infty)}^{\mathbb{C}}\leqslant\left(\widetilde{A_{p^*}}\right)^{-(m-1)}. \tag{4.4}$$

Since, for complex scalars, the only known estimate was the mentioned in Remark 4.2, the above estimate is somewhat new. However, the proof is simple and follows the lines of the proof of Proposition 3.1.

As in the linear case, we can provide an extension of the Blei–Khinchine inequality, for the multilinear setting. Before, we need to introduce some notation. Let $m \in \mathbb{N}$, and $(a_{n_1 \ldots n_m})_{n_1,\ldots,n_m=1}^N$ be an array of scalars, and $0 < p < \infty$, and $M \geqslant 2$. We define

$$E_{m,M,p}\left((a_{n_1,\ldots,n_m})_{n_1,\ldots,n_m=1}^N\right) = \left(\left(\frac{1}{M}\right)^{Nm} \sum_{\left(t^{(1)},\ldots,t^{(m)}\right)\in\left(\Omega_M^N\right)^m} \left|\sum_{n_1,\ldots,n_m=1}^N a_{n_1\ldots n_m} e^{it_{n_1}^{(1)}} \cdot \cdots \cdot e^{it_{n_m}^{(m)}}\right|^p\right)^{\frac{1}{p}}.$$

Using the Dominated Convergence Theorem it is possible to prove that

$$\lim_{M\to\infty}\left(E_{m,M,p}\left((a_{i_1,\ldots,i_m})_{n_1,\ldots,n_m=1}^N\right)\right) = \left(\mathbb{E}_m\left|\sum_{n_1,\ldots,n_m=1}^N a_{n_1\ldots n_m}\varepsilon_{n_1}^{(1)} \cdot \cdots \cdot \varepsilon_{n_m}^{(m)}\right|^p\right)^{\frac{1}{p}}$$

and, as in the linear case, we can observe that if $m \geqslant 1$, and $(a_{n_1\ldots n_m})_{n_1,\ldots,n_m=1}^N$ is an array of scalars, and $0 < p < \infty$, and $M \geqslant 2$, then

$$E_{m,M,p}\left((a_{n_1,\ldots,n_m})_{n_1,\ldots,n_m=1}^N\right) = E_{m,M,p}\left((a_{n_1,\ldots,n_m} e^{is_{n_1}^{(1)}} \cdot \cdots \cdot e^{is_{n_m}^{(m)}})_{n_1,\ldots,n_m=1}^N\right)$$

for all $\mathbf{s} = (s^{(1)},\ldots,s^{(m)}) \in \left(\Omega_M^N\right)^m$.

Now, we enunciate the announced extension of the Blei–Khinchine inequality for the multilinear setting. The proof is similar to the one we have given in the linear case, and for this reason it will be omitted.

THEOREM 4.7. (Multiple Blei–Khinchine inequality) *For every $1 \leqslant p \leqslant 2$, and $m \geqslant 1$, and $M \geqslant 2$, there is a (optimal) constant $\mathscr{B}_{m,M,p}$ such that regardless of the array of scalars $(a_{n_1,\ldots,n_m})_{n_1,\ldots,n_m=1}^\infty$ we have*

$$\left(\sum_{n_1,\ldots,n_m=1}^N |a_{n_1\ldots n_m}|^2\right)^{1/2} \leqslant \mathscr{B}_{m,M,p} \cdot E_{m,M,p}\left((a_{n_1\ldots n_m})_{n_1,\ldots,n_m=1}^N\right),$$

for all $N \in \mathbb{N}$. Moreover,

$$\mathscr{B}_{m,M,p} \leqslant \mathscr{L}_{m+1,m+1,(p^*,\infty,\ldots,\infty)}^{\mathbb{C}} \cdot r_M^{-m},$$

where

$$r_M := \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{M}\right)\right)^{\frac{1}{2}}.$$

We observe that making $M \to \infty$, Theorem 4.7 recovers Theorem 4.5, for $1 \leqslant p \leqslant 2$, and the estimate

$$\mathscr{B}_{m,M,p} \leqslant \mathscr{L}_{m+1,m+1,(p^*,\infty,\ldots,\infty)}^{\mathbb{C}} \cdot r_M^{-m}$$

becomes

$$\left(\widetilde{A_p}\right)^{-m} = (S_{m,p})^{-1} \leqslant \mathscr{L}_{m+1,m+1,(p^*,\infty,\ldots,\infty)}^{\mathbb{C}}. \tag{4.5}$$

Combining the inequalities (4.3), (4.4), and (4.5) we conclude that for all $m \geqslant 2$, and for all $p \in [2,\infty]$, the optimal constants in (4.2), for the complex case are

$$\mathscr{L}^{\mathbb{C}}_{m,m,(p,\infty,\ldots,\infty)} = \cdots = \mathscr{L}^{\mathbb{C}}_{2,m,(p,\infty,\ldots,\infty)} = \mathscr{L}^{\mathbb{C}}_{1,m,(p,\infty,\ldots,\infty)} = \left(\widetilde{A_{p^*}}\right)^{-(m-1)},$$

where the notation is as in the Khinchine inequality for Steinhaus variables (Theorem 2.1).

In short, we have proved the following:

THEOREM 4.8. *For all $m \geqslant 2$, for all $k \in \{1,\ldots,m\}$, and for all $p \in [2,\infty]$, the optimal constants in the complex mixed Littlewood inequalities are*

$$\mathscr{L}^{\mathbb{C}}_{k,m,(p,\infty,\ldots,\infty)} = \left(\widetilde{A_{p^*}}\right)^{-(m-1)}.$$

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REFERENCES

[1] N. ALBUQUERQUE, F. BAYART, D. PELLEGRINO AND J. B. SEOANE-SEPÚLVEDA, *Optimal Hardy–Littlewood type inequalities for polynomials and multilinear operators*, Israel J. Math. **211** (2016), no. 1, 197–220.

[2] G. ARAÚJO, D. PELLEGRINO AND D. SILVA, *On the upper bounds for the constants of the Hardy–Littlewood inequality*, J. Funct. Anal. **264** (2014), 1878–1888.

[3] S. ARUNACHALAM, S. CHAKRABORTY, M. KOUCKÝ AND N. SAURABH, R. DE WOLF, *Improved bounds on Fourier entropy and Min-entropy*, (English summary) 37th International Symposium on Theoretical Aspects of Computer Science, Art. 45–19 pp., LIPIcs. Leibniz Int. Proc. Inform., 154, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2020.

[4] F. BAYART, *Multiple summing maps: coordinatewise summability, inclusion theorems and p-Sidon sets*, J. Funct. Anal. **274** (2018), no. 4, 1129–1154.

[5] F. BAYART, D. PELLEGRINO AND J. SEOANE-SEPULVEDA, *The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{\frac{\log n}{n}}$*, Adv. Math. **264** (2014), 726–746.

[6] R. BLEI, *Analysis in Integer and Fractional Dimensions*, Cambridge Studies in Advanced Mathematics, 71. Cambridge University Press, Cambridge, 2001. xx+556 pp.

[7] H. F. BOHNENBLUST AND E. HILLE, *On the absolute convergence of Dirichlet series*, Ann. of Math. (2) **32** (1931), 600–622.

[8] G. BOTELHO AND D. FREITAS, *Summing multilinear operators by blocks: The isotropic and anisotropic cases*, J. Math. Anal. Appl. **490** (2020), no. 1, 124203, 21 pp.

[9] W. CAVALCANTE AND D. NÚÑEZ-ALARCÓN, *Remarks on the Hardy–Littlewood inequality for m-homogeneous polynomials and m-linear forms*, Quaest. Math. **39** (2016), no. 8, 1101–1113.

[10] J. DIESTEL, H. JARCHOW AND A. TONGE, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, Cambridge, 1995. xvi+474 pp.

[11] V. DIMANT AND P. SEVILLA-PERIS, *Summation of coefficients of polynomials on ℓ_p spaces*, Publ. Mat. **60** (2016), no. 2, 289–310.

[12] D. J. H. GARLING, *Inequalities: A Journey into Linear Analysis*, Cambridge University Press, 2007.

[13] U. HAAGERUP, *The best constants in the Khinchine inequality*, Studia Math. **70** (1982) 231–283.

[14] G. HARDY AND J. E. LITTLEWOOD, *Bilinear forms bounded in space $[p,q]$*, Quart. J. Math. 5 (1934), 241–254.

[15] A. KHINTCHINE, *Über dyadische Brüche*, (German) Math. Z. **18** (1923), no. 1, 109–116.

[16] H. KÖNIG, *On the best constants in the Khintchine inequality for Steinhaus variables*, Israel J. Math. **203** (2014), 23–57

[17] J. E. LITTLEWOOD, *On bounded bilinear forms in an infinite number of variables*, The Quarterly Journal of Mathematics **1** (1930), 164–174.
[18] M. MAIA AND J. SANTOS, *On the mixed* (ℓ_1, ℓ_2)*-Littlewood inequalities and interpolation*, Math. Inequal. Appl. (3) **21**, (2018), 721–727.
[19] M. MASTYŁO AND E. A. SÁNCHEZ PÉREZ, *Factorization theorems for some new classes of multilinear operators*, Asian J. Math. **24** (2020), no. 1, 1–29.
[20] M. MASTYŁO AND E. B. SILVA, *Interpolation of compact bilinear operators*, Bull. Math. Sci. **10** (2020), no. 2, 2050002, 26 pp.
[21] A. MONTANARO, *Some applications of hypercontractive inequalities in quantum information theory*, J. Math. Phys. **53** (2012), no. 12, 122206, 15 pp.
[22] T. NOGUEIRA, D. NÚÑEZ-ALARCÓN AND D. PELLEGRINO, *Optimal constants for a mixed Littlewood type inequality*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111** (2017), no. 4, 1083–1092.
[23] D. NÚÑEZ-ALARCÓN AND D. M. SERRANO-RODRÍGUEZ, *The best constants in the multiple Khintchine inequality*, Linear Multilinear Algebra **67** (2019), no. 11, 2325–2344.
[24] D. PELLEGRINO, *The optimal constants of the mixed* (ℓ_1, ℓ_2)*-Littlewood inequality*, J. Number Theory **160** (2016), 11–18.
[25] D. PELLEGRINO, J. SANTOS AND J. SEOANE-SEPULVEDA, *Some techniques on nonlinear analysis and applications*, Adv. Math. 229 (2012), no. 2, 1235–1265.
[26] D. PELLEGRINO AND E. TEIXEIRA, *Towards sharp Bohnenblust-Hille constants*, Commun. Contemp. Math. **20** (2018), no. 3, 1750029, 33 pp.
[27] T. PRACIANO-PEREIRA, *On bounded multilinear forms on a class of* ℓ_p *spaces*, J. Math. Anal. Appl. **81** (1981), 561–568.
[28] H. QUEFFÉLEC, *H. Bohr's vision of ordinary Dirichlet series: old and new results*, J. Anal. **3** (1995), 43–60.
[29] F. VIEIRA COSTA JUNIOR, *The optimal multilinear Bohnenblust-Hille constants: a computational solution for the real case*, Numer. Funct. Anal. Optim. **39** (2018), no. 15, 1656–1668.

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