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REFINEMENTS OF SOME CLASSICAL INEQUALITIES ON TIME SCALES

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Abstract. In this paper we obtain refinement of the Hölder's inequality and its related inequalities including integral Minkowski's inequality and some Hardy type inequalities on time scales.

1. Introduction

Inequalities are used everywhere in mathematics. Among the inequalities, some of the most important inequalities are given by Hölder, Minkowski, Hardy, Hilbert, and Pólya–Knopp. These inequalities are given both for sums and integrals. After the development of time scales theory [1, 3, 4, 5, 6], these inequalities are also investigated on time scales (see Chapter 10, [2]). In this paper, we give some refinements of these inequalities on time scales.

A time scale is usually denoted by $\mathbb{T}$ and is defined as a nonempty closed subset of the real numbers. For the basic time scales calculus, we refer the readers [3, 4]. Multiple integration on time scales is given by Martin Bohner and Gusein Sh. Guseinov [5, 6]. They compare the Lebesgue Δ-integral with the Riemann Δ-integral.

Let $p \in \mathbb{N}$ be fixed. For time scales, $\mathbb{T}_i$, $i \in \{1,\ldots,p\}$, let

$$\Lambda^p = \mathbb{T}_1 \times \ldots \times \mathbb{T}_p = \{x = (x_1,\ldots,x_p) : \, x_i \in \mathbb{T}_i, \; 1 \leqslant i \leqslant p\} \tag{1}$$

an p-dimensional time scale. Suppose that μ_Δ is the σ-additive Lebesgue Δ-measure on Λ^p and $\mathscr{M}$ is the collection of Δ-measurable subsets of Λ^p. If $\mathscr{A} \in \mathscr{M}$, $(\mathscr{A}, \mathscr{M}, \mu_\Delta)$ is a time scale measure space, and $s : \mathscr{A} \to \mathbb{R}$ is a Δ-measurable function, then the corresponding Δ-integral of s over $\mathscr{A}$ is denoted by (see [6, (3.18)])

$$\int_{\mathscr{A}} s(x_1,\ldots,x_p)\Delta_1 x_1 \ldots \Delta_p x_p, \quad \int_{\mathscr{A}} s(x)\Delta x, \quad \int_{\mathscr{A}} s \mathrm{d}\mu_\Delta, \quad \text{or} \quad \int_{\mathscr{A}} s(x)\mathrm{d}\mu_\Delta(x).$$

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All theorems of the general Lebesgue integration theory also hold for Lebesgue Δ-integrals on Λ^p (see [6, Section 3]).

Specifically, if the interval $[a,b) \subset \mathbb{T}$ contains only isolated points, then

$$\int_a^b s(x)\mathrm{d}\mu_\Delta(x) = \sum_{[a,b)} (\sigma(x) - x)s(x).$$

In the following theorem we present Fubini's theorem on time scales (see [2, Theorem 1.8]), it is used in the proofs of our main results.

THEOREM 1. *Let* $(\mathscr{A}, \mathscr{M}, \mu_\Delta)$ *and* $(\mathscr{B}, \mathscr{L}, \nu_\Delta)$ *be two finite-dimensional time scale measure spaces. If* $s : \mathscr{A} \times \mathscr{B} \to \mathbb{R}$ *is a* Δ*-integrable function and if we define the functions*

$$\varphi(y) = \int_{\mathscr{A}} s(x,y)\mathrm{d}\mu_\Delta(x) \quad \textit{for a.e.} \quad y \in \mathscr{B}$$

and

$$\psi(x) = \int_{\mathscr{B}} s(x,y)\mathrm{d}\nu_\Delta(y) \quad \textit{for a.e.} \quad x \in \mathscr{A},$$

then φ *is* Δ*-integrable on* $\mathscr{B}$ *and* ψ *is* Δ*-integrable on* $\mathscr{A}$ *and*

$$\int_{\mathscr{A}} \mathrm{d}\mu_\Delta(x) \int_{\mathscr{B}} s(x,y)\mathrm{d}\nu_\Delta(y) = \int_{\mathscr{B}} \mathrm{d}\nu_\Delta(y) \int_{\mathscr{A}} s(x,y)\mathrm{d}\mu_\Delta(x). \tag{2}$$

2. Refinement of Hölder's inequality on time scales

Hölder's inequality on time scales is given in the following theorem.

THEOREM 2. (see [1]) *For* $m \neq 1$*, define* $n = m/(m-1)$*. Let* $(\mathscr{A}, \mathscr{M}, \mu_\Delta)$ *be a time scale measure space. Assume* w*,* s*,* t *are nonnegative functions such that* ws^m*,* wt^n*,* wst *are* Δ*-integrable on* $\mathscr{A}$*. If* $m > 1$*, then*

$$\int_{\mathscr{A}} w(x)s(x)t(x)\mathrm{d}\mu_\Delta(x) \leqslant \left(\int_{\mathscr{A}} w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m} \times \left(\int_{\mathscr{A}} w(x)t^n(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n}. \tag{3}$$

A new improvement of Hölder's inequality for sums and integrals is given by İşcan in [8].

THEOREM 3. *Let* $l \in \mathbb{N}$*. For* $r \in \{1,\ldots,l\}$*, let* $s_r, t_r > 0$*. If* $m,n > 1$ *such that* $\frac{1}{m} + \frac{1}{n} = 1$*, then we have*

$$\sum_{r=1}^{l} s_r t_r \leqslant \frac{1}{l}\left\{\left(\sum_{r=1}^{l} r s_r^m\right)^{1/m}\left(\sum_{r=1}^{l} r t_r^n\right)^{1/n} + \left(\sum_{r=1}^{l}(l-r)s_r^m\right)^{1/m}\left(\sum_{r=1}^{l}(l-r)t_r^n\right)^{1/n}\right\}. \tag{4}$$

THEOREM 4. *Let $m,n>1$ such that $\frac{1}{m}+\frac{1}{n}=1$. If $s,t:[a,b]\to\mathbb{R}$ are such that st, s^m, and $t^n(x)$ are integrable functions on [a,b], then we have*

$$\begin{aligned}
&\int_a^b s(x)t(x)dx \qquad (5)\\
&\leqslant \frac{1}{b-a}\left\{\left(\int_a^b (b-x)s^m(x)dx\right)^{1/m}\left(\int_a^b (b-x)t^n(x)dx\right)^{1/n}\right.\\
&\quad\left.+\left(\int_a^b (x-a)s^m(x)dx\right)^{1/m}\left(\int_a^b (x-a)t^n(x)dx\right)^{1/n}\right\}.
\end{aligned}$$

Now our first result is the refinement of the Hölder's inequality on time scales as well as generalization of the improved version of Hölder's inequality for sums and integrals.

THEOREM 5. *Let $m,n>1$ such that $\frac{1}{m}+\frac{1}{n}=1$ and let $(\mathscr{A},\mathscr{M},\mu_\Delta)$ be a time scale measure space. If θ, ϑ, w, s, t are nonnegative functions on $\mathscr{A}$ such that $\theta+\vartheta=1$ and θws^m, ϑws^m, θwt^n, ϑwt^n, θwst, ϑwst, wst are Δ-integrable, then we obtain*

$$\begin{aligned}
&\int_{\mathscr{A}} w(x)s(x)t(x)\mathrm{d}\mu_\Delta(x)\leqslant A \qquad (6)\\
&\qquad\leqslant\left(\int_{\mathscr{A}} w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m}\left(\int_{\mathscr{A}} w(x)t^n(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n}
\end{aligned}$$

where

$$\begin{aligned}
A=&\left(\int_{\mathscr{A}}\theta(x)w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m}\left(\int_{\mathscr{A}}\theta(x)w(x)t^n(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n}\\
&+\left(\int_{\mathscr{A}}\vartheta(x)w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m}\left(\int_{\mathscr{A}}\vartheta(x)w(x)t^n(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n}.
\end{aligned}$$

Proof. By using Hölder's inequality on time scales (3), we get

$$\begin{aligned}
&\int_{\mathscr{A}} w(x)s(x)t(x)\mathrm{d}\mu_\Delta(x)\\
&=\int_{\mathscr{A}}\left(\theta(x)w(x)s(x)t(x)+\vartheta(x)w(x)s(x)t(x)\right)\mathrm{d}\mu_\Delta(x)\\
&=\int_{\mathscr{A}}\theta(x)w(x)s(x)t(x)\mathrm{d}\mu_\Delta(x)+\int_{\mathscr{A}}\vartheta(x)w(x)s(x)t(x)\mathrm{d}\mu_\Delta(x)\\
&\leqslant\left(\int_{\mathscr{A}}\theta(x)w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m}\left(\int_{\mathscr{A}}\theta(x)w(x)t^n(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n}\\
&\quad+\left(\int_{\mathscr{A}}\vartheta(x)w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m}\left(\int_{\mathscr{A}}\vartheta(x)w(x)t^n(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n},
\end{aligned}$$

which shows the first inequality in (6).

The second inequality in (6) is proved by using the discrete Hölder's inequality:

$$u_1v_1+u_2v_2 \leqslant (u_1^m+u_2^m)^{\frac{1}{m}}(v_1^n+v_2^n)^{\frac{1}{n}}, \qquad u_1,u_2,v_1,v_2 \geqslant 0.$$

Putting in the above inequality

$$u_1=\left(\int_{\mathscr{A}}\theta(x)w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m}, \quad u_2=\left(\int_{\mathscr{A}}\vartheta(x)w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m},$$
$$v_1=\left(\int_{\mathscr{A}}\theta(x)w(x)t^n(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n}, \quad v_2=\left(\int_{\mathscr{A}}\vartheta(x)w(x)t^n(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n},$$

we get the second inequality in (6). □

REMARK 1. From (1), let $p=1$, $\Lambda^p=\mathbb{N}$, $[a,b)=\{1,2,\dots,l\}\subset\mathbb{N}$, $\theta(x)=\frac{x}{l}$, $\vartheta(x)=\frac{l-x}{l}$, $s(x)=s_x$, $t(x)=t_x$, and $w(x)=1$. Then (6) implies the inequality (4).

If $\Lambda^p=\mathbb{R}$, $(p=1)$, $[a,b]\subseteq\mathbb{R}$, $\theta(x)=\frac{b-x}{b-a}$, $\vartheta(x)=\frac{x-a}{b-a}$, and $w(x)=1$, then (6) implies the inequality (5).

By taking $t(x)=1$ for all $x\in\mathscr{A}$ in Theorem 5, we obtain the following useful result for a power mean.

COROLLARY 1. *Let $m,n>1$ such that $\frac{1}{m}+\frac{1}{n}=1$ and let $(\mathscr{A},\mathscr{M},\mu_\Delta)$ be a time scale measure space. If θ, ϑ, w, and s are nonnegative functions on $\mathscr{A}$ such that $\theta+\vartheta=1$ and θws^m, ϑws^m, θw, ϑw, θws, ϑws, ws are Δ-integrable, then we obtain*

$$\begin{aligned}
&\left(\frac{\int_{\mathscr{A}}w(x)s(x)\mathrm{d}\mu_\Delta(x)}{\int_{\mathscr{A}}w(x)\mathrm{d}\mu_\Delta(x)}\right)^m\\
&\quad\leqslant\left(\frac{\left(\int_{\mathscr{A}}\theta(x)w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m}\left(\int_{\mathscr{A}}\theta(x)w(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n}}{\int_{\mathscr{A}}w(x)\mathrm{d}\mu_\Delta(x)}\right.\\
&\qquad\left.+\frac{\left(\int_{\mathscr{A}}\vartheta(x)w(x)s^m(x)\mathrm{d}\mu_\Delta(x)\right)^{1/m}\left(\int_{\mathscr{A}}\vartheta(x)w(x)\mathrm{d}\mu_\Delta(x)\right)^{1/n}}{\int_{\mathscr{A}}w(x)\mathrm{d}\mu_\Delta(x)}\right)^m\\
&\quad\leqslant\frac{\int_{\mathscr{A}}w(x)s^m(x)\mathrm{d}\mu_\Delta(x)}{\int_{\mathscr{A}}w(x)\mathrm{d}\mu_\Delta(x)}.
\end{aligned}$$

3. Refinement of Minkowski's, Hardy's and Hilbert's inequalities on time scales

Following theorem is the refinement of the integral Minkowski's inequality on time scales given in [2, Theorem 9.1]. In the sequel, we assume that all occurring integrals are finite.

THEOREM 6. *Let* $(\mathscr{A},\mathscr{M},\mu_\Delta)$ *and* $(\mathscr{B},\mathscr{L},\nu_\Delta)$ *be time scale measure spaces and let* u, v, *and* s *be nonnegative functions on* $\mathscr{A}$, $\mathscr{B}$, *and* $\mathscr{A}\times\mathscr{B}$, *respectively. If* $m \geqslant 1$, θ *and* ϑ *are nonnegative functions on* $\mathscr{A}$ *such that* $\theta+\vartheta=1$, *then*

$$\left[\int_{\mathscr{A}}\left(\int_{\mathscr{B}} s(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^m u(x)\mathrm{d}\mu_\Delta(x)\right]^{\frac{1}{m}} \tag{7}$$
$$\leqslant \left(\int_{\mathscr{A}}\left(\int_{\mathscr{B}} s(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^m u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{m}{m-1}} B$$
$$\leqslant \int_{\mathscr{B}}\left(\int_{\mathscr{A}} s^m(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{m}} v(y)\mathrm{d}\nu_\Delta(y)$$

holds provided that all integrals in (7) *exist, where*

$$B=\int_{\mathscr{B}}\left(\int_{\mathscr{A}}\theta(x)s^m(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{m}}\times$$
$$\times\left(\int_{\mathscr{A}}\theta(x)\left(\int_{\mathscr{B}} s(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^m u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{m-1}{m}} v(y)\mathrm{d}\nu_\Delta(y)$$
$$+\int_{\mathscr{B}}\left(\int_{\mathscr{A}}\vartheta(x)s^m(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{m}}\times$$
$$\times\left(\int_{\mathscr{A}}\vartheta(x)\left(\int_{\mathscr{B}} s(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)^m u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{m-1}{m}} v(y)\mathrm{d}\nu_\Delta(y).$$

Proof. Let

$$H(x)=\int_{\mathscr{B}} s(x,y)v(y)\mathrm{d}\nu_\Delta(y).$$

By using Fubini's theorem (Theorem 1) and Hölder's inequality (Theorem 2) on time scales, we have

$$\int_{\mathscr{A}} H^m(x)u(x)\mathrm{d}\mu_\Delta(x)=\int_{\mathscr{A}} H(x)H^{m-1}(x)u(x)\mathrm{d}\mu_\Delta(x)$$
$$=\int_{\mathscr{A}}\left(\int_{\mathscr{B}} s(x,y)v(y)\mathrm{d}\nu_\Delta(y)\right)H^{m-1}(x)u(x)\mathrm{d}\mu_\Delta(x)$$
$$=\int_{\mathscr{B}}\left(\int_{\mathscr{A}} s(x,y)H^{m-1}(x)u(x)\mathrm{d}\mu_\Delta(x)\right)v(y)\mathrm{d}\nu_\Delta(y)$$
$$\leqslant\int_{\mathscr{B}}\left(\left(\int_{\mathscr{A}}\theta(x)s^m(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{m}}\left(\int_{\mathscr{A}}\theta(x)H^m(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{m-1}{m}}\right.$$
$$\left.+\left(\int_{\mathscr{A}}\vartheta(x)s^m(x,y)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{1}{m}}\left(\int_{\mathscr{A}}\vartheta(x)H^m(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{m-1}{m}}\right)v(y)\mathrm{d}\nu_\Delta(y).$$

$$\leqslant \int_{\mathscr{B}} \left(\int_{\mathscr{A}} s^m(x,y)u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{m}} \left(\int_{\mathscr{A}} H^m(x)u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{m-1}{m}} v(y)\mathrm{d}\nu_\Delta(y)$$
$$= \int_{\mathscr{B}} \left(\int_{\mathscr{A}} s^m(x,y)u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{1}{m}} v(y)\mathrm{d}\nu_\Delta(y) \left(\int_{\mathscr{A}} H^m(x)u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{m-1}{m}}.$$

Now dividing by $\left(\int_{\mathscr{A}} H^m(x)u(x)\mathrm{d}\mu_\Delta(x)\right)^{\frac{m-1}{m}}$ we obtain the required inequalities in (7). □

REMARK 2. If we take $\mathbb{T}_i = \mathbb{R}$ in (1), then (7) implies the refinement of the integral Minkowski inequality for Lebesgue integrals.

$$\left[\int_{\mathscr{A}} \left(\int_{\mathscr{B}} s(x,y)v(y)d\nu(y) \right)^m u(x)d\mu(x) \right]^{\frac{1}{m}}$$
$$\leqslant \left[\int_{\mathscr{A}} \left(\int_{\mathscr{B}} s(x,y)v(y)d\nu(y) \right)^m u(x)d\mu(x) \right]^{\frac{m}{m-1}} \left[\int_{\mathscr{B}} \left(\int_{\mathscr{A}} \theta(x)s^m(x,y)u(x)d\mu(x \right)^{\frac{1}{m}} \times \right.$$
$$\times \left(\int_{\mathscr{A}} \theta(x) \left(\int_{\mathscr{B}} s(x,y)v(y)d\nu(y) \right)^m u(x)\mathrm{d}\mu_\Delta(x) \right)^{\frac{m-1}{m}} v(y)d\nu(y)$$
$$+ \int_{\mathscr{B}} \left(\int_{\mathscr{A}} \vartheta(x)s^m(x,y)u(x)d\mu(x) \right)^{\frac{1}{m}} \times$$
$$\left. \times \left(\int_{\mathscr{A}} \vartheta(x) \left(\int_{\mathscr{B}} s(x,y)v(y)d\nu(y) \right)^m u(x)d\mu(x) \right)^{\frac{m-1}{m}} v(y)d\nu(y). \right]$$
$$\leqslant \int_{\mathscr{B}} \left(\int_{\mathscr{A}} s^m(x,y)u(x)d\mu(x) \right)^{\frac{1}{m}} v(y)d\nu(y)$$

In this section, we let

(a) $(\mathscr{A}, \mathscr{M}, \mu_\Delta)$ and $(\mathscr{B}, \mathscr{L}, \nu_\Delta)$ be finite dimensional time scale measure spaces.

(b) $k : \mathscr{A} \times \mathscr{B} \to \mathbb{R}$ is a non-negative kernel and

$$J(x) = \int_{\mathscr{B}} k(x,y)\mathrm{d}\nu_\Delta(y) < \infty, \quad x \in \mathscr{A}.$$

(c) $u : \mathscr{A} \to \mathbb{R}_+$ is a μ_Δ-integrable function and

$$w(y) = \int_{\mathscr{A}} \frac{k(x,y)u(x)}{J(x)}\mathrm{d}\mu_\Delta(x), \quad y \in \mathscr{B}.$$

Now we obtain the refinement of the Hardy's inequality on time scales given in [2, Theorem 10.1].

THEOREM 7. *Let (a), (b) and (c) be satisfied. If $m>1$ and s, θ, and ϑ are nonnegative functions on $\mathscr{B}$ such that $\theta+\vartheta=1$, then*

$$\begin{aligned}
&\int_{\mathscr{A}} u(x)\left(\frac{1}{J(x)}\int_{\mathscr{B}} k(x,y)s(y)\mathrm{d}\nu_\Delta(y)\right)^m \mathrm{d}\mu_\Delta(x) \\
&\leqslant \int_{\mathscr{A}} \frac{u(x)}{J^m(x)}\left(\mathscr{K}_1(x)+\mathscr{K}_2(x)\right)^m \mathrm{d}\mu_\Delta(x) \\
&\leqslant \int_{\mathscr{B}} w(y)s^m(y)\mathrm{d}\nu_\Delta(y),
\end{aligned} \tag{8}$$

hold, where

$$\mathscr{K}_1(x)=\left(\int_{\mathscr{B}}\theta(y)k(x,y)s^m(y)\mathrm{d}\nu_\Delta(y)\right)^{1/m}\left(\int_{\mathscr{B}}\theta(y)k(x,y)\mathrm{d}\nu_\Delta(y)\right)^{1/n}$$

and

$$\mathscr{K}_2(x)=\left(\int_{\mathscr{B}}\vartheta(y)k(x,y)s^m(y)\mathrm{d}\nu_\Delta(y)\right)^{1/m}\left(\int_{\mathscr{B}}\vartheta(y)k(x,y)\mathrm{d}\nu_\Delta(y)\right)^{1/n}.$$

Proof. By using the Corollary 1 and Fubini theorem on time scales, we obtain

$$\begin{aligned}
&\int_{\mathscr{A}} u(x)\left(\frac{1}{J(x)}\int_{\mathscr{B}} k(x,y)s(y)\mathrm{d}\nu_\Delta(y)\right)^m \mathrm{d}\mu_\Delta(x) \\
&= \int_{\mathscr{A}} u(x)\left(\frac{\int_{\mathscr{B}} k(x,y)s(y)\mathrm{d}\nu_\Delta(y)}{\int_{\mathscr{B}} k(x,y)\mathrm{d}\nu_\Delta(y)}\right)^m \mathrm{d}\mu_\Delta(x) \\
&\leqslant \int_{\mathscr{A}}\left(\frac{\left(\int_{\mathscr{B}}\theta(y)k(x,y)s^m(y)\mathrm{d}\nu_\Delta(y)\right)^{1/m}\left(\int_{\mathscr{B}}\theta(y)k(x,y)\mathrm{d}\nu_\Delta(y)\right)^{1/n}}{\int_{\mathscr{B}} k(x,y)\mathrm{d}\nu_\Delta(y)}\right. \\
&\quad\left.+\frac{\left(\int_{\mathscr{B}}\vartheta(y)k(x,y)s^m(y)\mathrm{d}\nu_\Delta(y)\right)^{1/m}\left(\int_{\mathscr{B}}\vartheta(y)k(x,y)\mathrm{d}\nu_\Delta(y)\right)^{1/n}}{\int_{\mathscr{B}} k(x,y)\mathrm{d}\nu_\Delta(y)}\right)^m u(x)\mathrm{d}\mu_\Delta(x) \\
&\leqslant \int_{\mathscr{A}} \frac{\int_{\mathscr{B}} k(x,y)s^m(y)\mathrm{d}\nu_\Delta(y)}{\int_{\mathscr{B}} k(x,y)\mathrm{d}\nu_\Delta(y)} u(x)\mathrm{d}\mu_\Delta(x) \\
&= \int_{\mathscr{A}} \frac{u(x)}{J(x)}\int_{\mathscr{B}} k(x,y)s^m(y)\mathrm{d}\nu_\Delta(y)\mathrm{d}\mu_\Delta(x) \\
&= \int_{\mathscr{B}} s^m(y)\int_{\mathscr{A}} \frac{k(x,y)u(x)}{J(x)}\mathrm{d}\mu_\Delta(x)\mathrm{d}\nu_\Delta(y) \\
&= \int_{\mathscr{B}} w(y)s^m(y)\mathrm{d}\nu_\Delta(y). \quad \square
\end{aligned}$$

COROLLARY 2. *Let (a) and (b) be satisfied with $\mathscr{A}=\mathscr{B}=[a_1,b_1)_{\mathbb{T}}\times\ldots\times[a_p,b_p)_{\mathbb{T}}$. Let $m>1$, $\mathscr{U}$ be a nonnegative function on $\mathscr{A}$ and s, θ, ϑ be nonnegative functions on $\mathscr{B}$ such that $\theta+\vartheta=1$ and*

$$\mathscr{V}(y)=\int_{\mathscr{A}}\frac{y_1\ldots y_p k(x,y)\mathscr{U}(x)}{\sigma(x_1)\ldots\sigma(x_p)J(x)}\mathrm{d}\mu_\Delta x<\infty.$$

Then the inequalities

$$\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}\mathscr{U}(x)\left((A_k s)(x)\right)^m\frac{\mathrm{d}\mu_\Delta x_1\ldots\mathrm{d}\mu_\Delta x_p}{\sigma(x_1)\ldots\sigma(x_p)} \tag{9}$$
$$\leqslant\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}\frac{\mathscr{U}(x)}{J^m(x)}\left(\mathscr{K}_3(x)+\mathscr{K}_4(x)\right)^m\frac{\mathrm{d}\mu_\Delta x_1\ldots\mathrm{d}\mu_\Delta x_p}{\sigma(x_1)\ldots\sigma(x_p)}$$
$$\leqslant\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}\mathscr{V}(y)s^m(y)\frac{\mathrm{d}\nu_\Delta y_1\ldots\mathrm{d}\nu_\Delta y_p}{y_1\ldots y_p},$$

hold provided that all integrals in (9) *exist, where*

$$(A_k s)(x)=\frac{1}{J(x)}\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}k(x,y)s(y)\mathrm{d}\nu_\Delta y_1\ldots\mathrm{d}\nu_\Delta y_p,$$

$$\mathscr{K}_3(x)=\left(\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}\theta(y)k(x,y)s^m(y)\mathrm{d}\nu_\Delta y_1\ldots\mathrm{d}\nu_\Delta y_p\right)^{1/m}\left(\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}\theta(y)k(x,y)\mathrm{d}\nu_\Delta y_1\ldots\mathrm{d}\nu_\Delta y_p\right)^{1/n}$$

and

$$\mathscr{K}_4(x)=\left(\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}\vartheta(y)k(x,y)s^m(y)\mathrm{d}\nu_\Delta y_1\ldots\mathrm{d}\nu_\Delta y_p\right)^{1/m}\left(\int_{a_1}^{b_1}\cdots\int_{a_p}^{b_p}\vartheta(y)k(x,y)\mathrm{d}\nu_\Delta y_1\ldots\mathrm{d}\nu_\Delta y_p\right)^{1/n}.$$

Proof. The inequalities in (9) follows from Theorem 7 by substituting

$$u(x)=\frac{\mathscr{U}(x)}{\sigma(x_1)\ldots\sigma(x_p)}.\quad\square$$

In the following corollary, we present a refinement of Hardy–Hilbert type inequality.

COROLLARY 3. *Let* (b) *be satisfied with* $\mathscr{A}=\mathscr{B}=[0,\infty)_{\mathbb{T}}$. *If* $m>1$, s, θ, ϑ *be nonnegative functions on* $[0,\infty)_{\mathbb{T}}$ *such that* $\theta+\vartheta=1$ *and*

$$J_1(x)=\int_0^\infty\frac{\left(\frac{y}{x}\right)^{-1/m}}{x+y}\mathrm{d}\nu_\Delta(y),\quad J_2(y)=\int_0^\infty\frac{\left(\frac{y}{x}\right)^{1-1/m}}{x+y}\mathrm{d}\mu_\Delta(x),$$

then

$$\int_0^\infty J_1^{1-m}(x)\left(\int_0^\infty\frac{t(y)}{x+y}\mathrm{d}\nu_\Delta(y)\right)^m\mathrm{d}\mu_\Delta(x) \tag{10}$$
$$\leqslant\int_0^\infty J_1^{1-m}(x)\left(\mathscr{K}_5(x)+\mathscr{K}_6(x)\right)^m\mathrm{d}\mu_\Delta(x)$$
$$\leqslant\int_0^\infty J_2(y)t^m(y)\mathrm{d}\nu_\Delta(y)$$

hold provided that all integrals in (10) *exist, where*

$$\mathscr{K}_5(x) = \left(\int_0^\infty \theta(y)\frac{\left(\frac{y}{x}\right)^{1-1/m}}{x+y}t^m(y)\mathrm{d}\nu_\Delta(y)\right)^{1/m}\left(\int_0^\infty \theta(y)\frac{\left(\frac{y}{x}\right)^{-1/m}}{x+y}\mathrm{d}\nu_\Delta(y)\right)^{1/n}$$

and

$$\mathscr{K}_6(x) = \left(\int_0^\infty \vartheta(y)\frac{\left(\frac{y}{x}\right)^{1-1/m}}{x+y}t^m(y)\mathrm{d}\nu_\Delta(y)\right)^{1/m}\left(\int_0^\infty \vartheta(y)\frac{\left(\frac{y}{x}\right)^{-1/m}}{x+y}\mathrm{d}\nu_\Delta(y)\right)^{1/n}.$$

Proof. By substituting $s(y) = t(y)y^{1/m}$, $u(x) = \frac{J_1(x)}{x}$ and

$$k(x,y) = \begin{cases} \frac{\left(\frac{y}{x}\right)^{-1/m}}{x+y}, & \text{if} \quad x \neq 0,\ y \neq 0,\ x+y \neq 0, \\ 0, & \text{otherwise} \end{cases}$$

in Theorem 7, we get

$$\begin{aligned} w(y) &= \int_0^\infty \frac{k(x,y)u(x)}{J_1(x)}\mathrm{d}\mu_\Delta(x) = \int_0^\infty \frac{k(x,y)\mathrm{d}\mu_\Delta(x)}{x} \\ &= \frac{1}{y}\int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-1/m}}{x+y}\mathrm{d}\mu_\Delta(x) = \frac{J_2(y)}{y}. \end{aligned}$$

Now the inequalities in (10) follows from (8). □

REMARK 3. If we take $\mathbb{T} = \mathbb{R}$, then

$$\int_0^\infty \frac{\left(\frac{y}{x}\right)^{-1/m}}{x+y}dy = \int_0^\infty \frac{\left(\frac{y}{x}\right)^{1-1/m}}{x+y}dx = \frac{\pi}{\sin(\pi/m)}. \tag{11}$$

Now the inequalities in (10) implies

$$\begin{aligned} \int_0^\infty \left(\int_0^\infty \frac{t(y)}{x+y}dy\right)^m dx &\leqslant \int_0^\infty (\mathscr{K}_7(x) + \mathscr{K}_8(x))^m dx \\ &\leqslant \left(\frac{\pi}{\sin(\pi/m)}\right)^m \int_0^\infty t^m(y)dy, \end{aligned}$$

which is the refinement of classical Hilbert inequality [7], where

$$\mathscr{K}_7(x) = \left(\int_0^\infty \theta(y)\frac{\left(\frac{y}{x}\right)^{1-1/m}}{x+y}t^m(y)dy\right)^{1/m}\left(\int_0^\infty \theta(y)\frac{\left(\frac{y}{x}\right)^{-1/m}}{x+y}dy\right)^{1/n}$$

and

$$\mathscr{K}_8(x) = \left(\int_0^\infty \vartheta(y)\frac{\left(\frac{y}{x}\right)^{1-1/m}}{x+y}t^m(y)dy\right)^{1/m}\left(\int_0^\infty \vartheta(y)\frac{\left(\frac{y}{x}\right)^{-1/m}}{x+y}dy\right)^{1/n}.$$

REMARK 4. From Theorem 7 we can also obtain refinements of several other inequalities of Hardy, Hilbert, and Pólya-Knopp type as a special case. Further, results in this paper are obtained for delta integrals, but these results can also be obtained for other time scales integrals, e.g., nabla integrals, diamond-α integrals and diamond integrals, in a similar way.

Our results can also be more generalized for k number of function instead of two functions θ and ϑ. For example, let m, n, w, s and t be defined as in Theorem 5 and α_i $(i \in 1,2,\ldots,k)$ be nonnegative functions defined on $\mathscr{A}$ such that $\sum_{i=1}^{k} \alpha_i = 1$, then the inequality (6) holds with

$$A = \sum_{i=1}^{k} \left(\int_{\mathscr{A}} \alpha_i(x) w(x) s^m(x) \mathrm{d}\mu_\Delta(x) \right)^{1/m} \left(\int_{\mathscr{A}} \alpha_i(x) w(x) t^n(x) \mathrm{d}\mu_\Delta(x) \right)^{1/n}$$

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