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WEIGHTED INEQUALITIES FOR MULTILINEAR OPERATORS ACTING BETWEEN GENERALIZED ZYGMUND SPACES ASSUMING MUSIELAK–ORLICZ BUMPS CONDITIONS

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Abstract. We study continuity properties for multilinear operators between generalized Zygmund spaces of $L\log L$ type, in the variable exponent setting with different weights. In order to attack this goal we consider generalized bump conditions on the weights involved.

We shall be dealing with two different classes of operators. The former deals with operators dominated by multilinear sparse forms and the latter are potential operators and their commutators. These classes includes the multilinear Calderón-Zygmund operators, the bilinear Hilbert transform, the multilinear fractional integral operator and the multilinear Bessel potential, among others. The symbols of the commutators belong to some generalized spaces that include bounded mean oscillation spaces and the classical Lipschitz spaces.

1. Introduction

The main purpose of this paper is to give sufficient conditions on a family of weights that guarantee weighted norm inequalities for multilinear versions of operators from harmonic analysis between generalized Zygmund spaces of $L\log L$ type. In order to obtain this objective we consider certain conditions on the multilinear weights which are perturbations of the of the well known classes given in the literature [28, 3, 22].

We shall be dealing with two different classes of operator. The former deals with operators dominated by multilinear sparse forms. This includes the multilinear Calderón-Zygmund operators (CZO's) and the bilinear Hilbert transform, among others. The second class is the family of potential operators and their commutators. Examples of operators of this type are provided by the multilinear fractional integral operator and the multilinear Bessel potential. The symbols of the commutators belong to a generalized Lipschitz spaces that include bounded mean oscillation spaces (BMO) and the classical Lipschitz spaces.

In [34], Sawyer and Wheeden obtained power bump type conditions on a pair of weights in order to prove boundedness results for the fractional integral operator I_α, between Lebesgue spaces with different weights. These type of conditions appear as suitable analogues for the Muckenhoupt conditions that characterize the boundedness of

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I_α for the case of one weight (see [29]). Motivated by the results above, in [31], Pérez considered weaker norms than those involved in the conditions in [34], and obtained two-weighted boundedness estimates for potential type operators. Later in [23], two weighted norm inequalities in the spirit of those in [31] were proved for the higher order commutators associated to potential operators with BMO symbols. Recently these results were extended to the context of spaces with variable exponents in [25] and [26].

On the other hand, in [10] the author studied a similar problem for CZO's and their commutators with BMO symbols. In that paper, Cruz Uribe and Pérez conjectured that weaker conditions involving Young functions, are sufficient to obtain the desired results. This conjecture have been studied extensively, for a complete history we refer the reader to [9, 8, 7, 19] and [11] for the references that they contain. The problem considered in [10] was approached in the general setting of variable exponents in [27].

Motivated by the work in [21], K. Moen ([28]) considered the multilinear fractional integral operator and proved two weighted $L^p - L^q$ estimates, generalizing to the multilinear context some results given in [31]. Later, Bernardis, Gorosito and Pradolini ([3]) extend the result to multilinear potential operators and their commutators with BMO symbols.

One of our main results generalizes the main theorem in [3] not only by considering power bump type conditions involving Musielak-Orlicz spaces but also by dealing with variable Lebesgue spaces. Moreover, the classes of the symbols in our results is wider than the corresponding considered in [3].

Related with the results involving operators controled by sparse forms, our results consider power bump type conditions involving Musielak-Orlicz spaces and extend those from [10] to the multilinear context and the general setting of the generalized Zygmund spaces of $L\log L$ type.

As far as we know the main results of this work are new even in the classical setting.

The paper is organized as follows. In Section 2 we introduce basic definitions and known results to state and prof our main results. In Section 3 we present the classes of multilinear operators wich are our focus of study and our main results associated to it. Finally, in Section 4 and 5 we prove our main results.

2. Preliminaries

2.1. Musielak-Orlicz spaces

With $\mathscr{F}$ we denote the set of all Lebesgue real valued, measurable functions on $\mathbb{R}^n$.

A convex function $\psi : [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$, $\lim_{t\to 0^+} \psi(t) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$ is called a Φ-function.

A real function $\Psi : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is said to be a generalized Φ-function (GΦ-function), and we denote $\Psi \in \Phi(\mathbb{R}^n)$, if $\Psi(x,t)$ is Lebesgue-measurable in x for every $t \geqslant 0$ and $\Psi(x,\cdot)$ is a Φ-function for every $x \in \mathbb{R}^n$.

If $\Psi \in \Phi(\mathbb{R}^n)$, then the set

$$L^{\Psi}(\mathbb{R}^n) = \left\{ f \in \mathscr{F} : \int_{\mathbb{R}^n} \Psi(x, |f(x)|)\, dx < \infty \right\}$$

defines a Banach function space equipped with the Luxemburg norm given by

$$\|f\|_{\Psi(\cdot,\cdot)} = \inf\left\{ \lambda > 0 : \int_{\mathbb{R}^n} \Psi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leqslant 1 \right\}.$$

The space $L^{\Psi}(\mathbb{R}^n)$ is called a Musielak-Orlicz (MO) space (or generalized Orlicz space). The MO spaces provide the framework for a variety of different function spaces, including classical (weighted) Lebesgue and Orlicz spaces, generalized Zygmund spaces of $L\log L$ type and variable exponent Lebesgue spaces. We refer the reader to [17, 13, 6] for a detailed description of these spaces or some particular cases of these and their properties. Below we shall describe some definitions and results in these spaces relevant for the present work.

Let $\Psi \in \Phi(\mathbb{R}^n)$, then for any $x \in \mathbb{R}^n$ we denote by $\Psi^*(x,\cdot)$ the conjugate function of $\Psi(x,\cdot)$ which is defined by

$$\Psi^*(x,u) = \sup_{t \geqslant 0} (tu - \Psi(x,t)), \quad u \geqslant 0.$$

Also we can define Ψ^{-1}, the generalized inverse function of Ψ by

$$\Psi^{-1}(x,t) = \inf\{u \geqslant 0 : \Psi(x,u) \geqslant t\}, \quad x \in \mathbb{R}^n, t \geqslant 0.$$

The following result is a generalization of the classical Hölder inequality to the MO spaces.

LEMMA 1. *Let* $\Psi \in \Phi(\mathbb{R}^n)$, *then*

$$\int_{\mathbb{R}^n} f(x)g(x)\, dx \lesssim \|f\|_{\Psi(\cdot,\cdot)} \|g\|_{\Psi^*(\cdot,\cdot)} \tag{1}$$

for all $f \in L^{\Psi}(\mathbb{R}^n)$ *and* $g \in L^{\Psi^*}(\mathbb{R}^n)$.

For $\Psi \in G\Phi(\mathbb{R}^n)$ wich satisfies that every simple function belongs to $L^{\Psi^*}(\mathbb{R}^n)$, we have the following norm conjugate formula,

$$\|f\|_{\Psi(\cdot,\cdot)} \simeq \sup_{\|g\|_{\Psi^*(\cdot,\cdot)} \leqslant 1} \int_{\mathbb{R}^n} |f(x)g(x)|\, dx \tag{2}$$

for every function $f \in L^{\Psi}(\mathbb{R}^n)$ (see [[14], Corollary 2.7.5]).

For $\Psi \in \Phi(\mathbb{R}^n)$ and $r > 0$, a rescaling of Ψ is given by

$$r\Psi(x,t) = \Psi(x,t^r). \tag{3}$$

It follows directly from the definition of the Luxemburg norm that,

$$\|f\|_{r\Psi(\cdot,\cdot)}^r = \|f^r\|_{\Psi(\cdot,\cdot)}. \tag{4}$$

2.1.1. Generalized Zygmund space of $L\log L$ type

We say that $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ if $p(\cdot):\mathbb{R}^n\to[1,\infty)$ is measurable function. We denote

$$p^-=\inf_{x\in\mathbb{R}^n}p(x)\qquad\text{and}\qquad p^+=\sup_{x\in\mathbb{R}^n}p(x).$$

Let $p'(\cdot)$ the *conjugate exponent of* $p(\cdot)$ given by $p'(\cdot)=p(\cdot)/(p(\cdot)-1)$. It is not hard to prove that $(p')^-=(p^+)'$ and $(p')^+=(p^-)'$.

For $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and $q(\cdot):\mathbb{R}^n\to\mathbb{R}$ such that $q^+<\infty$, we define the function

$$\varphi_{p(\cdot),q(\cdot)}(x,t)=t^{p(x)}(\log(e+t))^{q(x)} \tag{5}$$

for $t\geqslant 0$ and $x\in\mathbb{R}^n$, with the convention $\infty\cdot 0=0$. To guarantee the convexity property of $\varphi_{p(\cdot),q(\cdot)}$ we suppose that the two exponents satisfies the inequality

$$2(p(x)-1)+q(x)\geqslant 0,$$

for all $x\in\mathbb{R}^n$. Then $\varphi_{p(\cdot),q(\cdot)}\in\Phi(\mathbb{R}^n)$.

The *generalized Zygmund space of* $L\log L$ *type*, is the MO space associated to $\varphi_{p(\cdot),q(\cdot)}$, $L^{\varphi_{p(\cdot),q(\cdot)}}(\mathbb{R}^n)$. We shall denote this space $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

When $q(\cdot)\equiv 0$, $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)=L^{p(\cdot)}(\mathbb{R}^n)$ is the well known variable Lebesgue space. We denote $\|f\|_{L^{p(\cdot)}(\log L)^0}=\|f\|_{p(\cdot)}$ (see [6] and [14] for more information).

By $[L^{p(\cdot)}(\log L)^{q(\cdot)}]_{\text{loc}}(\mathbb{R}^n)$ we denote the space of the functions f such that $f\in L^{p(\cdot)}(\log L)^{q(\cdot)}(K)$ for every compact set $K\subset\mathbb{R}^n$.

A locally integrable function w defined in $\mathbb{R}^n$ which is positive almost everywhere is called a weight. For a given weight w, we define the *weighted generalized Zygmund space of* $L\log L$ *type* $[L^{p(\cdot)}(\log L)^{q(\cdot)}]_w(\mathbb{R}^n)$ as the set of the measurable functions f defined on $\mathbb{R}^n$ such that $fw\in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$. When $q(\cdot)\equiv 0$, we denote $[L^{p(\cdot)}(\log L)^{q(\cdot)}]_w(\mathbb{R}^n)=L^{p(\cdot)}_w(\mathbb{R}^n)$.

A stardar prove show that if $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$, $q(\cdot):\mathbb{R}^n\to[0,\infty)$ with $p^+,q^+<\infty$ and $w\in[L^{p(\cdot)}(\log L)^{q(\cdot)}]_{\text{loc}}(\mathbb{R}^n)$, then the set of bounded functions with compact support is dense in $[L^{p(\cdot)}(\log L)^{q(\cdot)}]_w(\mathbb{R}^n)$.

Simple calculus shows that $\varphi^*_{p(\cdot),q(\cdot)}(x,t)\simeq t^{p'(x)}(\log(e+t))^{-q(x)/(p(x)-1)}$. Then, from (2) we can deduce the following result.

LEMMA 2. *Let* $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ *with* $p^->1$, $q:\mathbb{R}^n\to[0,\infty)$ *with* $q^+<\infty$ *and* w *a weight, then*

$$\|f\|_{[L^{p(\cdot)}(\log L)^{q(\cdot)}]_w}\simeq\sup_g\int_{\mathbb{R}^n}|f(x)g(x)|\,dx, \tag{6}$$

holds for every measurable function f, *where the supremun is taken over all functions* g *such that* $\|gw^{-1}\|_{L^{p'(\cdot)}(\log L)^{-q(\cdot)/(p(\cdot)-1)}}\leqslant 1$.

The following classes of exponents appear in connection with the boundedness properties of different operators from harmonic analysis on the spaces defined above. We say that $p(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$ if $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and satisfy the following inequalities

$$\left|\frac{1}{p(x)}-\frac{1}{p(y)}\right|\leqslant\frac{C}{\log(e+1/|x-y|)},\ x,y\in\mathbb{R}^n$$

and

$$\left|\frac{1}{p(x)}-\frac{1}{p_\infty}\right| \leqslant \frac{C}{\log(e+|x|)},\ x\in\mathbb{R}^n \tag{7}$$

for some positive constants C and p_∞. It is easy to see that the inequality (7) implies that $\lim_{|x|\to\infty} 1/p(x) = 1/p_\infty$.

Let $q(\cdot):\mathbb{R}^n \to \mathbb{R}$, we say that $q(\cdot)\in\mathscr{P}^{loglog}(\mathbb{R}^n)$, if is bounded, i.e. it satisfies $-\infty < q^- \leqslant q^+ < \infty$, and there exists a positive constant C such that

$$|q(x)-q(y)| \leqslant \frac{C}{\log(e+\log(e+1/|x-y|))}, \quad \forall x,y\in\mathbb{R}^n.$$

In [24], the authors proved that $p(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$ with $1<p^-\leqslant p^+<\infty$ and $q(\cdot)\in\mathscr{P}^{loglog}(\mathbb{R}^n)$ are sufficient conditions in order that the Hardy-Littlewood maximal operator M is continuous in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

2.1.2. Variable Lebesgue spaces

When we deal with variable Lebesgue spaces, we have the following known results that we shall be using throughout this paper.

LEMMA 3. ([14], Lemma 3.2.20) *Let $s(\cdot),p(\cdot),q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ be such that $1/s(\cdot) = 1/p(\cdot)+1/q(\cdot)$. Then*

$$\|fg\|_{s(\cdot)} \lesssim \|f\|_{p(\cdot)}\|g\|_{q(\cdot)}. \tag{8}$$

Particularly, if $s(\cdot)\equiv 1$, the inequality above gives

$$\int_{\mathbb{R}^n} |f(y)g(y)|\,dy \lesssim \|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} \tag{9}$$

which is an extension of the classical Hölder inequality.

LEMMA 4. ([14], Lemma 3.2.6) *Let $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and s be a constant such that $s\geqslant 1/p^-$. Then $\||f|^s\|_{p(\cdot)} = \|f\|^s_{sp(\cdot)}$.*

LEMMA 5. ([14], see Corollary 4.5.9) *Let $p(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$. Then $\|\chi_Q\|_{p(\cdot)}\|\chi_Q\|_{p'(\cdot)} \simeq |Q|$, for every cubes $Q\subset\mathbb{R}^n$.*

Moreover, we have the following result.

COROLLARY 1. *Let $p(\cdot),d(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot)\leqslant d(\cdot)$. Suppose that $1/p(\cdot) = 1/\beta(\cdot)+1/d(\cdot)$ then, for every cube $Q\subset\mathbb{R}^n$,*

$$\|\chi_Q\|_{p(\cdot)} \simeq \|\chi_Q\|_{\beta(\cdot)}\|\chi_Q\|_{d(\cdot)}.$$

LEMMA 6. ([26], Lemma 3.7) *Let k be a positive integer and $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ such that $1 < p^- \leqslant p^+ < \infty$. Let $a \in T_\infty$ and $b \in \mathscr{L}_a$. Then for every $Q \in \mathscr{Q}$,*

$$\frac{\left\|\chi_Q(b-b_Q)^k\right\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \lesssim a(Q)^k \|b\|_{\mathscr{L}_a}^k.$$

LEMMA 7. ([26], Lemma 3.8) *Let $a \in T_\infty$ and $b \in \mathscr{L}_a$, then the following inequality*

$$|b_{3Q} - b_Q| \lesssim a(3Q) \|b\|_{\mathscr{L}_a}$$

holds for every $Q \in \mathscr{Q}$.

LEMMA 8. ([26], Lemma 3.9) *Let $d(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ with $d_\infty \leqslant d(\cdot) \leqslant d^+ < \infty$ and $\delta(\cdot)$ be defined as in (29) and $b \in \mathbb{L}(\delta(\cdot))$. Let Q be a cube in $\mathbb{R}^n$ and $z \in kQ$ for some positive integer k. Then*

$$|b(z) - b_Q| \lesssim \|\chi_Q\|_{n/\delta(\cdot)}.$$

The following lemma can be deduced from [[14], Corollary 7.3.21].

LEMMA 9. ([14]) *Let $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ and $\mathscr{G} \subset \mathscr{Q}$ a disjoint family. Then*

$$\left\|\sum_{Q\in\mathscr{G}} \chi_Q \frac{\|f\chi_Q\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}}\right\|_{p(\cdot)} \simeq \left\|\sum_{Q\in\mathscr{G}} f\chi_Q\right\|_{p(\cdot)}$$

for every $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$.

The following lemma gives a doubling property for the functional define by $\mathtt{f}(Q) := \|\chi_Q\|_{L^{p(\cdot)}}$ with $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$.

LEMMA 10. ([33], Equation (2.11)) *If $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ with $p^+ < \infty$, then there exists a positive constant C_p such that the inequality*

$$\|\chi_{2Q}\|_{p(\cdot)} \leqslant C_p \|\chi_Q\|_{p(\cdot)} \tag{10}$$

holds for every cube $Q \subset \mathbb{R}^n$.

Let $\gamma > 0$, by iteration of inequality (10) it is not difficult to prove that

$$\left\|\chi_{\gamma Q}\right\|_{p(\cdot)} \lesssim \|\chi_Q\|_{p(\cdot)} \tag{11}$$

holds for every cube $Q \subset \mathbb{R}^n$, with an appropriate constant depending on γ and C_p.

Let $p(\cdot), q(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leqslant q(\cdot)$, then

$$\frac{\|\chi_Q f\|_{p(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \lesssim \frac{\|\chi_Q f\|_{q(\cdot)}}{\|\chi_Q\|_{q(\cdot)}}, \qquad f \in L^1_{\text{loc}}(\mathbb{R}^n). \tag{12}$$

Indeed, let $\beta(\cdot)$ be defined by $1/\beta(\cdot) = 1/p(\cdot) - 1/q(\cdot)$. Then $\beta(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ and, by Hölder's inequality (8) and Corollary 1 we obtain (12).

2.1.3. Maximal operators

A corresponding maximal operator associated to $\Psi \in \Phi(\mathbb{R}^n)$ is

$$M_{\Psi(\cdot,\cdot)} f(x) = \sup_{Q \ni x} \frac{\|\chi_Q f\|_{\Psi(\cdot,\cdot)}}{\|\chi_Q\|_{\Psi(\cdot,\cdot)}} \tag{13}$$

and, the fractional type version of this maximal operator is given by

$$M_{\beta(\cdot),\Psi(\cdot,\cdot)} f(x) = \sup_{Q \ni x} \|\chi_Q\|_{\beta(\cdot)} \frac{\|\chi_Q f\|_{\Psi(\cdot,\cdot)}}{\|\chi_Q\|_{\Psi(\cdot,\cdot)}}, \tag{14}$$

where $\beta(\cdot) \in \mathscr{P}(\mathbb{R}^n)$.

For the case of a rescaling of Ψ, taking into account (4), the maximal operator satisfies

$$M_{r\Psi(\cdot,\cdot)} f(x) = \sup_{Q \ni x} \left(\frac{\|\chi_Q f^r\|_{\Psi(\cdot,\cdot)}}{\|\chi_Q\|_{\Psi(\cdot,\cdot)}} \right)^{1/r}. \tag{15}$$

If $\Psi(x,t) = t^{s(x)}$, then $M_\Psi = M_{s(\cdot)}$ was introduced in [14] and $M_{\beta(\cdot),\Psi} = M_{\beta(\cdot),s(\cdot)}$ was defined in [25].

Notice that, when $s(\cdot) \equiv 1$ and $\beta(\cdot) \equiv n/\alpha$, $M_{s(\cdot)} = M$ and $M_{\beta(\cdot),s(\cdot)} = M_\alpha$ where M and M_α are the Hardy-Littlewood maximal function and its fractional version, respectively.

The next boundedness result for $M_{\beta(\cdot),s(\cdot)}$ was proved in [25] in generalized Zygmund space of $L\log L$ type.

THEOREM 1. *Let $p(\cdot), r(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ such that $p(\cdot) \leqslant r(\cdot) \leqslant r^+ < \infty$ and $q(\cdot) \in \mathscr{P}^{loglog}(\mathbb{R}^n)$ a non-negative function. Suppose that $\beta(\cdot)$ is the exponent define by $1/\beta(\cdot) = 1/p(\cdot) - 1/r(\cdot)$ and $s(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ satisfies $(p/s)^- > 1$. Then*

$$M_{\beta(\cdot),s(\cdot)} : L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{r(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n).$$

REMARK 1. For the case $s(\cdot) \equiv S$, where S is a constant, if $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ with $1 \leqslant S < p^- \leqslant p^+ < \infty$ and $q(\cdot) \in \mathscr{P}^{loglog}(\mathbb{R}^n)$, from the result of [24] it can be deduced that $M_S : L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n)$.

2.2. Sparse family

We now introduce the dyadic structures we will working with. These definitions and a substantial treatise on dyadic calculus can be found in [20].

We say that a collection of cubes $\mathscr{D}$ in $\mathbb{R}^n$ is a *dyadic grid* if it satisfies the following properties:

1. If $Q \in \mathscr{D}$, then $\ell(Q) = 2^k$ for some $k \in \mathbb{Z}$.

2. If $P, Q \in \mathscr{D}$, then $P \cap Q \in \{P, Q, \emptyset\}$.

3. For every $k \in \mathbb{Z}$, the cubes $\mathscr{D}_k = \{Q \in \mathscr{D} : \ell(Q) = 2^k\}$ form a partition of $\mathbb{R}^n$.

We shall use the following proposition that contain the so called 2^n dyadic lattices trick. The origin of this result is obscure. A very careful history of this result is given by Cruz-Uribe in [5] (see the footnote following Theorem 3.4) where the credit is given to Okikiolu [30]. We state the result from [[18], Proof of Theorem 1.10].

PROPOSITION 1. *There are 2^n dyadic grids $\mathscr{D}_t$, such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_t \in \mathscr{D}_t$ satisfying $Q \subset Q_t$ and $\ell(Q_t) \leqslant 6\ell(Q)$.*

Given a dyadic grid $\mathscr{D}$, a set $\mathscr{S} \subset \mathscr{D}$ is *sparse* if there exist $\eta \in (0,1)$ such that

(S1) For every $Q \in \mathscr{S}$ there exist $E(Q) \subset Q$ such that $\eta|Q| \leqslant |E(Q)|$.

(S2) The sets $E(Q)$ are pairwise disjoint.

The classic example of a dyadic grid is the standard dyadic grid on $\mathbb{R}^n$ and an example of a sparce family can obtain by a careful construction of Calderón-Zygmund cubes associated with an L^1_{loc} function at an infinite number of levels (for details see [32, 5]).

3. Statement of the main results

3.1. Operators dominated by multilinear sparse forms

In this subsection we present a class of operators related to a class of multilinear sparse forms, and state the main results associated with these operators.

Given a dyadic grid $\mathscr{D}$, a sparse family $\mathscr{S} \subset \mathscr{D}$, and $\vec{r} = (r_1, \ldots, r_{m+1})$ with $r_i \geqslant 1$, for every $1 \leqslant i \leqslant m+1$, let us consider the multilinear sparse form $\Lambda_{\mathscr{S},\vec{r}}$ introduced in [22] as

$$\Lambda_{\mathscr{S},\vec{r}}(h, f_1, \ldots, f_m) = \sum_{Q \in \mathscr{S}} |Q| \left(\frac{1}{|Q|} \int_Q h(x)^{r_{m+1}}\, dx \right)^{1/r_{m+1}} \prod_{i=1}^{m} \left(\frac{1}{|Q|} \int_Q f_i(x)^{r_i}\, dx \right)^{1/r_i},$$

(for the definition of dyadic grid and sparse family see Subsection 2.2).

Our goal is to give weighted boundedness results for operators which are controlled by multilinear sparse forms $\Lambda_{\mathscr{S},\vec{r}}$. We denote $T \in D(\Lambda_{\vec{r}})$ if T is an operator such that for every $h, f_1, \ldots, f_m$ non-negative bounded functions with compact support on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} h|T((f_1, \ldots, f_m))|\, dx \lesssim \sup_{\mathscr{S}} \Lambda_{\mathscr{S},\vec{r}}(h, f_1, \ldots, f_m), \tag{16}$$

where the sup is taken over all sparse families and $\lesssim$ means that there exists a positive constant C such that (16) holds with $\lesssim$ replaced by $\leqslant C$.

We now present some operators satisfying the assumption (16). The first example is the multilinear Calderón-Zygmund operator. Let T be an m-linear operator satisfying

$$T(f_1, \ldots, f_m)(x) = \int_{\mathbb{R}^{nm}} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m$$

whenever $f_1,\dots,f_m \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ and $x \notin \cup_{j=1}^m \operatorname{supp} f_j$. We say that T is a *multilinear Calderón-Zygmund operator* if it can be extended as a bounded operator from $L^{p_1} \times \dots \times L^{p_m}$ to L^p for some $1 < p_1,\dots,p_m < \infty$ with $1/p_1 + \dots + 1/p_m = 1/p$. The kernel K satisfies two conditions: the *size estimate* and the *smoothness condition*. The size estimate is

$$|K(y_0\dots,y_m)| \lesssim \frac{1}{\left(\sum_{i,j=0}^m |y_i - y_j|\right)^{nm}}.$$

The smoothness condition assume

$$\begin{aligned}&|K(y_0,\dots,y_j,\dots,y_m) - K(y_0,\dots,y_j',\dots,y_m)| \\ &\lesssim \omega\left(\frac{|y_j - y_j'|}{\sum_{i,j=0}^m |y_i - y_j|}\right)\frac{1}{\left(\sum_{i,j=0}^m |y_i - y_j|\right)^{nm}},\end{aligned}$$

for all $0 \leqslant j \leqslant m$, whenever $|y_j - y_j'| \leqslant \frac{1}{2}\max_{0\leqslant k\leqslant m} |y_j - y_k|$, where ω is a modulus of continuity, i.e. a positive nondecreasing continuous and doubling function.

If T is a multilinear Calderón-Zygmund operator, independently and simultaneously, in [4] and [20], the authors proved the following pointwise sparse bound that is stronger and imply form bounds like (16). Let $\mathscr{D}$ a dyadic grid, $\mathscr{S} \subset \mathscr{D}$ a sparse family and

$$T_{\mathscr{S}}(f_1,\dots,f_m)(x) = \sum_{Q\in\mathscr{S}} \chi_Q(x) \prod_{i=1}^m |f_i|_Q.$$

Then there exists 3^n dyadic grids $\mathscr{D}_i$ and associated sparse families $\mathscr{S}_i \subset \mathscr{D}_i$ such the inequality

$$|T(f_1,\dots,f_m)| \lesssim \sum_{i=1}^{3^n} T_{\mathscr{S}_i}(f_1,\dots,f_m) \tag{17}$$

holds for every $f_1,\dots,f_m \in \mathscr{C}_c^\infty(\mathbb{R}^n)$. Hence (17) shows that (16) holds with $\vec{r} = (1,\dots,1)$.

The second example is a class of rough bilinear singular integrals studied by A. Barron [1]. Suppose $\Omega \in L^q(S^{2n-1})$ for some $q > 1$ with $\int_{S^{2n-1}} \Omega\, d\sigma = 0$, the *rough bilinear operator* is define by

$$T_\Omega(f_1,f_2)(x) = \text{p.v.}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_1(x-y_1)f_2(x-y_2)\frac{\Omega((y_1-y_2)/(y_1,y_2)}{|(y_1,y_2)|^{2n}}\,dy_1dy_2.$$

In [1] the author prove that (16) holds for $\vec{r} = (r,r,r)$ with any $1<r<\infty$.

The last and the most prominent example is the *bilinear Hilbert transform* defined as

$$BH(f,g)(x) = \text{p.v.}\int_{\mathbb{R}} f(x-t)g(x+t)\,\frac{dt}{t}.$$

In [[12], Theorem 2] (see also [2]), this operator and some other bilinear multipliers have been shown to satisfy (16) with $\vec{r} = (r_1,r_2,r_3)$ satisfying $1 < r_1,r_2,r_3 < \infty$ and

$$\frac{1}{\min\{r_1,2\}} + \frac{1}{\min\{r_2,2\}} + \frac{1}{\min\{r_3,2\}} < 2.$$

The next theorem gives a continuity property for $T \in D(\Lambda_{\vec{r}})$ acting between generalized Zygmund space of $L\log L$ type with different weights. For notational convenience, we write $L^{p_i(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) = L^{p_i(\cdot)}(\log L)^{q(\cdot)}$, and by $\mathscr{Q}$ we denote the set of cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes.

To state the result we give the following definition. We say that a pair of GΦ-functions (Υ,Ψ) satisfy condition $\mathscr{A}\mathscr{V}$, and we denote $(\Upsilon,\Psi) \in \mathscr{A}\mathscr{V}$, if it satisfies

$$\frac{1}{|Q|}\int_Q f(x)g(x)\,dx \lesssim \frac{\|f\chi_Q\|_{\Upsilon(\cdot,\cdot)}}{\|\chi_Q\|_{\Upsilon(\cdot,\cdot)}}\frac{\|g\chi_Q\|_{\Psi(\cdot,\cdot)}}{\|\chi_Q\|_{\Psi(\cdot,\cdot)}}. \tag{18}$$

We shall give later some examples of GΦ-functions satisfying condition $\mathscr{A}\mathscr{V}$.

THEOREM 2. *Let $\vec{r} = r_1,\ldots,r_{m+1} \geqslant 1$ and $T \in D(\Lambda_{\vec{r}})$. Let $q(\cdot) \in \mathscr{P}^{loglog}(\mathbb{R}^n)$ be a non-negative function and $p_1(\cdot),\ldots,p_m(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ with $p_i^- > 1$ and $1/p(\cdot) = \sum_{i=1}^m 1/p_i(\cdot)$ such that*

$$r_i < p_i^- \leqslant p_i^+ < \infty \text{ for } 1 \leqslant i \leqslant m \quad \text{and} \quad 1 < p^- \leqslant p^+ < r'_{m+1}.$$

Let (Υ_i,Ψ_i), $1 \leqslant i \leqslant m+1$ pairs of GΦ-functions satisfying condition $\mathscr{A}\mathscr{V}$,

$$M_{r_i\Psi_i(\cdot,\cdot)} : L^{p_i(\cdot)}(\log L)^{q(\cdot)} \hookrightarrow L^{p_i(\cdot)}(\log L)^{q(\cdot)} \text{ for } 1 \leqslant i \leqslant m \tag{19}$$

and

$$M_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)} : L^{p'(\cdot)}(\log L)^{-q(\cdot)/(p(\cdot)-1)} \hookrightarrow L^{p'(\cdot)}(\log L)^{-q(\cdot)/(p(\cdot)-1)}. \tag{20}$$

Suppose that $(v_1,\ldots,v_m,w)$ is any $m+1$-tuple of weights such that v_i belongs to $[L^{p_i(\cdot)}(\log L)^{q(\cdot)}]_{\mathrm{loc}}$, $1 \leqslant i \leqslant m$, and that satisfies

$$\sup_{Q\in\mathscr{Q}} \frac{\|\chi_Q w^{r_{m+1}}\|^{1/r_{m+1}}_{\Upsilon_{m+1}(\cdot,\cdot)}}{\|\chi_Q\|^{1/r_{m+1}}_{\Upsilon_{m+1}(\cdot,\cdot)}} \prod_{i=1}^m \frac{\left\|\chi_Q v_i^{-1}\right\|^{1/r_i}_{\Upsilon_i(\cdot,\cdot)}}{\|\chi_Q\|^{1/r_i}_{\Upsilon_i(\cdot,\cdot)}} < \infty. \tag{21}$$

Then

$$T : \left[L^{p_1(\cdot)}(\log L)^{q(\cdot)}\right]_{v_1} \times \ldots \times \left[L^{p_m(\cdot)}(\log L)^{q(\cdot)}\right]_{v_m} \hookrightarrow \left[L^{p(\cdot)}(\log L)^{q(\cdot)}\right]_w.$$

We can also obtain continuity properties for $T \in D(\Lambda_{\vec{r}})$ acting between variable Lebesgue spaces associated to different exponents.

THEOREM 3. *Let $\vec{r} = r_1,\ldots,r_{m+1} \geqslant 1$ and $T \in D(\Lambda_{\vec{r}})$. Let $p_1(\cdot),\ldots,p_m(\cdot)$ and $d(\cdot)$ exponents in $\mathscr{P}^{log}(\mathbb{R}^n)$, with $p_i^- > 1$ and $1/p(\cdot) = \sum_{i=1}^m 1/p_i(\cdot)$ such that*

$$r_i < p_i^- \leqslant p_i^+ < \infty \quad \text{and} \quad 1 < p^- \leqslant p(\cdot) \leqslant d(\cdot) \leqslant d^+ < r'_{m+1},$$

Let (Υ_i,Ψ_i), $1 \leqslant i \leqslant m+1$, pairs of G$\Phi$-functions satisfying condition $\mathscr{A}\mathscr{V}$,

$$M_{r_i\Psi_i(\cdot,\cdot)} : L^{p_i(\cdot)} \hookrightarrow L^{p_i(\cdot)} \text{ for } 1 \leqslant i \leqslant m \tag{22}$$

and

$$M_{\beta(\cdot),r_{m+1}\Psi_{m+1}(\cdot,\cdot)} : L^{d'(\cdot)} \hookrightarrow L^{p'(\cdot)} \tag{23}$$

where $\beta(\cdot)$ *is defined by* $1/\beta(\cdot) = 1/p(\cdot) - 1/d(\cdot)$. *Suppose that* $(v_1,\ldots,v_m,w)$ *is any* $m+1$*-tuple of weights such that* v_i *belongs to* $L^{p_i(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \leqslant i \leqslant m$, *and that satisfies*

$$\sup_{Q\in\mathscr{Q}} \frac{\|\chi_Q\|_{d(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \frac{\|\chi_Q w^{r_{m+1}}\|^{1/r_{m+1}}_{\Upsilon_{m+1}(\cdot,\cdot)}}{\|\chi_Q\|^{1/r_{m+1}}_{\Upsilon_{m+1}(\cdot,\cdot)}} \prod_{i=1}^{m} \frac{\|\chi_Q v_i^{-1}\|^{1/r_i}_{\Upsilon_i(\cdot,\cdot)}}{\|\chi_Q\|^{1/r_i}_{\Upsilon_i(\cdot,\cdot)}} < \infty. \tag{24}$$

Then

$$T : L^{p_1(\cdot)}_{v_1} \times \ldots \times L^{p_m(\cdot)}_{v_m} \hookrightarrow L^{d(\cdot)}_{w}.$$

Let us now give some examples of GΦ-functions that satisfy the hypothesis of the theorems above. In order to check the examples see the details in [26].

EXAMPLE 1. Let $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ and $R, r \geqslant 1$ two constants such that $r < p^- \leqslant p^+ < \infty$ and

$$R > \frac{[(p/r)']^+}{[(p/r)']^-}.$$

If $s(\cdot) = R(p(\cdot)/r)'$, $\Upsilon(x,t) = t^{s(x)}$ and $\Psi(x,t) = t^{s'(x)}$, then $(\Upsilon,\Psi) \in \mathscr{AV}$. Also, note that $M_{r\Psi(\cdot,\cdot)} = M_{rs'(\cdot)}$. Then by Theorem 1, for some non-negative function $q(\cdot) \in \mathscr{P}^{loglog}(\mathbb{R}^n)$,

$$M_{r\Psi(\cdot,\cdot)} : L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^n).$$

EXAMPLE 2. Let $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$ and

$$\sigma > \frac{(p')^+}{(p')^-}.$$

If we define $\Upsilon_1(x,t) = t^{\sigma p'(x)}(\log(e+t))^{\sigma p'(x)}$ and $\Psi_1(x,t) = t^{(\sigma p')'(x)}$ then $(\Upsilon_1,\Psi_1) \in \mathscr{AV}$. Also,

$$M_{\Psi_1(\cdot,\cdot)} : L^{p(\cdot)}(\mathbb{R}^n) \to L^{p(\cdot)}(\mathbb{R}^n).$$

EXAMPLE 3. Let $d(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ with $1 < d^- \leqslant d^+ < \infty$ and

$$\eta > \frac{d^+}{d^-}.$$

If $\Upsilon_2(x,t) = t^{\eta d(x)}(\log(e+t))^{\eta d(x)}$ and $\Psi_2(x,t) = t^{(\eta d)'(x)}$ then $(\Upsilon_2,\Psi_2) \in \mathscr{AV}$. Moreover, if $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ satisfies $d'(\cdot) \leqslant p'(\cdot) \leqslant (p')^+ < \infty$ and $\beta(\cdot)$ is the exponent define by $1/\beta(\cdot) = 1/d'(\cdot) - 1/p'(\cdot)$, then

$$M_{\beta(\cdot),\Psi_2(\cdot,\cdot)} : L^{d'(\cdot)}(\mathbb{R}^n) \to L^{p'(\cdot)}(\mathbb{R}^n).$$

EXAMPLE 4. Let $p(\cdot)$, η and Ψ_2 as in the above example. Let $\mu(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ with $1 < \mu^- \leqslant \mu^+ < \infty$ such that

$$\frac{1}{\eta p'(\cdot)} - \frac{1}{\mu(\cdot)} > \varepsilon$$

for some constant $\varepsilon \in (0,1)$ and $\nu(\cdot) \in \mathscr{P}^{loglog}(\mathbb{R}^n)$. If we define

$$\Upsilon_2(x,t) = t^{\mu(x)}(\log(e+t))^{\nu(x)\mu(x)},$$

then $(\Upsilon_2, \Psi_2) \in \mathscr{A}\mathscr{V}$.

3.2. Multilinear potential operators and their commutators

We now consider the multilinear potential operator defined in [3] as

$$P_\Gamma(f_1,\dots f_m)(x) = \int_{\mathbb{R}^{nm}} \Gamma(x-y_1,\dots,x-y_m)\prod_{i=1}^m f_i(y_i)\,dy_1\dots dy_m,$$

where Γ is a non-negative function defined on $\mathbb{R}^{nm}$. We also deal with the commutator associated to this operator, given by

$$P_{\Gamma,\vec{b}}(f_1,\dots f_m)(x) = \sum_{j=1}^m P_{\Gamma,b_j}(f_1,\dots f_m)(x), \tag{25}$$

where

$$P_{\Gamma,b_j}(f_1,\dots f_m)(x) = b_j(x)P_\Gamma(f_1,\dots f_m)(x) - P_\Gamma(f_1,\dots,b_jf_j,\dots,f_m)(x).$$

In this subsection we present two weighted strong type inequalities for the operators above. As in [3] we assume that the function Γ satisfies a growth condition. More precisely, we say that a non-negative locally integrable function Γ defined in $\mathbb{R}^{nm}$ satisfies a $\mathfrak{R}$-condition (or that $\Gamma \in \mathfrak{R}$) if there exist two positive constants ε and δ such that the inequality

$$\sup_{w_1\dots w_m \in \mathscr{A}_{(2^k,1,0)}} \Gamma(w_1\dots w_m) \leqslant \frac{C}{2^{knm}}\int_{\mathscr{A}_{(2^k,\delta,\varepsilon)}} \Gamma(y_1\dots y_m)\,dy_1\dots dy_m$$

holds for every $k \in \mathbb{Z}$, where

$$\mathscr{A}_{(t,\delta,\varepsilon)} = \left\{ y_1,\dots,y_m : \delta(1-\varepsilon)t < \sum_{i=1}^m |y_i| \leqslant \delta(1+\varepsilon)2t \right\},\ t > 0. \tag{26}$$

Although the basic example of operators of this type is provided by the multilinear fractional integral operator defined by the kernel

$$\Gamma(w_1,\dots,w_m) = \left(\sum_{i=1}^m |w_i|\right)^{\alpha-nm},$$

for $0 < \alpha < nm$, another important example is the multilinear Bessel potential. For $\alpha > 0$ the kernel of this operator is given by

$$\Gamma_\alpha(x_1,\dots,x_m) = C_{\alpha,n,m}\int_0^\infty e^{-t}e^{-\frac{(\sum_{i=1}^m |x_i|)^2}{4t}}t^{\frac{\alpha-nm}{2}}\frac{dt}{t},$$

where $C_{\alpha,n,m} = \frac{1}{2^{nm}\gamma(\alpha/2)\pi^{nm/2}}$ and $\gamma(\cdot)$ is the gamma function. As in [3], Γ_α satisfies the $\mathfrak{R}$-condition.

We now define the functional related with the space where the symbol $\vec{b}$ belongs. We consider a functional $a:\mathscr{Q}\to[0,\infty)$. We say that a satisfies the T_∞ condition, and we denote by $a \in T_\infty$, if there exists a finite positive constant t_∞ such that for every $Q,Q' \in \mathscr{Q}$ such that $Q' \subset Q$,

$$a(Q') \leqslant t_\infty a(Q). \tag{27}$$

We denote the least constant t_∞ in (27) by $\|a\|_{t_\infty}$. Clearly, $\|a\|_{t_\infty} \geqslant 1$.

Let $0<\rho<\infty$ and $a \in T_\infty$. We say that a function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the generalized Lipschitz space $\mathscr{L}_a^\rho$ if

$$\sup_Q \frac{1}{a(Q)}\left(\frac{1}{|Q|}\int_Q |b-b_Q|^\rho\,dx\right)^{1/\rho} < \infty$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ and b_Q denote the average $\frac{1}{|Q|}\int_Q b$. We consider the vector of symbols $\vec{b}=(b_1,\dots,b_m)\in(\mathscr{L}_a^\rho)^m$.

We denote $\widetilde{\Gamma}$ the function definded by

$$\widetilde{\Gamma}(t) = \int_{|z|\leqslant t}\Gamma(z)\,dz.$$

THEOREM 4. *Let $p_1(\cdot),\dots,p_m(\cdot),r(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$, such that $p_i^- > 1$ and $1/p(\cdot) = \sum_{i=1}^m 1/p_i(\cdot)$ that satisfies*

$$1 < p^- \leqslant p(\cdot) \leqslant r(\cdot) \leqslant r^+ < \infty$$

and $\Gamma\in\mathfrak{R}$. Let $1\leqslant\rho<\infty$, $a\in T_\infty$ and $\vec{b}\in(\mathscr{L}_a^\rho)^m$. Suppose that $(v_1,\dots,v_m,w)$ is any $m+1$-tuple of weights such that $v_i\in L^{p_i(\cdot)}_{\mathrm{loc}}$ and, for some constants $R_i > (p_i')^+/(p_i')^-$ and $S > r^+/r^-$,

$$\sup_{Q\in\mathscr{Q}} a(Q)\widetilde{\Gamma}(\ell(Q))\frac{\|\chi_Q\|_{r(\cdot)}}{\|\chi_Q\|_{p(\cdot)}}\frac{\|\chi_Q w\|_{Sr(\cdot)}}{\|\chi_Q\|_{Sr(\cdot)}}\prod_{i=1}^m\frac{\|\chi_Q v_i^{-1}\|_{R_i p_i'(\cdot)}}{\|\chi_Q\|_{R_i p_i'(\cdot)}} < \infty. \tag{28}$$

Then

$$P_{\Gamma,\vec{b}} : L^{p_1(\cdot)}_{v_1}\times\dots\times L^{p_m(\cdot)}_{v_m} \hookrightarrow L^{r(\cdot)}_w.$$

Let us observe that, if $a(Q)=|Q|^{\delta/n}$, $0<\delta<1$, then $a\in T_\infty$. It is known that $\mathscr{L}_a^1 := \mathbb{L}(\delta)$ coincides with the classical Lipschitz spaces Λ_δ define as the set of functions b such that

$$|b(x)-b(y)| \lesssim |x-y|^\delta$$

for every $x,y \in \mathbb{R}^n$.

On the other hand, if $d(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$, $0 < \alpha < n$ such that $n/d^- \leqslant \alpha$ and $\delta(\cdot)$ is the exponent defined by

$$\frac{\delta(\cdot)}{n} = \frac{\alpha}{n} - \frac{1}{d(\cdot)}, \tag{29}$$

the functional $a(Q) = \|\chi_Q\|_{n/\delta(\cdot)}$ satisfies the T_∞ condition and $\mathscr{L}_a = \mathbb{L}(\delta(\cdot))$ is a variable version of the spaces $\mathbb{L}(\delta)$ defined above.

For $\Psi_1, \ldots, \Psi_m$ GΦ-functions, we define the following multilinear version of the maximal operator M_Ψ given in (13), as follows

$$\mathscr{M}_{\Psi_1(\cdot,\cdot),\ldots,\Psi_m(\cdot,\cdot)}(f_1, \ldots, f_m)(x) = \sup_{Q \ni x} \prod_{i=1}^{m} \frac{\|\chi_Q f_i\|_{\Psi_i(\cdot,\cdot)}}{\|\chi_Q\|_{\Psi_i(\cdot,\cdot)}}.$$

When $\Psi_1 \equiv \ldots \equiv \Psi_m \equiv 1$, the maximal operator $\mathscr{M}_{\Psi_1(\cdot,\cdot),\ldots,\Psi_m(\cdot,\cdot)} = \mathscr{M}$ was introduced in [21]. When $\Psi_i(x,t) = t^{s_i(x)}$, we denote $\mathscr{M}_{\Psi_1(\cdot,\cdot),\ldots,\Psi_m(\cdot,\cdot)} = \mathscr{M}_{s_1(\cdot),\ldots,s_m(\cdot)}$

An auxiliaty result for prove the Theorem 4 is the following that gives a variation of the classical Calderón-Zygmund decomposition, associated to the maximal operator $\mathscr{M}_{s_1(\cdot),\ldots,s_m(\cdot)}$ (for the result that describes the classical Calderón-Zygmund decomposition we refer the reader to [15, 16]). For a dyadic drid $\mathscr{D}$ we define

$$\mathscr{M}^{\mathscr{D}}_{s_1(\cdot),\ldots,s_m(\cdot)}(f_1, \ldots, f_m)(x) = \sup_{Q \in \mathscr{D}: Q \ni x} \prod_{i=1}^{m} \frac{\|\chi_Q f_i\|_{s_i(\cdot)}}{\|\chi_Q\|_{s_i(\cdot)}}.$$

PROPOSITION 2. *Let* $s_1(\cdot), \ldots, s_m(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ *with* $1/s(\cdot) = \sum_{i=1}^m 1/s_i(\cdot)$ *such that* $s(\cdot) \geqslant 1$ *and* $\mathscr{D}$ *be a dyadic grid. Suppose that* $f_1, \ldots, f_m$ *are measurable functions such that*

$$\lim_{|Q| \to \infty} \prod_{i=1}^{m} \frac{\|\chi_Q f_i\|_{s_i(\cdot)}}{\|\chi_Q\|_{s_i(\cdot)}} = 0. \tag{30}$$

Then the following are true:

1. *For each* $\lambda > 0$, *there exists a disjoint collection of maximal cubes* $\{Q_j\}_{j \in \mathbb{N}} \subset \mathscr{D}$ *such that*

$$E_\lambda = \left\{ x \in \mathbb{R}^n : \mathscr{M}^{\mathscr{D}}_{s_1(\cdot),\ldots,s_m(\cdot)}(f_1, \ldots, f_m)(x) > \lambda \right\} = \bigcup_{j \in \mathbb{N}} Q_j, \tag{31}$$

and for every j,

$$\lambda < \prod_{i=1}^{m} \frac{\|\chi_{Q_j} f_i\|_{s_i(\cdot)}}{\|\chi_{Q_j}\|_{s_i(\cdot)}} \leqslant C_s^{2m} \lambda. \tag{32}$$

2. *There exists a positive constant* σ *such that, if* $\alpha > \sigma$ *and for each* $k \in \mathbb{Z}$ *we consider* $\{Q_j^k\}_{j \in \mathbb{N}}$ *the collection of maximal dyadic cubes from* (1) *with*

$$\Omega_k = \left\{ x \in \mathbb{R}^n : \mathscr{M}^{\mathscr{D}}_{s_1(\cdot),\ldots,s_m(\cdot)}(f_1, \ldots, f_m)(x) > \alpha^k \right\} = \bigcup_j Q_j^k,$$

then $\mathscr{S} = \{Q_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ *is a sparse family.*

Particularly, if $b \in \mathbb{L}(\delta(\cdot))$, we can improve the Theorem 4 in the sense that we can introduce certain type of norms in the conditions on the weights involving GΦ-functions.

THEOREM 5. *Let* $p_1(\cdot),\ldots,p_m(\cdot),r(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ *such that* $p_i^- > 1$ *and* $1/p(\cdot) = \sum_{i=1}^m 1/p_i(\cdot)$ *that satisfies*

$$1 < p^- \leqslant p(\cdot) \leqslant r(\cdot) \leqslant r^+ < \infty$$

and $\Gamma \in \mathfrak{R}$. *Let* $\beta(\cdot)$ *be a function such that*

$$\frac{1}{\beta(\cdot)} = \frac{1}{p(\cdot)} - \frac{1}{r(\cdot)}.$$

Let $d(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ *and* $\delta(\cdot)$ *defined as in (29), such that* $d_\infty \leqslant d(\cdot)$ *and let* $\vec{b} \in (\mathbb{L}(\delta(\cdot)))^m$. *Let* (Υ_i,Ψ_i), $1 \leqslant i \leqslant m+1$, *pairs of G*$\Phi$*-functions satisfying condition* $\mathscr{AV}$,

$$\mathscr{M}_{\Psi_1(\cdot,\cdot),\ldots,\Psi_m(\cdot,\cdot)} : L^{p_1(\cdot)} \times \ldots \times L^{p_m(\cdot)} \to L^{p(\cdot)} \tag{33}$$

and

$$M_{\beta(\cdot),\Psi_{m+1}(\cdot,\cdot)} : L^{r'(\cdot)} \to L^{p'(\cdot)}, \quad i = 1,\ldots,m. \tag{34}$$

Suppose that $(v_1,\ldots,v_m,w)$ *is any* $m+1$*-tuple of weights such that* $v_i \in L^{p_i(\cdot)}_{\text{loc}}$ *and*

$$\sup_{Q\in\mathscr{Q}} \|\chi_Q\|_{n/\delta(\cdot)}\, \widetilde{\Gamma}(\ell(Q)) \frac{\|\chi_Q\|_{r(\cdot)}}{\|\chi_Q\|_{p(\cdot)}} \frac{\|\chi_Q w\|_{\Upsilon_{m+1}(\cdot,\cdot)}}{\|\chi_Q\|_{\Upsilon_{m+1}(\cdot,\cdot)}} \prod_{i=1}^m \frac{\|\chi_Q v_i^{-1}\|_{\Upsilon_i(\cdot,\cdot)}}{\|\chi_Q\|_{\Upsilon_i(\cdot,\cdot)}} < \infty. \tag{35}$$

Then

$$P_{\Gamma,\vec{b}} : L^{p_1(\cdot)}_{v_1} \times \ldots \times L^{p_m(\cdot)}_{v_m} \hookrightarrow L^{r(\cdot)}_w.$$

REMARK 2. Note that condition (35) with $\Upsilon_{m+1}(x,t) = t^{\sigma r(x)}(\log(e+t))^{\sigma r(x)}$ and $\Upsilon_i(x,t) = t^{\eta p'(x)}(\log(e+t))^{\eta p'(x)}$ is weaker than condition (28) since, if $\sigma < R$ and $\eta < S$, we have

$$\frac{\|\chi_Q w\|_{\Upsilon_{m+1}(\cdot,\cdot)}}{\|\chi_Q\|_{\Upsilon_{m+1}(\cdot,\cdot)}} \lesssim \frac{\|\chi_Q w\|_{Rr(\cdot)}}{\|\chi_Q\|_{Rr(\cdot)}} \quad \text{and} \quad \frac{\|\chi_Q v^{-1}\|_{\Upsilon_i(\cdot,\cdot)}}{\|\chi_Q\|_{\Upsilon_i(\cdot,\cdot)}} \lesssim \frac{\|\chi_Q v^{-1}\|_{Sp'(\cdot)}}{\|\chi_Q\|_{Sp'(\cdot)}}.$$

4. Proofs of theorems from subsection 3.1

In this section we present the proofs of Theorem 2 and Theorem 3.

Proof of Theorem 2. Since $v_i \in [L^{p_i(\cdot)}(\log L)^{q(\cdot)}]_{\text{loc}}$ implies that the set of bounded functions with compact support is dense in $[L^{p_i(\cdot)}(\log L)^{q(\cdot)}]_{v_i}(\mathbb{R}^n)$, it is enough to show that

$$\|T(f_1,\ldots,f_m)\|_{[L^{p(\cdot)}(\log L)^{q(\cdot)}]_w} \lesssim \prod_{i=1}^m \|f_i\|_{[L^{p_i(\cdot)}(\log L)^{q(\cdot)}]_{v_i}}$$

for each $f_1,\dots,f_m \geqslant 0$ a bounded function with compact support. This is in turn equivalent by duality to

$$\int_{\mathbb{R}^n} |T(f_1,\dots,f_m)(x)|w(x)g(x)\,dx \lesssim \prod_{i=1}^m \|f_i\|_{[L^{p_i(\cdot)}(\log L)^{q(\cdot)}]_{v_i}}$$

for all non-negative bounded functions with compact support $f_1,\dots,f_m$ and g with $\|g\|_{L^{p'(\cdot)}(\log L)^{-q(\cdot)/(p(\cdot)-1)}(\cdot)} \leqslant 1$. Let $f_1,\dots,f_m$ and g be functions with these properties. By (16) it is enough to prove that, for every spase family $\mathscr{S} \subset \mathscr{D}$ a dyadic grid,

$$\sum_{Q\in\mathscr{S}} |Q| \left(\frac{1}{|Q|}\int_Q g(x)^{r_{m+1}} w(x)^{r_{m+1}}\,dx\right)^{1/r_{m+1}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q f_i(x)^{r_i}\,dx\right)^{1/r_i} \lesssim \prod_{i=1}^m \|f_i\|_{[L^{p_i(\cdot)}(\log L)^{q(\cdot)}]_{v_i}}. \tag{36}$$

By condition $\mathscr{AV}$ we have

$$\begin{aligned}
&\sum_{Q\in\mathscr{S}} |Q| \left(\frac{1}{|Q|}\int_Q g(x)^{r_{m+1}} w(x)^{r_{m+1}}\,dx\right)^{1/r_{m+1}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q f_i(x)^{r_i}\,dx\right)^{1/r_i}\\
&\lesssim \sum_{Q\in\mathscr{S}} |Q| \left(\frac{\left\|\chi_{\gamma Q_j^k} g^{r_{m+1}}\right\|_{\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_{m+1}(\cdot,\cdot)}} \frac{\left\|\chi_{\gamma Q_j^k} w^{r_{m+1}}\right\|_{\Upsilon_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Upsilon_{m+1}(\cdot,\cdot)}}\right)^{1/r_{m+1}}\\
&\quad\times \prod_{i=1}^m \left(\frac{\left\|\chi_{\gamma Q_j^k} f_i^{r_i} v_i^{r_i}\right\|_{\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_i(\cdot,\cdot)}} \frac{\left\|\chi_{\gamma Q_j^k} v_i^{-1}\right\|_{\Upsilon_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Upsilon_i(\cdot,\cdot)}}\right)^{1/r_i}
\end{aligned} \tag{37}$$

Consequenly by the hypothesis on the weights (21) and (4) we have

$$\begin{aligned}
&\sum_{Q\in\mathscr{S}} |Q| \left(\frac{1}{|Q|}\int_Q g(x)^{r_{m+1}} w(x)^{r_{m+1}}\,dx\right)^{1/r_{m+1}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q f_i(x)^{r_i}\,dx\right)^{1/r_i}\\
&\lesssim \sum_{Q\in\mathscr{S}} |Q| \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}} \prod_{i=1}^m \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{r_i\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_i\Psi_i(\cdot,\cdot)}}.
\end{aligned}$$

Using that $\mathscr{S}$ is a sparse family and Hölder inequality (1) we obtain

$$\begin{aligned}
&\sum_{Q\in\mathscr{S}} |Q| \left(\frac{1}{|Q|}\int_Q g(x)^{r_{m+1}} w(x)^{r_{m+1}}\,dx\right)^{1/r_{m+1}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q f_i(x)^{r_i}\,dx\right)^{1/r_i}\\
&\lesssim \sum_{Q\in\mathscr{S}} |E(Q)| \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}} \prod_{i=1}^m \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{r_i\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_i\Psi_i(\cdot,\cdot)}}
\end{aligned}$$

$$\lesssim \int_{\mathbb{R}^n} M_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}(g)(y)\prod_{i=1}^m M_{r_i\Psi_i(\cdot,\cdot)}(f_i v_i)(y)\,dy$$
$$\lesssim \left\|M_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}(g)\right\|_{L^{p'(\cdot)}(\log L)^{-q(\cdot)/(p(\cdot)-1)}}\prod_{i=1}^m \left\|M_{r_i\Psi_i(\cdot,\cdot)}(f_i v_i)\right\|_{L^{p_i(\cdot)}(\log L)^{q(\cdot)}}.$$

Thus by conditions (19) and (20) we can conclude (36) and complete the proof of Theorem 2. □

Proof of Theorem 3. Proceeding in the same way as in the proof of Theorem 2 (see (37)) replacing the corresponding spaces we obtain

$$\sum_{Q\in\mathscr{S}}|Q|\left(\frac{1}{|Q|}\int_Q g(x)^{r_{m+1}}w(x)^{r_{m+1}}\,dx\right)^{1/r_{m+1}}\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q f_i(x)^{r_i}\,dx\right)^{1/r_i}$$
$$\lesssim \sum_{Q\in\mathscr{S}}|Q|\left(\frac{\left\|\chi_{\gamma Q_j^k}g^{r_{m+1}}\right\|_{\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_{m+1}(\cdot,\cdot)}}\frac{\left\|\chi_{\gamma Q_j^k}w^{r_{m+1}}\right\|_{\Upsilon_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Upsilon_{m+1}(\cdot,\cdot)}}\right)^{1/r_{m+1}}$$
$$\times\prod_{i=1}^m\left(\frac{\left\|\chi_{\gamma Q_j^k}f_i^{r_i}v_i^{r_i}\right\|_{\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_i(\cdot,\cdot)}}\frac{\left\|\chi_{\gamma Q_j^k}v_i^{-1}\right\|_{\Upsilon_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Upsilon_i(\cdot,\cdot)}}\right)^{1/r_i}$$

Consequenly the hypothesis on the weights (24) we have

$$\sum_{Q\in\mathscr{S}}|Q|\left(\frac{1}{|Q|}\int_Q g(x)^{r_{m+1}}w(x)^{r_{m+1}}\,dx\right)^{1/r_{m+1}}\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q f_i(x)^{r_i}\,dx\right)^{1/r_i}$$
$$\lesssim \sum_{Q\in\mathscr{S}}|Q|\frac{\|\chi_Q\|_{d(\cdot)}}{\|\chi_Q\|_{p(\cdot)}}\frac{\left\|\chi_{\gamma Q_j^k}g\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}}\prod_{i=1}^m\frac{\left\|\chi_{\gamma Q_j^k}f_i v_i\right\|_{r_i\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_i\Psi_i(\cdot,\cdot)}}.$$

By Corollary 1 the last sum is equivalent to

$$\sum_{Q\in\mathscr{S}}|Q|\,\|\chi_Q\|_{\beta(\cdot)}\frac{\left\|\chi_{\gamma Q_j^k}g\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}}\prod_{i=1}^m\frac{\left\|\chi_{\gamma Q_j^k}f_i v_i\right\|_{r_i\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_i\Psi_i(\cdot,\cdot)}}.$$

Using that $\mathscr{S}$ is a sparse family and Hölder inequality (9) we obtain

$$\sum_{Q\in\mathscr{S}}|Q|\left(\frac{1}{|Q|}\int_Q g(x)^{r_{m+1}}w(x)^{r_{m+1}}\,dx\right)^{1/r_{m+1}}\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q f_i(x)^{r_i}\,dx\right)^{1/r_i}$$
$$\lesssim \sum_{Q\in\mathscr{S}}|E(Q)|\,\|\chi_Q\|_{\beta(\cdot)}\frac{\left\|\chi_{\gamma Q_j^k}g\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_{m+1}\Psi_{m+1}(\cdot,\cdot)}}\prod_{i=1}^m\frac{\left\|\chi_{\gamma Q_j^k}f_i v_i\right\|_{r_i\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r_i\Psi_i(\cdot,\cdot)}}$$

$$\begin{aligned}
&\lesssim \int_{\mathbb{R}^n} M_{\beta(\cdot),r_{m+1}\Psi_{m+1}(\cdot,\cdot)}(g)(y)\prod_{i=1}^m M_{r_i\Psi_i(\cdot,\cdot)}(f_i v_i)(y)\,dy \\
&\lesssim \left\|M_{\beta(\cdot),r_{m+1}\Psi_{m+1}(\cdot,\cdot)}(g)\right\|_{p'(\cdot)}\prod_{i=1}^m \left\|M_{r_i\Psi_i(\cdot,\cdot)}(f_i v_i)\right\|_{p_i(\cdot)} \\
&\lesssim \|g\|_{d'(\cdot)}\prod_{i=1}^m \|f_i v_i\|_{p_i(\cdot)} = \prod_{i=1}^m \|f_i\|_{L^{p_i(\cdot)}_{v_i}},
\end{aligned}$$

where we have used conditions (22) and (23). This concludes the proof of Theorem 3. □

5. Proof of results from subsection 3.2

In this section we present the proofs of Theorem 4, Proposition 2 and Theorem 5. In order to give the proof of Theorem 4 we state and prove three auxiliary results.

LEMMA 11. *Let* $s_1(\cdot),\ldots,s_m(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$, *with* $1/s(\cdot)=\sum_{i=1}^m 1/s_i(\cdot)$ *such that* $s(\cdot)\geqslant 1$. *Let* $\nu\in\mathbb{Z}$ *and* $Q_0\in\mathscr{D}$. *If we define*

$$\mathscr{O}=\{Q: Q\in\mathscr{D}, Q\subset Q_0 \ y\ \ell(Q)=2^{-\nu}\},$$

then

$$\sum_{Q\in\mathscr{O}}\|g\chi_Q\|_{s'(\cdot)}\prod_{i=1}^m\|f_i\chi_Q\|_{s_i(\cdot)}\lesssim\|g\chi_{Q_0}\|_{s'(\cdot)}\prod_{i=1}^m\|f_i\chi_{Q_0}\|_{s_i(\cdot)} \tag{38}$$

for every $f_i\in L^{s_i(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ *and* $g\in L^{s'(\cdot)}_{\text{loc}}(\mathbb{R}^n)$.

Proof. Let $f_i\in L^{s_i(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ and $g\in L^{s'(\cdot)}_{\text{loc}}(\mathbb{R}^n)$. By Lemma 5 we have

$$\begin{aligned}
\sum_{Q\in\mathscr{O}}\|g\chi_Q\|_{s'(\cdot)}\prod_{i=1}^m\|f_i\chi_Q\|_{s_i(\cdot)} &\simeq \sum_{Q\in\mathscr{O}}|Q|\frac{\|g\chi_Q\|_{s'(\cdot)}}{\|\chi_Q\|_{s'(\cdot)}}\prod_{i=1}^m\frac{\|f_i\chi_Q\|_{s_i(\cdot)}}{\|\chi_Q\|_{s_i(\cdot)}} \\
&\simeq \int_{\mathbb{R}^n}\sum_{Q\in\mathscr{O}}\chi_Q(x)\frac{\|g\chi_Q\|_{s'(\cdot)}}{\|\chi_Q\|_{s'(\cdot)}}\prod_{i=1}^m\frac{\|f_i\chi_Q\|_{s_i(\cdot)}}{\|\chi_Q\|_{s_i(\cdot)}}\,dx \\
&\leqslant \int_{\mathbb{R}^n}\left(\sum_{Q\in\mathscr{O}}\chi_Q(x)\frac{\|g\chi_Q\|_{s'(\cdot)}}{\|\chi_Q\|_{s'(\cdot)}}\right)\prod_{i=1}^m\left(\sum_{Q\in\mathscr{O}}\chi_Q(x)\frac{\|f_i\chi_Q\|_{s_i(\cdot)}}{\|\chi_Q\|_{s_i(\cdot)}}\right)dx.
\end{aligned}$$

Hence, by Hölder's inequality (8) we obtain

$$\begin{aligned}
&\sum_{Q\in\mathscr{O}}\|g\chi_Q\|_{s'(\cdot)}\prod_{i=1}^m\|f_i\chi_Q\|_{s_i(\cdot)} \\
&\quad\lesssim\left\|\sum_{Q\in\mathscr{O}}\chi_Q\frac{\|g\chi_Q\|_{s'(\cdot)}}{\|\chi_Q\|_{s'(\cdot)}}\right\|_{s'(\cdot)}\prod_{i=1}^m\left\|\sum_{Q\in\mathscr{O}}\chi_Q\frac{\|f_i\chi_Q\|_{s_i(\cdot)}}{\|\chi_Q\|_{s_i(\cdot)}}\right\|_{s_i(\cdot)}.
\end{aligned}$$

Since $\mathscr{O}$ is a disjoint family, by Lemma 9, we conclude that

$$\sum_{Q\in\mathscr{O}}\|g\chi_Q\|_{s'(\cdot)}\prod_{i=1}^m\|f_i\chi_Q\|_{s_i(\cdot)}\lesssim\Big\|g\sum_{Q\in\mathscr{O}}\chi_Q\Big\|_{s'(\cdot)}\prod_{i=1}^m\Big\|f_i\sum_{Q\in\mathscr{O}}\chi_Q\Big\|_{s_i(\cdot)}$$
$$\lesssim\|g\chi_{Q_0}\|_{s'(\cdot)}\prod_{i=1}^m\|f_i\chi_{Q_0}\|_{s_i(\cdot)}\qquad\square$$

Recall that $\widetilde{\Gamma}$ is definded by

$$\widetilde{\Gamma}(t)=\int_{|z|\leqslant t}\Gamma(z)\,dz,$$

and we introduce the function $\overline{\Gamma}$ as

$$\overline{\Gamma}(t)=\sup_{y_1,\dots,y_m\in\mathscr{A}_{t,1,0}}\Gamma(y_1,\dots,y_m),$$

where $\mathscr{A}_{(t,\delta,\varepsilon)}$ is the set defined in (26).

LEMMA 12. *Let $\mu(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$, $\Gamma\in\mathfrak{R}$ and $Q_0\in\mathscr{D}$. Then, for every $h\in L^{\mu(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, we have*

$$\sum_{Q\in\mathscr{D}:Q\subset Q_0}\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}\frac{\|\chi_Q h\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}}\lesssim\widetilde{\Gamma}[\delta(1+\varepsilon)\ell(Q_0)]|Q_0|\frac{\|\chi_{Q_0}h\|_{\mu(\cdot)}}{\|\chi_{Q_0}\|_{\mu(\cdot)}},\tag{39}$$

where ε,δ are the constants provided by condition $\mathfrak{R}$.

Proof. Let $h\in L^{\mu(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$. Suppose that $\ell(Q_0)=2^{-d_0}$ with $d_0\in\mathbb{Z}$. By the equivalence (5) and Lemma 11 we have

$$\sum_{Q\in\mathscr{D}:Q\subset Q_0}\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}\frac{\|\chi_Q h\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}}$$
$$\simeq\sum_{d\geqslant d_0}2^{-dnm}\overline{\Gamma}(2^{-d-1})\sum_{Q\subset Q_0:\ell(Q)=2^{-d}}\|h\chi_Q\|_{\mu(\cdot)}\|\chi_Q\|_{\mu'(\cdot)}$$
$$\lesssim\|h\chi_{Q_0}\|_{\mu(\cdot)}\|\chi_{Q_0}\|_{\mu'(\cdot)}\sum_{d\geqslant d_0}2^{-dnm}\overline{\Gamma}(2^{-d-1}).\tag{40}$$

Note that, by condition $\mathfrak{R}$,

$$\sum_{d\geqslant d_0}2^{-dnm}\overline{\Gamma}(2^{-d-1})\lesssim\sum_{d\geqslant d_0}\int_{\delta(1-\varepsilon)2^{-d-1}<|y|\leqslant\delta(1+\varepsilon)2^{-d}}\Gamma(y)\,dy$$
$$\leqslant\int_{|y|\leqslant\delta(1+\varepsilon)2^{-d_0}}\Gamma(y)\left(\sum_{d\geqslant d_0}\chi_{\delta(1-\varepsilon)2^{-d-1}<|y|\leqslant\delta(1+\varepsilon)2^{-d}}(y)\right)dy$$
$$\lesssim\int_{|y|\leqslant\delta(1+\varepsilon)\ell(Q_0)}\Gamma(y)\,dy=\widetilde{\Gamma}[\delta(1+\varepsilon)\ell(Q_0)],$$

since the overlap is finite. Combining this and (40) yields inequality (39). $\square$

LEMMA 13. *Let k be a positive integer and $p(\cdot) \in \mathscr{P}^{log}(\mathbb{R}^n)$ such that $1 < p^- \leqslant p^+ < \infty$. Let $a \in T_\infty$ and $b \in \mathscr{L}_a$ with $\|b\|_{\mathscr{L}_a} \neq 0$. If $H \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$\frac{1}{|dQ|}\int_{dQ} |b(y)-b_Q|^k H(y)\,dy \lesssim a(dQ)^k \|b\|^k_{\mathscr{L}_a} \frac{\|\chi_{dQ}H\|_{p(\cdot)}}{\|\chi_{dQ}\|_{p(\cdot)}}, \tag{41}$$

for every $Q \in \mathscr{Q}$, where $d = 1$ or $d = 3$.

Proof. Suppose $d = 3$, the argument to prove the case $d = 1$ is similar. Let $Q \in \mathscr{Q}$. By Hölder inequality (9) and Lemma 5 we have

$$\frac{1}{|3Q|}\int_{3Q} |b(y)-b_Q|^k H(y)\,dy \lesssim \frac{\left\|\chi_{3Q}|b-b_Q|^k\right\|_{p'(\cdot)}}{\|\chi_{3Q}\|_{p'(\cdot)}} \frac{\|\chi_{3Q}H\|_{p(\cdot)}}{\|\chi_{3Q}\|_{p(\cdot)}}. \tag{42}$$

By Lemmas 6 ans 7, we can estimate the first factor of this product as follows

$$\begin{aligned}\frac{\left\|\chi_{3Q}|b-b_Q|^k\right\|_{p'(\cdot)}}{\|\chi_{3Q}\|_{p'(\cdot)}} &\lesssim \frac{\left\|\chi_{3Q}|b-b_{3Q}|^k\right\|_{p'(\cdot)}}{\|\chi_{3Q}\|_{p'(\cdot)}} + \frac{\left\|\chi_{3Q}|b_{3Q}-b_Q|^k\right\|_{p'(\cdot)}}{\|\chi_{3Q}\|_{p'(\cdot)}} \\ &\lesssim a(3Q)^k \|b\|^k_{\mathscr{L}_a}.\end{aligned}$$

Hence, combining (42) with the previous inequality we deduce (41). □

Proof of Theorem 4. Since $v_i \in L^{p_i(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ implies that the set of bounded functions with compact support is dense in $L^{p_i(\cdot)}_{v_i}(\mathbb{R}^n)$, it is enough to show that

$$\left\|P_{\Gamma,\vec{b}}(f_1,\dots,f_m)\right\|_{L^{r(\cdot)}_w} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i(\cdot)}_{v_i}}$$

for each $f_1,\dots,f_m \geqslant 0$ bounded functions with compact support. This is in turn equivalent by duality to

$$\int_{\mathbb{R}^n} |P_{\Gamma,\vec{b}}(f_1,\dots,f_m)(x)|w(x)g(x)\,dx \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i(\cdot)}_{v_i}}$$

for all non-negative bounded functions with compact support $f_1,\dots,f_m$ and g with $\|g\|_{r'(\cdot)} \leqslant 1$. Let $f_1,\dots,f_m$ and g be functions with these properties. By definition of commutators (see (25)) it is enough to prove that, for every $j = 1,\dots,m$,

$$\int_{\mathbb{R}^n} |P_{\Gamma,b_j}(f_1,\dots,f_m)(x)|w(x)g(x)\,dx \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i(\cdot)}_{v_i}}. \tag{43}$$

For each $t > 0$, we set $\overline{\Gamma}(t) = \sup_{y_1,\dots,y_m \in \mathscr{A}_{t,1,0}} \Gamma(y_1,\dots,y_m)$, where $\mathscr{A}_{(t,\delta,\varepsilon)}$ is the set defined in (26). It was proved in [[3], Proof of Lemma 4.1] that, for $x \in \mathbb{R}^n$, we can discretize the commutator as follows

$$
\begin{aligned}
|P_{\Gamma,b_j}(f_1,\dots,f_m)(x)| \leqslant \sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|b_j(x)-(b_j)_Q|\chi_Q(x)\prod_{i=1}^{m}\int_{3Q} f_i(y_i)\,dy_i \\
+\sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)\chi_Q(x)\prod_{i=1,i\neq j}^{m}\int_{3Q} f_i(y_i)\,dy_i \\
\times\int_{3Q}|b_j(y_j)-(b_j)_Q|f_j(y_j)\,dy_j.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|P_{\Gamma,b_j}(f_1,\dots,f_m)(x)|w(x)g(x)\,dx \\
&\leqslant \sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)\int_Q|b_j(x)-(b_j)_Q|w(x)g(x)\,dx\prod_{i=1}^{m}\int_{3Q} f_i(y_i)\,dy_i \\
&\quad+\sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)\int_Q w(x)g(x)\,dx\prod_{i=1,i\neq j}^{m}\int_{3Q} f_i(y_i)\,dy_i \\
&\quad\times\int_{3Q}|b_j(y_j)-(b_j)_Q|f_j(y_j)\,dy_j, \qquad (44)
\end{aligned}
$$

where $\mathscr{D}$ is the standard dyadic grid. Let us denote $s_i(\cdot)=R_ip_i'(\cdot)$ and $l(\cdot)=Jr(\cdot)$. Since $(p_i')^+<R_i(p_i')^-$ and $r^+<Jr^-$, we have $(s_i')^+=(s_i^-)'<p_i^-$ and $(l')^+=(l^-)'<(r^+)'$. Then we can take constants η_i and θ such that

$$(s_i')^+<\eta_i<p_i^- \quad \text{and} \quad (l')^+<\theta<(r^+)',$$

and $\omega_i(\cdot),\mu(\cdot)$ define by

$$\frac{1}{\omega_i(\cdot)}=\frac{1}{s_i(\cdot)}+\frac{1}{\eta_i} \quad \text{and} \quad \frac{1}{\mu(\cdot)}=\frac{1}{l(\cdot)}+\frac{1}{\theta}. \qquad (45)$$

Observe that $\omega_i(\cdot),\tau(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$ since $s(\cdot),l(\cdot)\in\mathscr{P}^{log}(\mathbb{R}^n)$. Thus, by (44), using Lemma 13 twice with $H=gw$, $p(\cdot)=\mu(\cdot)$, $d=1$ and $H=f_j$, $p(\cdot)=\omega_j(\cdot)$, $d=3$ respectively, we obtain that

$$
\begin{aligned}
&\int_{\mathbb{R}^n}|P_{\Gamma,b_j}(f_1,\dots,f_m)(x)|w(x)g(x)\,dx \\
&\lesssim \|b\|_{\mathscr{L}_a}\sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|a(Q)\frac{\|\chi_Q wg\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}}\prod_{i=1}^{m}\int_{3Q} f_i(y_i)\,dy_i \\
&\quad+\|b\|_{\mathscr{L}_a}\sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)\int_Q w(x)g(x)\,dx\prod_{i=1,i\neq j}^{m}\int_{3Q} f_i(y_i)\,dy_i \\
&\quad\times|Q|a(3Q)\frac{\|\chi_{3Q}f_j\|_{\omega_j(\cdot)}}{\|\chi_{3Q}\|_{\omega_j(\cdot)}} \qquad (46)
\end{aligned}
$$

Notice that, by inequalities (12) and (11), condition $a \in T_\infty$ and Proposition 1 we can estimate (46) as follows

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |P_{\Gamma,b_j}(f_1,\dots,f_m)(x)|w(x)g(x)\,dx \\
&\lesssim \|b\|_{\mathscr{L}_a} \sum_{3Q:Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(3Q)}{2}\right)|Q|^{m+1}a(3Q)\frac{\|\chi_{3Q}wg\|_{\mu(\cdot)}}{\|\chi_{3Q}\|_{\mu(\cdot)}}\prod_{i=1}^{m}\frac{\|\chi_{3Q}f_j\|_{\omega_i(\cdot)}}{\|\chi_{3Q}\|_{\omega_i(\cdot)}} \\
&\lesssim \|b\|_{\mathscr{L}_a} \sum_{t=1}^{2^n}\sum_{Q\in\mathscr{D}_t} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}a(Q)\frac{\|\chi_{Q}wg\|_{\mu(\cdot)}}{\|\chi_{Q}\|_{\mu(\cdot)}}\prod_{i=1}^{m}\frac{\|\chi_{Q}f_i\|_{\omega_i(\cdot)}}{\|\chi_{Q}\|_{\omega_i(\cdot)}}. \qquad (47)
\end{aligned}
$$

Consequently, it is enough to estimate

$$
\|b\|_{\mathscr{L}_a} \sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}a(Q)\frac{\|\chi_{Q}wg\|_{\mu(\cdot)}}{\|\chi_{Q}\|_{\mu(\cdot)}}\prod_{i=1}^{m}\frac{\|\chi_{Q}f_i\|_{\omega_i(\cdot)}}{\|\chi_{Q}\|_{\omega_i(\cdot)}},
$$

for every dyadic grid $\mathscr{D}$.

Let $\mathscr{D}$ be a dyadic grid. The next task is to replace the sum over $\mathscr{D}$, by the sum over cubes from a sparse family. Since $f_1,\dots,f_m$ are bounded functions with compact support, we have that

$$
\lim_{|Q|\to\infty}\prod_{i=1}^{m}\frac{\|\chi_{Q}f_i\|_{\omega_i(\cdot)}}{\|\chi_{Q}\|_{\omega_i(\cdot)}} \lesssim \prod_{i=1}^{m}\|f_i\|_\infty \lim_{|Q|\to\infty}\frac{\|\chi_{\mathrm{supp}\,f_i}\|_{\omega_i(\cdot)}}{\|\chi_{Q}\|_{\omega_i(\cdot)}} = 0.
$$

Let $\sigma > 0$ the constant provided by the Proposition 2 for $\omega_1(\cdot),\dots,\omega_m(\cdot)$, $f_1,\dots,f_m$ and $\mathscr{D}$. If $\alpha > \max\{\sigma,\kappa\}$, where κ is the constant involved in the inequality (32), there exist a sparse family $\{Q_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}} \subset \mathscr{D}$ that satisfies

$$
\alpha^k < \prod_{i=1}^{m}\frac{\left\|\chi_{Q_j^k}f_i\right\|_{\omega_i(\cdot)}}{\left\|\chi_{Q_j^k}\right\|_{\omega_i(\cdot)}} \leqslant \kappa\alpha^k < \alpha^{k+1}. \qquad (48)
$$

For $k\in\mathbb{Z}$ we define the set

$$
\mathscr{C}_k = \left\{Q\in\mathscr{D} : \alpha^k < \prod_{i=1}^{m}\frac{\|\chi_{Q}f_i\|_{\omega_i(\cdot)}}{\|\chi_{Q}\|_{\omega_i(\cdot)}} \leqslant \alpha^{k+1}\right\}.
$$

Then every cube $Q\in\mathscr{D}$ for wich

$$
\prod_{i=1}^{m}\frac{\|\chi_{Q}f_i\|_{\omega_i(\cdot)}}{\|\chi_{Q}\|_{\omega_i(\cdot)}} \neq 0
$$

belongs to exactly one $\mathscr{C}_k$. Furthermore, if $Q\in\mathscr{C}_k$, it follows that $Q\subset Q_j^k$ for some

$j \in \mathbb{N}$. Then we obtain that

$$
\begin{aligned}
&\|b_j\|_{\mathscr{L}_a} \sum_{Q\in\mathscr{D}} a(Q)\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}\frac{\|\chi_Q wg\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}}\prod_{i=1}^m \frac{\|\chi_Q f_i\|_{\omega_i(\cdot)}}{\|\chi_Q\|_{\omega_i(\cdot)}}\\
&\lesssim \|b_j\|_{\mathscr{L}_a}\sum_k \sum_{Q\in\mathscr{C}_k} a(Q)\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}\frac{\|\chi_Q gw\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}}\prod_{i=1}^m \frac{\|\chi_Q f_i\|_{\omega_i(\cdot)}}{\|\chi_Q\|_{\omega_i(\cdot)}}\\
&\lesssim \|b_j\|_{\mathscr{L}_a}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\alpha^{k+1}\sum_{Q\in\mathscr{C}_k:Q\subset Q_j^k} a(Q)\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}\frac{\|\chi_Q gw\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}}\\
&\lesssim \|b_j\|_{\mathscr{L}_a}\alpha\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\prod_{i=1}^m \frac{\left\|\chi_{Q_j^k} f_i\right\|_{\omega_i(\cdot)}}{\left\|\chi_{Q_j^k}\right\|_{\omega_i(\cdot)}} a(Q_j^k)\\
&\quad\times\sum_{Q\in\mathscr{C}_k:Q\subset Q_j^k}\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}\frac{\|\chi_Q gw\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}}\\
&\lesssim \|b_j\|_{\mathscr{L}_a}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} a(Q_j^k)\widetilde{\Gamma}[\delta(1+\varepsilon)\ell(Q_j^k)]\,|Q_j^k|\frac{\left\|\chi_{Q_j^k} gw\right\|_{\mu(\cdot)}}{\left\|\chi_{Q_j^k}\right\|_{\mu(\cdot)}}\prod_{i=1}^m \frac{\left\|\chi_{Q_j^k} f_i\right\|_{\omega_i(\cdot)}}{\left\|\chi_{Q_j^k}\right\|_{\omega_i(\cdot)}}
\end{aligned}
$$

where ε,δ are the constants provided by condition $\mathfrak{R}$, $\widetilde{\Gamma}(t)=\int_{|z|\leqslant t}\Gamma(z)\,dz$ and we have used Lemma 12. Let $\gamma=\delta(1+\varepsilon)$, then by monotony, using that $a\in T_\infty$ and inequality (11) we can follow our chain of inequalities with

$$
\lesssim \|b_j\|_{\mathscr{L}_a}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} a(\gamma Q_j^k)\widetilde{\Gamma}(\gamma\ell(Q_j^k))\,|\gamma Q_j^k|\frac{\left\|\chi_{\gamma Q_j^k} gw\right\|_{\mu(\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\mu(\cdot)}}\prod_{i=1}^m \frac{\left\|\chi_{\gamma Q_j^k} f_i\right\|_{\omega_i(\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\omega_i(\cdot)}}.
$$

Recalling the definition of the exponents (see (45)), Hölder's inequality (8), Corollary 1 and the hypothesis on the weights we obtain

$$
\begin{aligned}
&\|b_j\|_{\mathscr{L}_a}\sum_{Q\in\mathscr{D}} a(Q)\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}\frac{\|\chi_Q wg\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}}\prod_{i=1}^m \frac{\|\chi_Q f_i\|_{\omega_i(\cdot)}}{\|\chi_Q\|_{\omega_i(\cdot)}}\\
&\lesssim \|b_j\|_{\mathscr{L}_a}\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}} a(\gamma Q_j^k)\widetilde{\Gamma}(\gamma\ell(Q_j^k))\,|\gamma Q_j^k|\frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{\theta}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\theta}}\frac{\left\|\chi_{\gamma Q_j^k} w\right\|_{l(\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{l(\cdot)}}\\
&\qquad\times\prod_{i=1}^m \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{\eta_i}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\eta_i}}\frac{\left\|\chi_{\gamma Q_j^k} v_i^{-1}\right\|_{s_i(\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{s_i(\cdot)}}
\end{aligned}
$$

$$\lesssim \|b_j\|_{\mathscr{L}_a} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |Q_j^k| \frac{\left\|\chi_{\gamma Q_j^k}\right\|_{p(\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r(\cdot)}} \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{\theta}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\theta}} \prod_{i=1}^{m} \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{\eta_i}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\eta_i}}.$$

Let $\beta(\cdot)$ defined as in Corollary 1. Then, by this corollary, the last sum is equivalent to

$$\|b_j\|_{\mathscr{L}_a} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |Q_j^k| \left\|\chi_{\gamma Q_j^k}\right\|_{\beta(\cdot)} \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{\theta}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\theta}} \prod_{i=1}^{m} \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{\eta_i}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\eta_i}}.$$

Using that $\{Q_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ is a sparse family and Hölder inequality (9) we obtain that

$$\begin{aligned}
&\|b_j\|_{\mathscr{L}_a} \sum_{Q\in\mathscr{D}} a(Q)\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right) |Q|^{m+1} \frac{\|\chi_Q w g\|_{\mu(\cdot)}}{\|\chi_Q\|_{\mu(\cdot)}} \prod_{i=1}^{m} \frac{\|\chi_Q f_i\|_{\omega_i(\cdot)}}{\|\chi_Q\|_{\omega_i(\cdot)}} \\
&\lesssim \|b_j\|_{\mathscr{L}_a} \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |E(Q_j^k)| \left\|\chi_{\gamma Q_j^k}\right\|_{\beta(\cdot)} \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{\theta}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\theta}} \prod_{i=1}^{m} \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{\eta_i}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\eta_i}} \\
&\lesssim \|b_j\|_{\mathscr{L}_a} \int_{\mathbb{R}^n} M_{\beta(\cdot),\theta}(g)(y) \prod_{i=1}^{m} M_{\eta_i}(f_i v_i)(y)\, dy \\
&\lesssim \|b_j\|_{\mathscr{L}_a} \left\|M_{\beta(\cdot),\theta}(g)\right\|_{p'(\cdot)} \prod_{i=1}^{m} \left\|M_{\eta_i}(f_i v_i)\right\|_{p_i(\cdot)} \\
&\lesssim \|b_j\|_{\mathscr{L}_a} \prod_{i=1}^{m} \|f_i\|_{L^{p_i(\cdot)}_{v_i}},
\end{aligned}$$

where we have used that by Theorem 1,

$$M_{\eta_i} : L^{p_i(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_i(\cdot)}(\mathbb{R}^n)$$

since $p_i^- > \eta_i$, and

$$M_{\beta(\cdot),\theta} : L^{r'(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p'(\cdot)}(\mathbb{R}^n)$$

since $(r')^- > \theta$. This proves (43) and concludes the proof of Theorem 4. □

Proof of Proposition 2. To prove (1) we may assume $E_\lambda \neq \emptyset$ since otherwise there is nothing to prove. Let Λ_λ be the family of dyadic cubes such that

$$\Lambda_\lambda = \left\{ Q \in \mathscr{D} : \prod_{i=1}^{m} \frac{\|\chi_Q f_i\|_{s_i(\cdot)}}{\|\chi_Q\|_{s_i(\cdot)}} > \lambda \right\};$$

this is non-empty since $E_\lambda \neq \emptyset$. For each $Q \in \Lambda_\lambda$ there exists a maximal cube $Q' \in \Lambda_\lambda$ with $Q \subset Q'$, since (30). Let $\{Q_j\}_{j\in\mathbb{N}} \subset \Lambda_\lambda$ denote the family of such maximal cubes;

clearly they are pairwise disjoint. Also, let $\widehat{Q_j} \in \mathscr{D}$ such that $Q_j \subset \widehat{Q_j}$ and $\ell(\widehat{Q_j}) = 2\ell(Q_j)$, then $\widehat{Q_j} \subset 4Q_j$. By maximality and Lemma (10), we have that

$$\lambda < \prod_{i=1}^{m} \frac{\|\chi_{Q_j} f_i\|_{s_i(\cdot)}}{\|\chi_{Q_j}\|_{s_i(\cdot)}} \leqslant \prod_{i=1}^{m} \frac{\left\|\chi_{\widehat{Q_j}}\right\|_{s_i(\cdot)}}{\|\chi_{Q_j}\|_{s_i(\cdot)}} \frac{\left\|\chi_{\widehat{Q_j}} f_i\right\|_{s_i(\cdot)}}{\left\|\chi_{\widehat{Q_j}}\right\|_{s_i(\cdot)}}$$
$$\leqslant C_s^{2m} \prod_{i=1}^{m} \frac{\left\|\chi_{\widehat{Q_j}} f_i\right\|_{s_i(\cdot)}}{\left\|\chi_{\widehat{Q_j}}\right\|_{s_i(\cdot)}} \leqslant C_s^{2m} \lambda.$$

If $x \in E_\lambda$, there exists a cube $Q \in \mathscr{D}$ such that $Q \ni x$ and

$$\prod_{i=1}^{m} \frac{\|\chi_Q f_i\|_{s_i(\cdot)}}{\|\chi_Q\|_{s_i(\cdot)}} > \lambda.$$

Hence, $Q \subseteq Q_j$ for some $j \in \mathbb{N}$. Conversely, since $x \in Q_j$ for some $j \in \mathbb{N}$, by property (32),

$$\prod_{i=1}^{m} \frac{\|\chi_{Q_j} f_i\|_{s_i(\cdot)}}{\|\chi_{Q_j}\|_{s_i(\cdot)}} > \lambda.$$

Then $\mathscr{M}^{\mathscr{D}}_{\vec{s}(\cdot)} f_1, \dots, f_m(x) > \lambda$, that imply $x \in E_\lambda$.

To prove (2), let $\alpha > 1$ be a constant that will be chosen later. For each non negative $k \in \mathbb{Z}$, we consider the set

$$\Omega_k = \left\{x \in \mathbb{R}^n : \mathscr{M}^{\mathscr{D}}_{\vec{s}(\cdot)} f_1, \dots, f_m(x) > \alpha^k\right\} = \bigcup_j Q_j^k \tag{49}$$

where $\{Q_j^k\}_{j\in\mathbb{N}}$ is the collection of maximal dyadic cubes from (1) that satisfies

$$\alpha^k < \prod_{i=1}^{m} \frac{\left\|\chi_{Q_j^k} f_i\right\|_{s_i(\cdot)}}{\left\|\chi_{Q_j^k}\right\|_{s_i(\cdot)}} \leqslant C_s^{2m} \alpha^k. \tag{50}$$

Let $F_j^k = Q_j^k \setminus \Omega_{k+1}$. Since $\Omega_{k+1} \subset \Omega_k$ it is immediate that the sets F_j^k are pairwise disjoint. Note that

$$\frac{|F_j^k|}{|Q_j^k|} = \frac{|Q_j^k \setminus (Q_j^k \cap \Omega_{k+1})|}{|Q_j^k|} = 1 - \frac{|Q_j^k \cap \Omega_{k+1}|}{|Q_j^k|}. \tag{51}$$

We estimate $|Q_j^k \cap \Omega_{k+1}|$. If A denotes one of the constants involved in (5), using that

$$1 = \frac{1}{s'(\cdot)} + \frac{1}{s(\cdot)} = \frac{1}{s'(\cdot)} + \sum_{i=1}^{m} \frac{1}{s_i(\cdot)},$$

we obtain

$$\begin{aligned}\left|Q_j^k \cap \Omega_{k+1}\right| &= \sum_{l:Q_l^{k+1}\subseteq Q_j^k} \left|Q_l^{k+1}\right| \\ &\leqslant A \sum_{l:Q_l^{k+1}\subseteq Q_j^k} \left\|\chi_{Q_l^{k+1}}\right\|_{s'(\cdot)} \prod_{i=1}^m \left\|\chi_{Q_l^{k+1}}\right\|_{s_i(\cdot)}. \end{aligned} \tag{52}$$

Notice that

$$\alpha^{k+1} < \prod_{i=1}^m \frac{\left\|\chi_{Q_l^{k+1}} f_i\right\|_{s_i(\cdot)}}{\left\|\chi_{Q_l^{k+1}}\right\|_{s_i(\cdot)}} \quad \text{y} \quad \prod_{i=1}^m \frac{\left\|\chi_{Q_j^k} f_i\right\|_{s_i(\cdot)}}{\left\|\chi_{Q_j^k}\right\|_{s_i(\cdot)}} \leqslant C_s^{2m}\alpha^k. \tag{53}$$

Hence, by (53) and Lemma 11, we can estimate (52) as follow

$$\begin{aligned}\left|Q_j^k \cap \Omega_{k+1}\right| &< A\alpha^{-k-1} \sum_{l:Q_l^{k+1}\subseteq Q_j^k} \left\|\chi_{Q_l^{k+1}}\right\|_{s'(\cdot)} \prod_{i=1}^m \left\|\chi_{Q_l^{k+1}} f_i\right\|_{s_i(\cdot)} \\ &\leqslant A\alpha^{-k-1} C \left\|\chi_{Q_j^k}\right\|_{s'(\cdot)} \prod_{i=1}^m \left\|\chi_{Q_j^k} f_i\right\|_{s_i(\cdot)} \\ &\leqslant AC\alpha^{-1} C_s^{2m} \left\|\chi_{Q_j^k}\right\|_{s'(\cdot)} \prod_{i=1}^m \left\|\chi_{Q_j^k}\right\|_{s_i(\cdot)} \\ &\leqslant AC\alpha^{-1} C_s^{2m} B |Q_j^k| \\ &= \sigma\alpha^{-1}|Q_j^k|. \end{aligned}$$

Consequently, if $\alpha > \sigma$, by (51), we can conclude

$$\frac{|F_j^k|}{|Q_j^k|} > 1 - \frac{\sigma}{\alpha} > 0.$$

Hence $\mathscr{S} = \{Q_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ is a sparse family. □

Proof of Theorem 5. As in the proof of Theorem 4 it is enough to prove that, for every $j = 1,\dots,m$,

$$\int_{\mathbb{R}^n} |P_{\Gamma,b_j}(f_1,\dots,f_m)(x)| w(x) g(x)\, dx \lesssim \prod_{i=1}^m \|f_i\|_{L_{v_i}^{p_i(\cdot)}} \tag{54}$$

for all non-negative bounded functions with compact support $f_1,\dots,f_m$ and g with $\|g\|_{r'(\cdot)} \leqslant 1$. Let $f_1,\dots,f_m$ and g be functions with these properties. We use the same technique as in the proof of the Theorem 4 to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |P_{\Gamma,b_j}(f_1,\ldots,f_m)(x)|w(x)g(x)\,dx \\
&\quad \leqslant \sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)\int_Q |b_j(x)-(b_j)_Q|w(x)g(x)\,dx\prod_{i=1}^m\int_{3Q} f_i(y_i)\,dy_i \\
&\quad\quad + \sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)\int_Q w(x)g(x)\,dx\prod_{i=1,i\neq j}^m\int_{3Q} f_i(y_i)\,dy_i \\
&\quad\quad \times \int_{3Q}|b_j(y_j)-(b_j)_Q|f_j(y_j)\,dy_j,
\end{aligned} \tag{55}$$

where $\mathscr{D}$ is the standard dyadic grid. Hence, by Lemma 8,

$$\begin{aligned}
&\int_{\mathbb{R}^n} |P_{\Gamma,b_j}(f_1,\ldots,f_m)(x)|w(x)g(x)\,dx \\
&\quad \lesssim \sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)\|\chi_Q\|_{n/\delta(\cdot)}|Q|^{m+1}\frac{1}{|Q|}\int_Q w(x)g(x)\,dx\prod_{i=1}^m\frac{1}{|3Q|}\int_{3Q} f_i(y_i)\,dy_i.
\end{aligned} \tag{56}$$

The next task is to replace the sum over $\mathscr{D}$, by the sume over cubes from a sparse family. Since $f_1,\ldots,f_m$ are bounded functions with compact support, we have that

$$\lim_{|Q|\to\infty}\prod_{i=1}^m\frac{1}{|3Q|}\int_{3Q} f_i(y_i)\,dy_i \lesssim \prod_{i=1}^m\|f_i\|_\infty\lim_{|Q|\to\infty}\frac{|\operatorname{supp} f_i|}{|Q|}=0.$$

Let $\alpha>\max\{2^{nm},6^n\|\mathscr{M}\|\}$ where $\|\mathscr{M}\|$ is the constant from the $L^1\times\ldots\times L^1\to L^{1/m,\infty}$ inequality for $\mathscr{M}$. It was proved in [28] that there exists a sparse family $\{Q_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}}\subset\mathscr{D}$, such that for every $k\in\mathbb{Z}$,

$$\alpha^k<\prod_{i=1}^m\frac{1}{|3Q_j^k|}\int_{3Q_j^k} f_i(y_i)\,dy_i\leqslant\alpha^{k+1}.$$

Hence, proceeding as in the proof of Theorem 4 we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |P_{\Gamma,b_j}(f_1,\ldots,f_m)(x)|w(x)g(x)\,dx \\
&\quad \lesssim \sum_{Q\in\mathscr{D}} \overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)\|\chi_Q\|_{n/\delta(\cdot)}|Q|^{m+1}\frac{1}{|Q|}\int_Q w(x)g(x)\,dx\prod_{i=1}^m\frac{1}{|3Q|}\int_{3Q} f_i(y_i)\,dy_i \\
&\quad \lesssim \alpha\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\left\|\chi_{Q_j^k}\right\|_{n/\delta(\cdot)}\prod_{i=1}^m\frac{1}{|3Q_j^k|}\int_{3Q_j^k} f_i(y_i)\,dy_i \\
&\quad\quad \times\sum_{Q\in\mathscr{C}_k:Q\subset Q_j^k}\overline{\Gamma}\left(\frac{\ell(Q)}{2}\right)|Q|^{m+1}\frac{1}{|Q|}\int_Q w(x)g(x)\,dx
\end{aligned}$$

$$\lesssim \alpha \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \left\|\chi_{Q_j^k}\right\|_{n/\delta(\cdot)} \widetilde{\Gamma}[\delta(1+\varepsilon)\ell(Q_j^k)]\,|Q_j^k|$$
$$\times \frac{1}{|Q_j^k|}\int_{Q_j^k} w(x)g(x)\,dx \prod_{i=1}^m \frac{1}{|3Q_j^k|}\int_{3Q_j^k} f_i(y_i)\,dy_i$$

where ε,δ are the constants provided by condition $\mathfrak{R}$, $\widetilde{\Gamma}(t)=\int_{|z|\leqslant t}\Gamma(z)\,dz$ and we have used Lemma 12. Let $\gamma=\delta(1+\varepsilon)$; then we can follow our chain of inequalities with

$$\lesssim \alpha \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \left\|\chi_{\gamma Q_j^k}\right\|_{n/\delta(\cdot)} \widetilde{\Gamma}(\gamma\ell(Q_j^k))\,|\gamma Q_j^k| \frac{1}{|\gamma Q_j^k|}\int_{\gamma Q_j^k} w(x)g(x)\,dx \prod_{i=1}^m \frac{1}{|\gamma Q_j^k|}\int_{\gamma Q_j^k} f_i(y_i)\,dy_i.$$

By condition $\mathscr{AV}$ and by the hypothesis on the weights (35) we obtain

$$\int_{\mathbb{R}^n} |P_{\Gamma,b_j}(f_1,\dots,f_m)(x)|w(x)g(x)\,dx$$
$$\lesssim \alpha \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \left\|\chi_{\gamma Q_j^k}\right\|_{n/\delta(\cdot)} \widetilde{\Gamma}(\gamma\ell(Q_j^k))\,|\gamma Q_j^k| \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_{m+1}(\cdot,\cdot)}} \frac{\left\|\chi_{\gamma Q_j^k} w\right\|_{\Upsilon_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Upsilon_{m+1}(\cdot,\cdot)}}$$
$$\times \prod_{i=1}^m \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_i(\cdot,\cdot)}} \frac{\left\|\chi_{\gamma Q_j^k} v_i^{-1}\right\|_{\Upsilon_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Upsilon_i(\cdot,\cdot)}}$$
$$\lesssim \alpha \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |Q_j^k| \frac{\left\|\chi_{\gamma Q_j^k}\right\|_{p(\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{r(\cdot)}} \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_{m+1}(\cdot,\cdot)}} \prod_{i=1}^m \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_i(\cdot,\cdot)}}$$

By Corollary 1 the last sum is equivalent to

$$\alpha \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |Q_j^k| \left\|\chi_{\gamma Q_j^k}\right\|_{\beta(\cdot)} \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_{m+1}(\cdot,\cdot)}} \prod_{i=1}^m \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_i(\cdot,\cdot)}}.$$

Using that $\{Q_j^k\}_{j\in\mathbb{N},k\in\mathbb{Z}}$ is a sparse family and Hölder inequality (9) we obtain that

$$\int_{\mathbb{R}^n} |P_{\Gamma,b_j}(f_1,\dots,f_m)(x)|w(x)g(x)\,dx$$
$$\lesssim \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} |E(Q_j^k)| \left\|\chi_{\gamma Q_j^k}\right\|_{\beta(\cdot)} \frac{\left\|\chi_{\gamma Q_j^k} g\right\|_{\Psi_{m+1}(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_{m+1}(\cdot,\cdot)}} \prod_{i=1}^m \frac{\left\|\chi_{\gamma Q_j^k} f_i v_i\right\|_{\Psi_i(\cdot,\cdot)}}{\left\|\chi_{\gamma Q_j^k}\right\|_{\Psi_i(\cdot,\cdot)}}$$
$$\lesssim \int_{\mathbb{R}^n} M_{\beta(\cdot),\Psi_{m+1}(\cdot,\cdot)}(g)(y)\mathscr{M}_{\Psi_1(\cdot,\cdot),\dots,\Psi_m(\cdot,\cdot)}(f_1v_1,\dots,f_mv_m)(y)\,dy$$

$$\lesssim \left\|M_{\beta(\cdot),\Psi_{m+1}(\cdot,\cdot)}(g)\right\|_{p'(\cdot)} \left\|\mathscr{M}_{\Psi_1(\cdot,\cdot),\ldots,\Psi_m(\cdot,\cdot)}(f_1v_1,\ldots,f_mv_m)\right\|_{p(\cdot)}$$
$$\lesssim \prod_{i=1}^{m} \|f_i\|_{L^{p_i(\cdot)}_{v_i}},$$

where we have used conditions (33) and (34). This proves (54) and concludes the proof of Theorem 5. □

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