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BOUNDS FOR COMPLETELY MONOTONIC DEGREES OF REMAINDERS IN ASYMPTOTIC EXPANSIONS OF THE DIGAMMA FUNCTION

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Dedicated to Dr. Prof. Pietro Cerone retired at La Trobe University and Victoria University in Australia

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Abstract. Motivated by several conjectures posed in the paper "F. Qi and A.-Q. Liu, *Completely monotonic degrees for a difference between the logarithmic and psi functions*, J. Comput. Appl. Math. **361** (2019), 366–371; https://doi.org/10.1016/j.cam.2019.05.001", the authors bound several completely monotonic degrees of the remainders in the asymptotic expansions of the logarithm of the gamma function and in the asymptotic expansions of the logarithm of the digamma function.

1. Preliminaries

The classical Euler gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z)=\int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t,\quad \Re(z)>0.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, is called the digamma function and the derivatives $\psi^{(i)}(x)$ for $i\geqslant 0$ are called the polygamma functions. The digamma function $\psi(z)$ has the series expansion

$$\psi(1+z)=-\gamma+\sum_{n=1}^{\infty}\frac{z}{n(n+z)},\quad z\neq-1,-1,-3,\dots$$

in [1, p. 259, 6.3.16] and has the asymptotic formula

$$\psi(z)\sim\ln z-\frac{1}{2z}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2n}\frac{1}{z^{2n}},\quad z\to\infty \text{ in } |\arg z|<\pi$$

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in [1, p. 259, 6.3.18], were $\gamma = 0.57721566\ldots$ stands for Euler–Mascheroni's constant and the Bernoulli numbers B_{2n} are generated [23, 27, 33] by

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi. \tag{1.1}$$

For more information on $\Gamma(z)$ and $\psi^{(i)}(x)$, refer to [1, 21, 25, 40] and closely related references therein.

Recall from [32, Chapter 1] that, if a function h on an interval I has derivatives of all orders on I and

$$(-1)^n h^{(n)}(t) \geqslant 0, \quad t \in I, \quad n \in \{0\} \cup \mathbb{N},$$

then we call h a completely monotonic function on I. In other words, a function h is completely monotonic on an interval I if its odd derivatives are negative and its even derivatives are positive on I. Theorem 12b in [35, p. 161] states that a necessary and sufficient condition for h to be completely monotonic on the infinite interval $(0,\infty)$ is that

$$h(t) = \int_0^{\infty} e^{-ts} \,d\sigma(s), \quad s \in (0,\infty), \tag{1.2}$$

where $\sigma(s)$ is non-decreasing and the above integral converges for $s \in (0,\infty)$. In other words, a function is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform of a non-negative measure.

DEFINITION 1.1. Let $h(t)$ be a completely monotonic function on $(0,\infty)$ and let

$$h(\infty) = \lim_{t\to\infty} h(t) \geqslant 0.$$

If the function

$$t^{\alpha}[h(t) - h(\infty)] \tag{1.3}$$

is completely monotonic on $(0,\infty)$ if and only if $0 \leqslant \alpha \leqslant r \in \mathbb{R}$, then we say that $h(t)$ is of completely monotonic degree r; if the function in (1.3) is completely monotonic on $(0,\infty)$ for all $\alpha \in \mathbb{R}$, then we say that the completely monotonic degree of $h(t)$ is ∞.

The function in (1.3) can be essentially regarded as the ratio $\frac{h(t)-h(\infty)}{(1/t)^{\alpha}}$ between the completely monotonic function $h(t) - h(\infty)$ and the α power of the completely monotonic function $\frac{1}{t}$ on $(0,\infty)$. This is the reason why we designed in [8] a notation $\deg_{\mathrm{cm}}^{t}[h(t)]$ to denote the completely monotonic degree r of $h(t)$ with respect to $t \in (0,\infty)$. According to this idea, we can define the completely monotonic degree r of $h(t)$ with respect to $g(t)$, denoted by $\deg_{\mathrm{cm}}^{g(t)}[h(t)]$, as the largest number α such that the ratio $\frac{h(t)-h(\infty)}{[g(t)-g(\infty)]^{\alpha}}$ between the completely monotonic function $h(t)$ and the α power of the completely monotonic function $g(t)$ on $(0,\infty)$.

In [41, Definition 1.2], the integer part of the completely monotonic degree r of $h(t)$ with respect to $t \in (0,\infty)$, that is, the quantity $\lfloor \deg_{\mathrm{cm}}^{t}[h(t)] \rfloor$, was needlessly and unnecessarily called as the completely monotonic integer degree of the function $h(t)$, where the notation $\lfloor t\rfloor$ denotes the floor function whose value equals the largest integer less than or equal to t. Proposition 1.2 in [28] can be modified as that $\deg_{\mathrm{cm}}^{t}[h(t)]=r>0$ if and only if

$$h(t)-h(\infty)=\int_0^\infty\left[\frac{1}{\Gamma(r)}\int_0^s(s-\tau)^{r-1}\,\mathrm{d}\mu(\tau)\right]\mathrm{e}^{-ts}\,\mathrm{d}s$$

for $0<t<\infty$, where $\mu(\tau)$ is a bounded and non-decreasing measure on $(0,\infty)$. For more information on completely monotonic degrees and their properties, please refer to the papers [13, 15, 16, 17, 18, 24, 26] and closely related references therein.

2. Motivations and main results

In [2, pp. 374–375, Theorem 1] and [11, Theorem 1], the function

$$t^\alpha[\ln t-\psi(t)],\quad \alpha\in\mathbb{R}$$

was proved to be completely monotonic on $(0,\infty)$ if and only if $\alpha\leqslant 1$. This means that completely monotonic degree of $\ln t-\psi(t)$ on $(0,\infty)$ is

$$\deg_{\mathrm{cm}}^{t}[\ln t-\psi(t)]=1. \tag{2.1}$$

In [31, Theorem 1.7], the function

$$t^2[\psi(t)-\ln t]+\frac{t}{2}$$

was proved to be decreasing and convex on $(0,\infty)$ and, as $t\to\infty$, to tend to $-\frac{1}{12}$. In [5, Theorem 1], the function

$$t^2[\psi(t)-\ln t]+\frac{t}{2}+\frac{1}{12}$$

was verified to be completely monotonic on $(0,\infty)$. In [29, Theorem 2], the completely monotonic degree of

$$\phi(t)=\psi(t)-\ln t+\frac{1}{2t}+\frac{1}{12t^2}$$

with respect to $t\in(0,\infty)$ was proved to be

$$\deg_{\mathrm{cm}}^{t}[\phi(t)]=2. \tag{2.2}$$

In [2, Theorem 8], [14, Theorem 2], and [39], the functions

$$R_n(t)=(-1)^n\left[\ln\Gamma(t)-\left(t-\frac{1}{2}\right)\ln t+t-\frac{1}{2}\ln(2\pi)-\sum_{k=1}^{n}\frac{B_{2k}}{(2k)(2k-1)}\frac{1}{t^{2k-1}}\right]$$

for $n\geqslant 0$ were proved to be completely monotonic on $(0,\infty)$, where an empty sum is understood to be 0. This conclusion implies that the functions $(-1)^m R_n^{(m)}(t)$ for

$m,n \geqslant 0$ are completely monotonic on $(0,\infty)$. See also [12, Section 1.4] and [20, Theorem 3.1]. By the way, we call the function $(-1)^n R_n(t)$ for $n \geqslant 0$ remainders of the asymptotic formula for $\ln\Gamma(t)$. See [1, p. 257, 6.1.40] and [21, p. 140, 5.11.1].

In [2, Theorem 1], [11, Theorem 1], and [37, Theorem 3], completely monotonic degree $\deg^t_{\mathrm{cm}}[-R_0'(t)] = 1$ was verified once again.

In [18, Theorem 2.1], it was proved that

$$\deg^t_{\mathrm{cm}}\left[R_n(t)\right] \geqslant n, \quad n \geqslant 0. \tag{2.3}$$

In [5, Theorem 1], [29, Theorem 2], and [37, Theorem 4], completely monotonic degree $\deg^t_{\mathrm{cm}}[-R_1'(t)] = 2$ was proved once again.

In [37, Theorems 1 and 2], it was shown that

$$\deg^t_{\mathrm{cm}}\left[(-1)^2 R_0''(t)\right] = 2$$

and

$$\deg^t_{\mathrm{cm}}\left[(-1)^2 R_1''(t)\right] = 3.$$

In [22], Qi proved that

$$4 \leqslant \deg^t_{\mathrm{cm}}\left[(-1)^2 R_2''(t)\right] \leqslant 5.$$

Due to the above results, we modify Qi's conjectures posed in [29] as follows:

1. the completely monotonic degrees of $R_n(t)$ for $n \geqslant 0$ with respect to $t \in (0,\infty)$ satisfy
$$\deg^t_{\mathrm{cm}}[R_0(t)] = 0, \quad \deg^t_{\mathrm{cm}}[R_1(t)] = 1, \tag{2.4}$$
and
$$\deg^t_{\mathrm{cm}}[R_n(t)] = 2(n-1), \quad n \geqslant 2; \tag{2.5}$$
2. the completely monotonic degrees of $-R_n'(t)$ for $n \geqslant 0$ with respect to $t \in (0,\infty)$ satisfy
$$\deg^t_{\mathrm{cm}}[-R_0'(t)] = 1, \quad \deg^t_{\mathrm{cm}}[-R_1'(t)] = 2, \tag{2.6}$$
and
$$\deg^t_{\mathrm{cm}}[-R_n'(t)] = 2n-1, \quad n \geqslant 2; \tag{2.7}$$
3. the completely monotonic degrees of $(-1)^m R_n^{(m)}(t)$ for $m \geqslant 2$ and $n \geqslant 0$ with respect to $t \in (0,\infty)$ satisfy
$$\deg^t_{\mathrm{cm}}\left[(-1)^m R_0^{(m)}(t)\right] = m, \quad \deg^t_{\mathrm{cm}}\left[(-1)^m R_1^{(m)}(t)\right] = m+1, \tag{2.8}$$
and
$$\deg^t_{\mathrm{cm}}\left[(-1)^m R_n^{(m)}(t)\right] = m+2(n-1), \quad n \geqslant 2. \tag{2.9}$$

In this paper, we will confirm that Qi's conjectures expressed in (2.4) and (2.6) are true and, via a double inequality, partially confirm that the conjecture expressed in (2.7) is almost true. Our main results can be stated as the following theorem.

THEOREM 2.1. *For* $n \geqslant 0$, *the completely monotonic degrees of the remainder* $R_n(t)$ *with respect to* $t \in (0,\infty)$ *satisfy the two equalities in* (2.4).

For $n \geqslant 0$, *the completely monotonic degrees of the functions*

$$-R_n'(t) = (-1)^{n+1}\left[\psi(t) - \ln t + \frac{1}{2t} + \sum_{k=1}^{n} \frac{B_{2k}}{2k}\frac{1}{t^{2k}}\right]$$

with respect to $t \in (0,\infty)$ *satisfy the two equalities in* (2.6) *and a double inequality*

$$2n-1 \leqslant \deg_{\mathrm{cm}}^{t}[-R_n'(t)] < 2n, \quad n \geqslant 2. \tag{2.10}$$

3. Proof of Theorem 2.1

Now we start out to prove our main results stated in Theorem 2.1.

3.1. Proofs of equalities in (2.4)

From Binet's first formula

$$\ln\Gamma(t) = \left(t-\frac{1}{2}\right)\ln t - t + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{1}{e^u-1} - \frac{1}{u} + \frac{1}{2}\right)\frac{e^{-tu}}{u}\,\mathrm{d}u$$

for $t > 0$ in [3, p. 28, Theorem 1.6.3] and [19, p. 11], it is easy to see that

$$R_0(t) = \ln\Gamma(t) - \left(t-\frac{1}{2}\right)\ln t + t - \frac{1}{2}\ln(2\pi)$$

and

$$R_1(t) = -\left[\ln\Gamma(t) - \left(t-\frac{1}{2}\right)\ln t + t - \frac{1}{2}\ln(2\pi) - \frac{1}{12t}\right]$$

satisfy

$$\lim_{t\to\infty} R_0(t) = \lim_{t\to\infty} R_1(t) = 0.$$

The inequality (2.3) means that the completely monotonic degree of the remainder $R_n(t)$ for $n \geqslant 0$ with respect to $t \in (0,\infty)$ is at least n. In particular, we have

$$\deg^t[R_0(t)] \geqslant 0 \quad \text{and} \quad \deg^t[R_1(t)] \geqslant 1. \tag{3.1}$$

If $t^\theta R_0(t)$ and $t^\lambda R_1(t)$ were completely monotonic on $(0,\infty)$, then their first derivatives should be non-positive. As a result, using

$$R_0'(t) = \psi(t) - \ln t + \frac{1}{2t}$$

and

$$-R_1'(t) = \psi(t) - \ln t + \frac{1}{2t} + \frac{1}{12t^2},$$

we acquire

$$\theta \leqslant -\frac{tR_0'(t)}{R_0(t)} = \frac{t\left[\ln t - \psi(t)\right] - \frac{1}{2}}{\ln\Gamma(t) - \left(t - \frac{1}{2}\right)\ln t + t - \frac{1}{2}\ln(2\pi)} \to 0, \quad t \to 0^+$$

and

$$\lambda \leqslant -\frac{tR_1'(t)}{R_1(t)} = -\frac{t^2\left[\psi(t) - \ln t + \frac{1}{2t} + \frac{1}{12t^2}\right]}{t\left[\ln\Gamma(t) - \left(t - \frac{1}{2}\right)\ln t + t - \frac{1}{2}\ln(2\pi) - \frac{1}{12t}\right]} \to 1, \quad t \to 0^+,$$

where we used the limits

$$\lim_{t\to 0^+}(t[\ln t - \psi(t)]) = 1, \quad \text{(see [2, p. 374])}$$

$$\lim_{t\to 0^+}\left(t^2\left[\psi(t) - \ln t + \frac{1}{2t} + \frac{1}{12t^2}\right]\right) = \frac{1}{12}, \quad \text{(see [29, Theorem 1])}$$

$$\begin{aligned} R_0(t) &= \ln\frac{\Gamma(t+3)}{(t+2)(t+1)t} - \left(t - \frac{1}{2}\right)\ln t + t - \frac{1}{2}\ln(2\pi) \\ &= \ln\Gamma(t+3) - \ln(t+2) - \ln(t+1) - t\ln t - \frac{1}{2}\ln t + t - \frac{1}{2}\ln(2\pi) \\ &\to \infty, \quad t \to 0^+, \end{aligned}$$

and

$$\begin{aligned} tR_1(t) &= -t\left[R_0(t) - \frac{1}{12t}\right] \\ &= \frac{1}{12} - t\ln\Gamma(t+3) + t\ln\frac{t+2}{t+1} + \left(t + \frac{1}{2}\right)t\ln t - t^2 + \frac{\ln(2\pi)}{2}t \\ &\to \frac{1}{12}, \quad t \to 0^+. \end{aligned}$$

Consequently, it follows that

$$\deg^t[R_0(t)] \leqslant 0 \quad \text{and} \quad \deg^t[R_1(t)] \leqslant 1. \tag{3.2}$$

Combining those inequalities in (3.1) and (3.2) concludes those equalities in (2.4).

3.2. Proofs of equalities in (2.6)

Since $-R_1'(t) = \phi(t)$, and

$$\lim_{t\to\infty} t[\ln t - \psi(t)] = \frac{1}{2}, \quad \text{(see [2, p. 374])}$$

$$\lim_{t\to 0^+}\frac{t\left[\frac{1}{2t^2} + \frac{1}{t} - \psi'(t)\right]}{\frac{1}{2t} - \ln t + \psi(t)} = \lim_{t\to 0^+}\frac{\frac{1}{2} + t - t^2\left[\psi'(t+1) + \frac{1}{t^2}\right]}{\frac{1}{2} - t[\ln t - \psi(t)]} = 1,$$

by the equation (2.1), we conclude the first conclusion in (2.6).

The equation (2.2) established in [29, Theorem 2] is equivalent to the second conclusion in (2.6).

3.3. Proof of the double inequality (2.10)

Let

$$f_n(v)=(-1)^n\left[\frac{1}{v}-\frac{1}{2}\coth\frac{v}{2}+\sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}v^{2k-1}\right]$$

for $n\geqslant 0$, where the empty sum is understood to be 0. Then, by virtue of the formulas

$$\coth v=\frac{1+e^{-2v}}{1-e^{-2v}}=\frac{2}{1-e^{-2v}}-1,$$

$$\psi(z)=\ln z+\int_0^\infty\left(\frac{1}{v}-\frac{1}{1-e^{-v}}\right)e^{-zv}\,dv,\quad \Re(z)>0$$

in [21, p. 140, 5.9.13], and

$$\frac{1}{z^w}=\frac{1}{\Gamma(w)}\int_0^\infty v^{w-1}e^{-zv}\,dv,\quad \Re(z),\Re(w)>0$$

in [1, p. 255, 6.1.1], we derive that

$$\begin{aligned}(-1)^n\int_0^\infty f_n(v)e^{-tv}\,dv&=\int_0^\infty\left[\frac{1}{v}-\frac{1}{2}\coth\frac{v}{2}+\sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}v^{2k-1}\right]e^{-tv}\,dv\\&=\int_0^\infty\left[\frac{1}{v}-\frac{1}{1-e^{-v}}+\frac{1}{2}+\sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}v^{2k-1}\right]e^{-tv}\,dv\\&=\psi(t)-\ln t+\frac{1}{2t}+\sum_{k=1}^{n}\frac{B_{2k}}{2k}\frac{1}{t^{2k}}\end{aligned}$$

for $n\geqslant 0$. This means that

$$R_n'(t)=\int_0^\infty f_n(v)e^{-tv}\,dv,\quad n\geqslant 0. \tag{3.3}$$

The inequality (2.3) tells us that the remainder $R_n(t)$ for $n\geqslant 0$ is completely monotonic in $t\in(0,\infty)$. Then, by definition of completely monotonic functions, it is ready that the remainder $-R_n'(t)$ for $n\geqslant 0$ is completely monotonic on $(0,\infty)$.

Since the function $\frac{v}{e^v-1}-1+\frac{v}{2}$ is even in $v\in\mathbb{R}$, by virtue of the equation (1.1), it follows that

$$\begin{aligned}f_n(v)&=\frac{(-1)^n}{v}\left[1-\frac{v}{1-e^{-v}}+\frac{v}{2}+\sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}v^{2k}\right]\\&=\frac{(-1)^n}{v}\left[-\left(\frac{v}{e^v-1}-1+\frac{v}{2}\right)+\sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}v^{2k}\right]\\&=(-1)^{n+1}\sum_{k=n+1}^{\infty}\frac{B_{2k}}{(2k)!}v^{2k-1}\end{aligned}$$

for $|v| < 2\pi$ and $n \geqslant 0$. Accordingly, we have

$$\lim_{v\to 0} f_n^{(\ell)}(v) = (-1)^{n+1} \lim_{v\to 0} \sum_{k=n+1}^{\infty} \frac{B_{2k}}{(2k)!} \langle 2k-1\rangle_\ell v^{2k-\ell-1} = 0 \tag{3.4}$$

for $0 \leqslant \ell \leqslant 2n$ or $\ell = 2m$ with $m,n \geqslant 0$, where

$$\langle\alpha\rangle_n = \prod_{k=0}^{n-1}(\alpha-k) = \begin{cases} \alpha(\alpha-1)\cdots(\alpha-n+1), & n \geqslant 1 \\ 1, & n = 0 \end{cases}$$

is called the falling factorial of α. For more information on the falling and rising factorials, refer to the papers [30].

Recall from [9, Theorem 2.1], [10, Theorem 2.1], and [38, Theorem 3.1] that, when $\vartheta > 0$ and $t \neq -\frac{\ln\vartheta}{\theta}$ or when $\vartheta < 0$ and $t \in \mathbb{R}$, we have

$$\frac{d^k}{dt^k}\left(\frac{1}{\vartheta e^{\theta t}-1}\right) = (-1)^k \theta^k \sum_{p=1}^{k+1} (p-1)! S(k+1,p) \left(\frac{1}{\vartheta e^{\theta t}-1}\right)^p \tag{3.5}$$

for $k \geqslant 0$, where

$$S(k,p) = \frac{1}{p!} \sum_{q=1}^{p} (-1)^{p-q} \binom{p}{q} q^k, \quad 1 \leqslant p \leqslant k$$

are the Stirling numbers of the second kind. Taking $\vartheta = \theta = 1$ in (3.5) leads to

$$\frac{d^k}{dt^k}\left(\frac{1}{e^t-1}\right) = (-1)^k \sum_{p=1}^{k+1} (p-1)! S(k+1,p) \left(\frac{1}{e^t-1}\right)^p \tag{3.6}$$

for $k \geqslant 0$. Utilizing (3.6) results in

$$\lim_{v\to\infty} \frac{d^\ell}{dv^\ell}\left(\frac{1}{v} - \frac{1}{2}\coth\frac{v}{2}\right) = \lim_{v\to\infty} \frac{d^\ell}{dv^\ell}\left[\frac{1}{v} - \left(\frac{1}{e^v-1} + \frac{1}{2}\right)\right] = \begin{cases} 0, & \ell \geqslant 1; \\ -\frac{1}{2}, & \ell = 0. \end{cases} \tag{3.7}$$

Making use of (3.7) and

$$\lim_{v\to\infty}\left(v^m e^{-tv}\right) = 0, \quad t > 0, \quad m \geqslant 0$$

yields

$$\begin{aligned} \lim_{v\to\infty}\left[f_n^{(\ell)}(v) e^{-tv}\right] &= (-1)^n \lim_{v\to\infty}\left[\left(\frac{1}{v} - \frac{1}{2}\coth\frac{v}{2}\right)^{(\ell)} e^{-tv}\right. \\ &\quad \left. + \sum_{k=1}^{n} \frac{B_{2k}}{(2k)!}\langle 2k-1\rangle_\ell v^{2k-\ell-1} e^{-tv}\right] \\ &= 0 \end{aligned} \tag{3.8}$$

for $\ell, n \geqslant 0$ and $t > 0$. Consequently, by the limits (3.4) and (3.8), integrating by parts inductively $2n-1$ times in the equation (3.3) gives

$$\begin{aligned}
R_n'(t) &= -\frac{1}{t}\int_0^\infty f_n(v)\,\mathrm{d}\,\mathrm{e}^{-tv} \\
&= -\frac{1}{t}\left[f_n(v)\,\mathrm{e}^{-tv}\Big|_{v=0}^{\infty} - \int_0^\infty f_n'(v)\,\mathrm{e}^{-tv}\,\mathrm{d}v\right] \\
&= \frac{1}{t}\int_0^\infty f_n'(v)\,\mathrm{e}^{-tv}\,\mathrm{d}v \\
&= \frac{1}{t^{2n-1}}\int_0^\infty f_n^{(2n-1)}(v)\,\mathrm{e}^{-tv}\,\mathrm{d}v \\
&= \frac{(-1)^n}{t^{2n-1}}\int_0^\infty \left[\frac{1}{v} - \frac{1}{2}\coth\frac{v}{2} + \sum_{k=1}^{n}\frac{B_{2k}}{(2k)!}v^{2k-1}\right]^{(2n-1)} \mathrm{e}^{-tv}\,\mathrm{d}v \\
&= \frac{(-1)^n}{t^{2n-1}}\int_0^\infty K_{2n-1}(v)\,\mathrm{e}^{-tv}\,\mathrm{d}v
\end{aligned}$$

for $n \geqslant 1$, where

$$K_{2n-1}(v) = \frac{\mathrm{d}^{2n-1}}{\mathrm{d}v^{2n-1}}\left(\frac{1}{v} - \frac{1}{2}\coth\frac{v}{2}\right) + \frac{B_{2n}}{2n}, \quad v > 0, \quad n \geqslant 1. \tag{3.9}$$

Using the relation

$$\int_0^\infty \frac{x^{2n+1}\cos(ax)}{\mathrm{e}^x - 1}\,\mathrm{d}x = (-1)^n \frac{\mathrm{d}^{2n+1}}{\mathrm{d}a^{2n+1}}\left[\frac{\pi}{2}\coth(a\pi) - \frac{1}{2a}\right], \quad a > 0, \quad n \geqslant 0$$

in [6, p. 48, (D20)], [7, p. 506, 3.951.13], and [36, p. 2], we arrive at

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}v^{2n-1}}\left(\frac{1}{v} - \frac{1}{2}\coth\frac{v}{2}\right) = (-1)^n 2\int_0^\infty \frac{w^{2n-1}\cos(wv)}{\mathrm{e}^{2\pi w} - 1}\,\mathrm{d}w \tag{3.10}$$

for $v > 0$ and $n \geqslant 1$. Using the relation

$$\int_0^\infty \frac{t^{2k-1}}{\mathrm{e}^{2\pi t} - 1}\,\mathrm{d}t = (-1)^{k-1}\frac{B_{2k}}{4k}, \quad k \geqslant 1$$

in [3, p. 29, (1.6.4)], [4, p. 220], and [34, p. 19], we obtain

$$(-1)^n 2\int_0^\infty \frac{w^{2n-1}}{\mathrm{e}^{2\pi w} - 1}\,\mathrm{d}w = -\frac{B_{2n}}{2n}, \quad n \geqslant 1. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9) reveals

$$K_{2n-1}(v) = (-1)^{n-1} 2\int_0^\infty \frac{w^{2n-1}[1 - \cos(wv)]}{\mathrm{e}^{2\pi w} - 1}\,\mathrm{d}w, \quad v > 0, \quad n \geqslant 1.$$

Consequently, we have

$$t^{2n-1}[-R_n'(t)] = 2\int_0^\infty \left(\int_0^\infty \frac{w^{2n-1}[1 - \cos(wv)]}{\mathrm{e}^{2\pi w} - 1}\,\mathrm{d}w\right)\mathrm{e}^{-tv}\,\mathrm{d}v, \quad n \geqslant 1. \tag{3.12}$$

Hence, the function $t^{2n-1}[-R_n'(t)]$ is completely monotonic on $(0,\infty)$. This means that

$$\deg_{\mathrm{cm}}^{t}[-R_n'(t)] \geqslant 2n-1, \quad n \geqslant 1. \tag{3.13}$$

If the function $t^{\alpha}[-R_n'(t)]$ were completely monotonic on $(0,\infty)$, then its first derivative is negative, hence

$$\alpha < -\frac{t[-R_n'(t)]'}{-R_n'(t)} = -\frac{tR_n''(t)}{R_n'(t)}.$$

From

$$\lim_{t\to 0^+}\left[t^2\left(\psi'(t)-\frac{1}{t}-\frac{1}{2t^2}\right)\right] = \lim_{t\to 0^+}\left[t^2\left(\psi'(t+1)-\frac{1}{t}+\frac{1}{2t^2}\right)\right] = \frac{1}{2}$$

and

$$\lim_{t\to 0^+}\left[t\left(\psi(t)-\ln t+\frac{1}{2t}\right)\right] = \lim_{t\to 0^+}\left[t\left(\psi(t+1)-\ln t-\frac{1}{2t}\right)\right] = -\frac{1}{2},$$

it follows that

$$\begin{aligned}
\lim_{t\to 0^+}\left[-\frac{tR_n''(t)}{R_n'(t)}\right] &= -\lim_{t\to 0^+}\frac{t\left[\psi'(t)-\frac{1}{t}-\frac{1}{2t^2}-\sum_{k=1}^{n}\frac{B_{2k}}{t^{2k+1}}\right]}{\left[\psi(t)-\ln t+\frac{1}{2t}+\sum_{k=1}^{n}\frac{B_{2k}}{2k}\frac{1}{t^{2k}}\right]}\\
&= -\lim_{t\to 0^+}\frac{t^2\left[\psi'(t)-\frac{1}{t}-\frac{1}{2t^2}\right]-\sum_{k=1}^{n}\frac{B_{2k}}{t^{2k-1}}}{t\left[\psi(t)-\ln t+\frac{1}{2t}\right]+\sum_{k=1}^{n}\frac{1}{2k}\frac{B_{2k}}{t^{2k-1}}}\\
&= 2n.
\end{aligned}$$

This means that

$$\deg_{\mathrm{cm}}^{t}[-R_n'(t)] \leqslant 2n, \quad n \geqslant 1. \tag{3.14}$$

We now prove that the inequality (3.14) is strict for $n \geqslant 2$, or say, when $n \geqslant 2$ the equality in (3.14) can be removed off. This proof is provided by an anonymous referee. In the formula (3.12), the integrand is non-negative and it is therefore permitted to interchange the order of integration. Doing so we obtain

$$t^{2n-1}[-R_n'(t)] = 2\int_0^\infty\left(\int_0^\infty[1-\cos(wv)]\,\mathrm{e}^{-tv}\,\mathrm{d}\,v\right)\frac{w^{2n-1}}{\mathrm{e}^{2\pi w}-1}\,\mathrm{d}\,w \tag{3.15}$$

for $n\in\mathbb{N}$. Computing the inner integral in (3.15), which is a Laplace transform, we acquire

$$t^{2n}[-R_n'(t)] = 2\int_0^\infty\frac{w}{\mathrm{e}^{2\pi w}-1}\frac{w^{2n}}{t^2+w^2}\,\mathrm{d}\,w \tag{3.16}$$

for $n\in\mathbb{N}$. The right hand side of the relation (3.16) is a smooth function of $t>0$. Straightforwardly differentiating under the integral in (3.16) with respect to $t>0$ gives us

$$\left(t^{2n}[-R_n'(t)]\right)'' = 4\int_0^\infty\frac{w}{\mathrm{e}^{2\pi w}-1}\frac{3t^2-w^2}{(t^2+w^2)^3}w^{2n}\,\mathrm{d}\,w \tag{3.17}$$

for $n \in \mathbb{N}$. We want to let $t \to 0^+$ in (3.17) using dominated convergence, and thus need an integral majorant

$$\left|\frac{3t^2 - w^2}{(t^2+w^2)^3} w^{2n}\right| \leqslant 3\frac{t^2+w^2}{(t^2+w^2)^3} w^{2n} \leqslant 3\frac{w^{2n}}{(t^2+w^2)^2} \leqslant 3w^{2n-4}.$$

For $n \geqslant 2$, we thus have a suitable bound of the integrand and the Lebesgues theorem can be applied and gives

$$\lim_{t\to 0^+}\left[\left(t^{2n}[-R_n'(t)]\right)''\right] = -4\int_0^\infty \frac{w}{e^{2\pi w}-1} w^{2n-4}\,\mathrm{d}w < 0.$$

Since the second derivative is negative at $t = 0^+$, it must also be negative for all $t > 0$ sufficiently close to 0. Consequently, for $n \geqslant 2$, the inequality (3.14) is refined as

$$\deg_{\mathrm{cm}}^{t}[-R_n'(t)] < 2n, \quad n \geqslant 2. \tag{3.18}$$

Combining (3.18) with (3.13) gives the double inequality (2.10). The proof of Theorem 2.1 is complete.

4. Remarks

Finally we list several remarks on main results and Qi's conjectures in this paper.

REMARK 4.1. As in the derivation of (3.9), integrating by parts $R_n'(t)$ inductively $2n \geqslant 2$ times in (3.3) yields

$$t^{2n}[-R_n'(t)] = \int_0^\infty K_{2n}(v)\,e^{-tv}\,\mathrm{d}v$$

for $n \geqslant 1$, where, by (3.6),

$$\begin{aligned} K_{2n}(v) &= \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\left(\frac{1}{v} - \frac{1}{2}\coth\frac{v}{2}\right) \\ &= \frac{\mathrm{d}^{2n}}{\mathrm{d}v^{2n}}\left(\frac{1}{v} - \frac{1}{2} - \frac{1}{e^v - 1}\right) \\ &= \frac{(2n)!}{v^{2n+1}} - \sum_{p=1}^{2n+1}(p-1)!S(2n+1,p)\left(\frac{1}{e^v-1}\right)^p \end{aligned}$$

for $v > 0$ and $n \geqslant 1$. It is easy to see that

$$\begin{aligned} K_2(v) &= \frac{2}{v^3} - \frac{2}{(e^v-1)^3} - \frac{3}{(e^v-1)^2} - \frac{1}{e^v-1} \\ &= \frac{2e^{3v} - e^{2v}(v^3+6) - e^v(v^3-6) - 2}{(e^v-1)^3 v^3}, \end{aligned}$$

$$
\begin{aligned}
\big[(e^{v}-1)^3v^3K_2(v)\big]' &= e^{v}\big[6e^{2v}-v^3-3v^2+6-e^{v}(2v^3+3v^2+12)\big]\\
&\to 0, \quad v\to 0^+,\\
\big(\big[(e^{v}-1)^3v^3K_2(v)\big]'e^{-v}\big)' &= 12e^{2v}-e^{v}(2v^3+9v^2+6v+12)-3v(v+2)\\
&\to 0, \quad v\to 0^+,\\
\big(\big[(e^{v}-1)^3v^3K_2(v)\big]'e^{-v}\big)'' &= 24e^{2v}-e^{v}(2v^3+15v^2+24v+18)-6(v+1)\\
&\to 0, \quad v\to 0^+\\
\big(\big[(e^{v}-1)^3v^3K_2(v)\big]'e^{-v}\big)^{(3)} &= 48e^{2v}-e^{v}(2v^3+21v^2+54v+42)-6\\
&\to 0, \quad v\to 0^+\\
\big(\big[(e^{v}-1)^3v^3K_2(v)\big]'e^{-v}\big)^{(4)} &= 96e^{v}\bigg(e^{v}-1-v-\frac{9}{16}\frac{v^2}{2!}-\frac{1}{8}\frac{v^3}{3!}\bigg)\\
&> 0
\end{aligned}
$$

for $v\in(0,\infty)$. Therefore, the function $K_2(v)$ is positive on $(0,\infty)$. By Theorem 12b in [35, p. 161] stated in (1.2), we conclude that the function $t^2[-R_1'(t)]$ is completely monotonic on $(0,\infty)$, that is,

$$
\deg^t_{\mathrm{cm}}[-R_1'(t)] \geqslant 2.
$$

This supplies an alternative and partial proof for the second equality in (2.6).

REMARK 4.2. Making use of (3.12) and integrating by parts lead to

$$
t^{2n}[-R_n'(t)] = 2\int_0^\infty\bigg[\int_0^\infty \frac{w^{2n}\sin(wv)}{e^{2\pi w}-1}\,\mathrm{d}w\bigg]e^{-tv}\,\mathrm{d}v, \quad n\geqslant 1. \tag{4.1}
$$

When $n=1$, the second equality in (2.6) means that the function $t^2[-R_n'(t)]$ is completely monotonic on $(0,\infty)$. By Theorem 12b in [35, p. 161] stated in (1.2), from the equality in (4.1), we conclude

$$
\int_0^\infty \frac{u^2\sin(su)}{e^u-1}\,\mathrm{d}u \geqslant 0, \quad s\in(0,\infty).
$$

Since the right hand side of the double inequality (2.10) in Theorem 2.1 is strict, when $n\geqslant 2$, the function $t^{2n}[-R_n'(t)]$ is not completely monotonic on $(0,\infty)$. Again by Theorem 12b in [35, p. 161] stated in (1.2), again from the equality in (4.1), we conclude that the function

$$
s\in(0,\infty)\mapsto \int_0^\infty \frac{u^{2n}\sin(su)}{e^u-1}\,\mathrm{d}u, \quad n\geqslant 2 \tag{4.2}
$$

attains negative values somewhere on $(0,\infty)$. Consequently, the function

$$
s\in(0,\infty)\mapsto \int_0^\infty \frac{u[1-\cos(su)]}{e^u-1}\,\mathrm{d}u
$$

is increasing on $(0,\infty)$ and the function

$$s\in(0,\infty)\mapsto\int_0^\infty\frac{u^{2n-1}[1-\cos(su)]}{e^u-1}\,du,\quad n\geqslant 2,$$

is decreasing on some subinterval $I\subset(0,\infty)$.

From the negativity somewhere on $(0,\infty)$ of the function in (4.2) for $n\geqslant 2$ and from the formula

$$\int_0^\infty\frac{x^{2n}\sin(ax)}{e^x-1}\,dx=(-1)^n\frac{d^{2n}}{da^{2n}}\left[\frac{\pi}{2}\coth(a\pi)-\frac{1}{2a}\right],\quad a>0,\quad n\geqslant 0$$

listed in [6, p. 48, (D19)], [7, p. 506, 3.951.12], and [36, p. 1], we see that the functions

$$s\in(0,\infty)\mapsto(-1)^n\left(\coth s-\frac{1}{s}\right)^{(2n)}\quad\text{and}\quad s\in(0,\infty)\mapsto(-1)^n\left(\frac{1}{e^s-1}-\frac{1}{s}\right)^{(2n)}$$

for $n\geqslant 2$ attain negative values on some subinterval $I\subset(0,\infty)$.

REMARK 4.3. From (3.10), we conclude that

$$\begin{aligned}
\int_0^\infty\frac{w^{2n-1}[1-\cos(wv)]}{e^{2\pi w}-1}\,dw&=\int_0^\infty\frac{w^{2n-1}}{e^{2\pi w}-1}\,dw-\int_0^\infty\frac{w^{2n-1}\cos(wv)}{e^{2\pi w}-1}\,dw\\
&=\frac{1}{(2\pi)^{2n}}\int_0^\infty\frac{w^{2n-1}}{e^w-1}\,dw-\frac{(-1)^n}{2}\frac{d^{2n-1}}{dv^{2n-1}}\left(\frac{1}{v}-\frac{1}{2}\coth\frac{v}{2}\right)\\
&=\frac{\Gamma(2n)\zeta(2n)}{(2\pi)^{2n}}-\frac{(-1)^n}{2}\frac{d^{2n-1}}{dv^{2n-1}}\left(\frac{1}{v}-\frac{1}{2}-\frac{1}{e^v-1}\right)\\
&=\frac{\Gamma(2n)\zeta(2n)}{(2\pi)^{2n}}-\frac{(-1)^n}{2}\left[\frac{(2n-1)!}{v^{2n}}-\sum_{p=1}^{2n}(p-1)!S(2n,p)\left(\frac{1}{e^v-1}\right)^p\right],
\end{aligned}$$

where $\zeta(z)$ denotes the Riemann zeta function [21, Chapter 25]. Consequently, we obtain three inequalities

$$\frac{(-1)^n}{2}\left(\frac{1}{v}-\frac{1}{2}\coth\frac{v}{2}\right)^{(2n-1)}<\frac{\Gamma(2n)\zeta(2n)}{(2\pi)^{2n}},$$

$$\frac{(-1)^n}{2}\left(\frac{1}{v}-\frac{1}{e^v-1}\right)^{(2n-1)}<\frac{\Gamma(2n)\zeta(2n)}{(2\pi)^{2n}},$$

and

$$\frac{(-1)^n}{2}\left[\frac{(2n-1)!}{v^{2n}}-\sum_{p=1}^{2n}(p-1)!S(2n,p)\left(\frac{1}{e^v-1}\right)^p\right]<\frac{\Gamma(2n)\zeta(2n)}{(2\pi)^{2n}}$$

for $n\geqslant 1$ and $v\in(0,\infty)$.

REMARK 4.4. From the proof of Theorem 2.1, we can deduce that

$$\lim_{t\to\infty} R_n'(t) = 0, \qquad \lim_{t\to\infty} \left[t^{2n-1} R_n'(t)\right] = 0,$$
$$\lim_{t\to\infty} \left[t^{2n+3} R_n'(t)\right] = (-1)^{n+1} \frac{B_{2n+2}}{2n+2}, \qquad \lim_{t\to 0^+} \left[t^{2n+1} R_n'(t)\right] = (-1)^{n} \frac{B_{2n}}{2n}$$

for $n \in \mathbb{N}$.

REMARK 4.5. The conjectures in (2.5), (2.8), and (2.9) posed by Qi in [29] are still kept open. Theorem 2.1 does not give a full answer to the conjecture in (2.7), but it still demonstrates that these open conjectures posed by Qi should be true.

REMARK 4.6. This paper is a revised version of electronic preprints at the web sites

`https://hal.archives-ouvertes.fr/hal-02415224`

and

`https://arxiv.org/abs/1912.07989`.

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