Mathematical
Inequalities
& Applications
Volume 25, Number 2 (2022), 307–318 doi:10.7153/mia-2022-25-18

VOLTERRA TYPE INTEGRAL OPERATOR ACTING BETWEEN FOCK SPACES

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(*Communicated by J. Pečarić*)

Abstract. The boundedness of the Volterra type integral operator acting from the space $\mathscr{F}_\alpha^\infty(\mathbb{C}^N)$ to the space $\mathscr{F}_\beta^p(\mathbb{C}^N)$ is characterized. This characterization also indicates the compactness of this operator.

1. Introduction

Let N be a fixed positive integer and $H(\mathbb{C}^N)$ denote the space of all entire functions on the N-dimensional complex Euclidean space. For each $\alpha > 0$ and $0 < p \leqslant \infty$, the Fock spaces $\mathscr{F}_\alpha^p(\mathbb{C}^N)$ are defined by

$$\mathscr{F}_\alpha^p(\mathbb{C}^N) = \left\{ f \in H(\mathbb{C}^N) \, : \, \|f\|_{p,\alpha}^p = \int_{\mathbb{C}^N} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dV(z) < \infty \right\}$$

and

$$\mathscr{F}_\alpha^\infty(\mathbb{C}^N) = \left\{ f \in H(\mathbb{C}^N) \, : \, \|f\|_{\infty,\alpha} = \sup_{z\in\mathbb{C}^N} |f(z)| e^{-\frac{\alpha}{2}|z|^2} < \infty \right\}.$$

Here dV denotes the ordinary Lebesgue measure on $\mathbb{C}^N$. Throughout this paper, the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leqslant CB$. Moreover, if both $A \lesssim B$ and $B \lesssim A$ hold, then one says that $A \approx B$.

For any $f \in H(\mathbb{C}^N)$ the radial derivative $\mathscr{R}f$ of f is defined by

$$\mathscr{R}f(z) = \sum_{j=1}^N z_j \frac{\partial f}{\partial z_j}(z).$$

For given $g \in H(\mathbb{C}^N)$, the Volterra type integral operator V_g is defined as following:

$$V_g f(z) = \int_0^1 f(tz) \mathscr{R}g(tz) \frac{dt}{t} \qquad (f \in H(\mathbb{C}^N),\ z \in \mathbb{C}^N).$$

This can be regarded as a multivariable version of the operator $f \mapsto \int_0^z f(w)g'(w)dw$ in the one variable case. This type operator has been studied by many researchers. As

Mathematics subject classification (2020): 30H20, 32A15, 47B38.
Keywords and phrases: Fock spaces, Volterra type integral operator.

has been shown in a series of studies on integral operators (see, e.g., [8, 9, 10, 11] and the related references therein), the radial derivative operator indicates the relation $\mathscr{R}[V_g f](z) = f(z)\mathscr{R}g(z)$. Combining this relation with the equivalence condition for the spaces $\mathscr{F}^p_\alpha(\mathbb{C}^N)$ via the radial derivative operator, we can investigate the properties of the operator V_g. In fact, Z. Hu [4] has given completely characterizations for the boundedness and the compactness of $V_g : \mathscr{F}^p_\alpha(\mathbb{C}^N) \to \mathscr{F}^q_\alpha(\mathbb{C}^N)$ for the both cases $0 < p \leqslant q < \infty$ and $0 < q < p < \infty$. O. Constantin [1] has considered the case $N = 1$. The author [15, 16] has investigated the case $p = q = \infty$, namely the boundedness and the compactness of $V_g : \mathscr{F}^\infty_\alpha(\mathbb{C}^N) \to \mathscr{F}^\infty_\alpha(\mathbb{C}^N)$. They do not consider the case $V_g : \mathscr{F}^\infty_\alpha(\mathbb{C}^N) \to \mathscr{F}^p_\alpha(\mathbb{C}^N)$ for $0 < p < \infty$. By means of their characterizations for the spaces $\mathscr{F}^p_\alpha(\mathbb{C}^N)$, however, we can find one of sufficient conditions for the boundedness of $V_g : \mathscr{F}^\infty_\alpha(\mathbb{C}^N) \to \mathscr{F}^p_\alpha(\mathbb{C}^N)$. In fact, we can easily obtain the following result.

PROPOSITION 1. Let $0 < p < \infty$, $\alpha, \beta > 0$ and $g \in H(\mathbb{C}^N)$. If the z-variable function $\frac{|\mathscr{R}g(z)|}{(1+|z|)^2} e^{\frac{\alpha-\beta}{2}|z|^2}$ is in $L^p(\mathbb{C}^N, dV)$, then V_g is bounded from $\mathscr{F}^\infty_\alpha(\mathbb{C}^N)$ into $\mathscr{F}^p_\beta(\mathbb{C}^N)$.

Proof. Since $V_g f(0) = 0$, Lemma 4 in Section 2 gives

$$\begin{aligned} \|V_g f\|^p_{p,\beta} &\approx \int_{\mathbb{C}^N} \frac{|f(z)|^p |\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{-\frac{p\beta}{2}|z|^2} dV(z) \\ &= \int_{\mathbb{C}^N} |f(z)|^p |e^{-\frac{p\alpha}{2}|z|^2} \frac{|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|z|^2} dV(z) \\ &\lesssim \|f\|^p_{\infty,\alpha} \int_{\mathbb{C}^N} \frac{|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|z|^2} dV(z). \end{aligned}$$

This implies that the condition $\frac{|\mathscr{R}g(z)|}{(1+|z|)^2} e^{\frac{\alpha-\beta}{2}|z|^2} \in L^p(\mathbb{C}^N, dV)$ is sufficient for the boundedness of $V_g : \mathscr{F}^\infty_\alpha(\mathbb{C}^N) \to \mathscr{F}^p_\beta(\mathbb{C}^N)$. □

Our purpose in this short paper is to prove that the condition $\frac{|\mathscr{R}g(z)|}{(1+|z|)^2} e^{\frac{\alpha-\beta}{2}|z|^2} \in L^p(\mathbb{C}^N, dV)$ characterize not only the boundedness of V_g but also its compactness. The following is the main result.

THEOREM 1. *Let* $0 < p < \infty$, $\alpha, \beta > 0$ *and* $g \in H(\mathbb{C}^N)$. *Then the following conditions are equivalent*:

(a) $V_g : \mathscr{F}^\infty_\alpha(\mathbb{C}^N) \to \mathscr{F}^p_\beta(\mathbb{C}^N)$ *is bounded,*

(b) $V_g : \mathscr{F}^\infty_\alpha(\mathbb{C}^N) \to \mathscr{F}^p_\beta(\mathbb{C}^N)$ *is compact,*

(c) $\frac{|\mathscr{R}g(z)|}{(1+|z|)^2} e^{\frac{\alpha-\beta}{2}|z|^2} \in L^p(\mathbb{C}^N, dV)$.

The proof of the theorem is given in Section 3. Since the direction (b)⇒(a) is trivial, it is enough to prove (a)⇒(c) and (c)⇒(b). In order to prove (a)⇒(c), we will use the result about a positive Borel measure based on the concept of the lattice in $\mathbb{C}^N$. In the proof of (c)⇒(b), we show that the essential norm of V_g is equal to 0. In both proofs, Hu [4] and our [16] results about characterizations for the spaces $\mathscr{F}^p_\alpha(\mathbb{C}^N)$ play a central role.

When $N=1$, we see that $\mathscr{R}g(z)=zg'(z)$ for $g\in H(\mathbb{C})$ and $z\in\mathbb{C}$. Thus we also obtain the result for $N=1$ as follows.

COROLLARY 1. *Let $0<p<\infty$, $\alpha,\beta>0$ and $g\in H(\mathbb{C})$. Then the following conditions are equivalent*:

(a) $V_g:\mathscr{F}^\infty_\alpha(\mathbb{C})\to\mathscr{F}^p_\beta(\mathbb{C})$ *is bounded,*

(b) $V_g:\mathscr{F}^\infty_\alpha(\mathbb{C})\to\mathscr{F}^p_\beta(\mathbb{C})$ *is compact,*

(c) $\frac{|g'(z)|}{1+|z|}e^{\frac{\alpha-\beta}{2}|z|^2}\in L^p(\mathbb{C},dV)$.

If $\alpha=\beta$ in Corollary 1, then the boundedness and the compactness of $V_g:\mathscr{F}^\infty_\alpha(\mathbb{C})\to\mathscr{F}^p_\alpha(\mathbb{C})$ are equivalent to the condition $\frac{|g'(z)|}{1+|z|}\in L^p(\mathbb{C},dV)$. On the other hand, T. Mengestie [6, Theorem 2.3] shows that $V_g:\mathscr{F}^\infty_\alpha(\mathbb{C})\to\mathscr{F}^p_\alpha(\mathbb{C})$ is bounded or compact if and only if

$$\int_{\mathbb{C}}dV(w)\int_{\mathbb{C}}\frac{|g'(z)|^p}{(1+|z|)^p}e^{-\frac{p\alpha}{2}|z-w|^2}dV(z)<\infty.$$

Our result simplifies this condition.

Let X and Y be two Banach spaces and $A:X\to Y$ be a bounded linear operator. The essential norm of the operator is defined as

$$\|A\|_e=\inf_{K\in\mathscr{K}(X,Y)}\|A-K\|_{X\to Y},$$

where $\mathscr{K}(X,Y)$ is the family of all compact operators from X to Y. Essential norms of some integral type operators on spaces of holomorphic functions have been studied, for example, in [2, 7, 9, 13, 14].

2. Preliminaries

For $a\in\mathbb{C}^N$ and $r>0$, $B(a,r)$ denotes the Euclidean open ball with center at a and radius r. The following lemma is a modification of well-known results in theory on Fock spaces. However we include a proof of it for completeness.

LEMMA 1. *Let $0<p,\alpha<\infty$. For each $f\in H(\mathbb{C}^N)$, $R>0$ and $z\in\mathbb{C}^N$, there exists a positive constant $C=C(N,p,\alpha,R)$ depends on N,p,α and R such that*

$$\frac{|f(z)|^p}{(1+|z|)^{2p}}e^{-\frac{p\alpha}{2}|z|^2}\leqslant C\int_{B(z,R)}\frac{|f(w)|^p}{(1+|w|)^{2p}}e^{-\frac{p\alpha}{2}|w|^2}dV(w).$$

Proof. Take an $r \in (0,R)$. The subharmonicity of $|f|^p$ gives

$$|f(0)|^p \leqslant \int_{\partial B(0,1)} |f(r\zeta)|^p d\sigma(\zeta),$$

where $\partial B(0,1)$ is the boundary of the unit ball $B(0,1)$ and $d\sigma$ is the normalized Lebesgue measure on $\partial B(0,1)$. Multiplying both sides by $2Nr^{2N-1}e^{-\frac{p\alpha}{2}r^2}$ and integrating with respect to r from 0 to R, we obtain

$$|f(0)|^p R^{2N} e^{-\frac{p\alpha}{2}R^2} \leqslant \frac{N!}{\pi^N} \int_{B(0,R)} |f(w)|^p e^{-\frac{p\alpha}{2}|w|^2} dV(w). \tag{1}$$

Now we consider the function

$$F_z^f(w) = f(w+z)e^{\alpha\langle w,z\rangle - \frac{\alpha}{2}|z|^2} \qquad (w \in \mathbb{C}^N).$$

Then $F_z^f \in H(\mathbb{C}^N)$ and $|f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} = |F_z^f(0)|^p$. Since $\frac{1}{1+|z|} < \frac{1+R}{1+|w+z|}$ for $w \in B(0,R)$, it follows from (1) that

$$\begin{aligned}
\frac{|f(z)|^p}{(1+|z|)^{2p}} e^{-\frac{p\alpha}{2}|z|^2} &= \frac{|F_z^f(0)|^p}{(1+|z|)^{2p}} \\
&\leqslant \frac{N! e^{\frac{p\alpha}{2}R^2}}{(\pi R^2)^N} \int_{B(0,R)} \frac{|F_z^f(w)|^p}{(1+|z|)^{2p}} e^{-\frac{p\alpha}{2}|w|^2} dV(w) \\
&= \frac{N! e^{\frac{p\alpha}{2}R^2}(1+R)^{2p}}{(\pi R^2)^N} \int_{B(0,R)} \frac{|f(w+z)|^p}{(1+|w+z|)^{2p}} e^{-\frac{p\alpha}{2}|w+z|^2} dV(w). \quad (2)
\end{aligned}$$

An application of a change of variables formula to (2) implies the desired estimation for $f \in H(\mathbb{C}^N)$. $\square$

We cite some result on a positive Borel measure in terms of a lattice. For given $r > 0$, a sequence $\{a_k\}$ in $\mathbb{C}^N$ is called an r-lattice if it satisfies the following conditions:

(i) $\mathbb{C}^N = \cup_{k=1}^{\infty} B(a_k, r)$,

(ii) $B(a_k, r/2) \cap B(a_j, r/2) = \emptyset$ if $k \neq j$,

(iii) For any $R > 0$ there is a positive integer M depending only on r and R, such that every point in $\mathbb{C}^N$ belongs to at most M of the balls $B(a_k, R)$.

The following result is appeared in [3, Lemma 2.3].

LEMMA 2. *Let $r > 0$ and $\{a_k\}$ be an r-lattice in $\mathbb{C}^N$. For a positive Borel measure μ the following two conditions are equivalent*:

(a) $\mu(B(\cdot, r)) \in L^1(\mathbb{C}^N, dV)$,

(b) $\{\mu(B(a_k, r))\} \in l^1$.

We shall need Khinchine's inequality based on the Rademacher functions on $[0,1]$. Recall that the Rademacher functions $\{r_j(t)\}_{j\geqslant 0}$ on $[0,1]$ are defined by

$$r_0(t)=\begin{cases} 1 \ (0\leqslant t-[t]<\frac{1}{2}), \\ -1 \ (\frac{1}{2}\leqslant t-[t]<1), \end{cases}$$
$$r_j(t)=r_0(2^j t) \qquad (j\geqslant 1).$$

Here $[t]$ denotes the largest integer not greater than t. The following result is well-known as Khinchine's inequality.

LEMMA 3. *Let $0<p<\infty$. There are constants $0<A_p\leqslant B_p<\infty$ such that for any positive integer m and any complex numbers $\{c_j\}_{j=1}^m$,*

$$A_p\left(\sum_{j=1}^m |c_j|^2\right)^{\frac{p}{2}} \leqslant \int_0^1 \left|\sum_{j=1}^m c_j r_j(t)\right|^p dt \leqslant B_p\left(\sum_{j=1}^m |c_j|^2\right)^{\frac{p}{2}}.$$

For a multi-index $\gamma=(\gamma_1,\ldots,\gamma_N)$ where each γ_j is a nonnegative integer, we write $|\gamma|=\sum_{j=1}^N \gamma_j$ and

$$\frac{\partial^{|\gamma|}f}{\partial z^\gamma}=\frac{\partial^{|\gamma|}f}{\partial z_1^{\gamma_1}\cdots\partial z_N^{\gamma_N}}$$

for $f\in H(\mathbb{C}^N)$. Furthermore we write $\mathscr{R}^m f(z)=\mathscr{R}[\mathscr{R}^{m-1}f](z)$ inductively. The Fock spaces $\mathscr{F}_\alpha^p(\mathbb{C}^N)$ $(0<p\leqslant\infty)$ have equivalent characterizations in terms of these higher order derivatives. The following two lemmas are helpful in proving main parts of Theorem 1.

LEMMA 4. *Let $0<p,\alpha<\infty$ and m be a positive integer. Then the following three quantities are equivalent:*

(a) $\|f\|_{p,\alpha}$,

(b) $\sum_{|\gamma|\leqslant m-1}\left|\frac{\partial^{|\gamma|}f}{\partial z^\gamma}(0)\right|+\left\{\sum_{|\gamma|=m}\int_{\mathbb{C}^N}\left|\frac{\partial^{|\gamma|}f}{\partial z^\gamma}(z)\frac{e^{-\frac{\alpha}{2}|z|^2}}{(1+|z|)^m}\right|^p dV(z)\right\}^{\frac{1}{p}}$,

(c) $|f(0)|+\left\{\int_{\mathbb{C}^N}\left|\frac{|\mathscr{R}^m f(z)|}{(1+|z|)^{2m}}e^{-\frac{\alpha}{2}|z|^2}\right|^p dV(z)\right\}^{\frac{1}{p}}$.

Proof. See Theorem 2.1 in [4]. □

LEMMA 5. *Let $\alpha>0$, m be a positive integer and $f\in H(\mathbb{C}^N)$. Then the following conditions are equivalent for all $f\in H(\mathbb{C}^N)$:*

(a) $f\in\mathscr{F}_\alpha^\infty(\mathbb{C}^N)$,

(b) $\max_{|\gamma|=m}\sup_{z\in\mathbb{C}^N}\left|\frac{\partial^{|\gamma|}f}{\partial z^\gamma}(z)\right|\frac{e^{-\frac{\alpha}{2}|z|^2}}{(1+|z|)^m}<\infty$,

(c) $\sup_{z\in\mathbb{C}^N} \frac{|\mathscr{R}^m f(z)|}{(1+|z|)^{2m}} e^{-\frac{\alpha}{2}|z|^2} < \infty.$

Furthermore, we have

$$\|f\|_{\infty,\alpha} \approx \sum_{|\gamma|\leqslant m-1} \left|\frac{\partial^{|\gamma|} f}{\partial z^\gamma}(0)\right| + \max_{|\gamma|=m} \sup_{z\in\mathbb{C}^N} \left|\frac{\partial^{|\gamma|} f}{\partial z^\gamma}(z)\right| \frac{e^{-\frac{\alpha}{2}|z|^2}}{(1+|z|)^m}$$
$$\approx |f(0)| + \sup_{z\in\mathbb{C}^N} \frac{|\mathscr{R}^m f(z)|}{(1+|z|)^{2m}} e^{-\frac{\alpha}{2}|z|^2}.$$

Proof. See Theorem 1 in [16]. □

Finally, we quote a result on a composition operator on $\mathscr{F}^\infty_\alpha(\mathbb{C}^N)$. We will need the following result in the proof of the direction (c)⇒(b) of Theorem 1.

LEMMA 6. *Let $\alpha > 0$. Suppose that $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ is an entire mapping which satisfies $|\varphi(z)| < |z|$ for all $z \in \mathbb{C}^N$ and $e^{\frac{\alpha}{2}(|\varphi(z)|^2-|z|^2)} \to 0$ as $|z| \to \infty$. Then the composition operator $C_\varphi : f \mapsto f\circ\varphi$ induced by φ is compact from $\mathscr{F}^\infty_\alpha(\mathbb{C}^N)$ into itself.*

Proof. Combining the condition $|\varphi(z)| < |z|$ with Theorem 1 in [12], we see that $C_\varphi : \mathscr{F}^\infty_\alpha(\mathbb{C}^N) \to \mathscr{F}^\infty_\alpha(\mathbb{C}^N)$ is bounded. Hence this lemma is a special case of Theorem 8 in [12]. We omit the detail of the proof. □

3. Proof of result

The proof of $(a) \Rightarrow (c)$. First we introduce the following contemporary notation:

$$d\mu_g(z) := \frac{|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|z|^2} dV(z) \quad (z\in\mathbb{C}^N).$$

Then for any $r > 0$ we obtain

$$\begin{aligned}\int_{\mathbb{C}^N} \mu_g(B(z,r))dV(z) &= \int_{\mathbb{C}^N} dV(z) \int_{\mathbb{C}^N} \chi_{B(z,r)}(w) \frac{|\mathscr{R}g(w)|^p}{(1+|w|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|w|^2} dV(w)\\ &= \int_{\mathbb{C}^N} dV(z) \int_{\mathbb{C}^N} \chi_{B(w,r)}(z) \frac{|\mathscr{R}g(w)|^p}{(1+|w|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|w|^2} dV(w)\\ &= \frac{(\pi r^2)^N}{N!} \int_{\mathbb{C}^N} \frac{|\mathscr{R}g(w)|^p}{(1+|w|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|w|^2} dV(w).\end{aligned}$$

This relation indicates that the condition (c) is equivalent to $\mu_g(B(\cdot,r))$ is in $L^1(\mathbb{C}^N,dV)$. Hence, by Lemma 2, it is enough to prove that $\{\mu_g(B(a_k,r))\} \in l^1$ for an r-lattice $\{a_k\}$

in $\mathbb{C}^N$. Take an r-lattice $\{a_k\}$ in $\mathbb{C}^N$ and let $\{r_k\}$ be the Rademacher functions on $[0,1]$. We consider the function F_t defined by

$$F_t(z) = \sum_{k=1}^{\infty} r_k(t) e^{\alpha\langle z,a_k\rangle - \frac{\alpha}{2}|a_k|^2} \qquad (z \in \mathbb{C}^N, t \in [0,1]).$$

As proved in [5, Theorem 8.2] or [3, Lemma 2.4], we see that $F_t \in \mathscr{F}_\alpha^\infty(\mathbb{C}^N)$ and $\|F_t\|_{\infty,\alpha} \lesssim 1$ uniformly in t. Thus the boundedness of $V_g : \mathscr{F}_\alpha^\infty(\mathbb{C}^N) \to \mathscr{F}_\beta^p(\mathbb{C}^N)$ and Lemma 4 show

$$\begin{aligned}\int_{\mathbb{C}^N} |F_t(z)|^p e^{-\frac{p\alpha}{2}|z|^2} d\mu_g(z) &= \int_{\mathbb{C}^N} \frac{|F_t(z)|^p |\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{-\frac{p\beta}{2}|z|^2} dV(z) \\ &\lesssim \|V_g F_t\|_{p,\beta}^p \lesssim 1. \end{aligned} \tag{3}$$

On the other hand, we put $c_k = e^{\alpha\langle z,a_k\rangle - \frac{\alpha}{2}(|z|^2+|a_k|^2)}$ in Lemma 3. Then Lemma 3 and Fubini's theorem give

$$\begin{aligned}\int_{\mathbb{C}^N} \left(\sum_{k=1}^{\infty} e^{-\alpha|z-a_k|^2} \right)^{\frac{p}{2}} d\mu_g(z) &\lesssim \int_{\mathbb{C}^N} d\mu_g(z) \int_0^1 \left| \sum_{k=1}^{\infty} e^{\alpha\langle z,a_k\rangle - \frac{\alpha}{2}(|z|^2+|a_k|^2)} r_k(t) \right|^p dt \\ &= \int_0^1 dt \int_{\mathbb{C}^N} |F_t(z)|^p e^{-\frac{p\alpha}{2}|z|^2} d\mu_g(z) \end{aligned} \tag{4}$$

By relations (3) and (4) we obtain

$$\int_{\mathbb{C}^N} \left(\sum_{k=1}^{\infty} e^{-\alpha|z-a_k|^2} \right)^{\frac{p}{2}} d\mu_g(z) \lesssim 1. \tag{5}$$

For any $R > r$ the property (iii) of the r-lattice $\{a_k\}$ implies that there is a positive integer M which depends only on r and R such that

$$\sum_{j=1}^{\infty} \int_{B(a_j,R)} \left(\sum_{k=1}^{\infty} e^{-\alpha|z-a_k|^2} \right)^{\frac{p}{2}} d\mu_g(z) \leqslant M \int_{\mathbb{C}^N} \left(\sum_{k=1}^{\infty} e^{-\alpha|z-a_k|^2} \right)^{\frac{p}{2}} d\mu_g(z).$$

Hence we obtain

$$\begin{aligned}\int_{\mathbb{C}^N} \left(\sum_{k=1}^{\infty} e^{-\alpha|z-a_k|^2} \right)^{\frac{p}{2}} d\mu_g(z) &\geqslant \frac{1}{M} \sum_{j=1}^{\infty} \int_{B(a_j,R)} \left(\sum_{k=1}^{\infty} e^{-\alpha|z-a_k|^2} \right)^{\frac{p}{2}} d\mu_g(z) \\ &\geqslant \frac{1}{M} \sum_{j=1}^{\infty} \int_{B(a_j,r)} \left(\sum_{k=1}^{\infty} e^{-\alpha|z-a_k|^2} \right)^{\frac{p}{2}} d\mu_g(z) \\ &\geqslant \frac{1}{M} \sum_{j=1}^{\infty} \int_{B(a_j,r)} \left(e^{-\alpha|z-a_j|^2} \right)^{\frac{p}{2}} d\mu_g(z) \\ &\geqslant \frac{e^{-\frac{p\alpha r^2}{2}}}{M} \sum_{j=1}^{\infty} \mu_g(B(a_j,r)). \end{aligned}$$

Combining this with (5) we see that the sequence $\{\mu_g(B(a_k,r))\}$ is in l^1. Hence we have the desired claim, which completes the proof of (a)⇒(c). □

The proof of $(c)\Rightarrow(b)$. Suppose that the z-variable function $\frac{|\mathscr{R}g(z)|}{(1+|z|)^2}e^{\frac{\alpha-\beta}{2}|z|^2}$ is in $L^p(\mathbb{C}^N,dV)$. Then by Proposition 1 we see that $V_g:\mathscr{F}^\infty_\alpha(\mathbb{C}^N)\to\mathscr{F}^p_\beta(\mathbb{C}^N)$ is bounded. In order to deduce the compactness of V_g, we will show that the essential norm $\|V_g\|_e$ is equal to 0.

For a positive integer k, we consider the following entire mapping:

$$\varphi_k(z)=\left(\frac{k}{k+1}z_1,\dots,\frac{k}{k+1}z_N\right)\qquad (z\in\mathbb{C}^N).$$

Then φ_k satisfies that $|\varphi_k(z)|<|z|$ and

$$e^{\frac{\alpha}{2}(|\varphi_k(z)|^2-|z|^2)}=e^{-\frac{\alpha(2k+1)|z|^2}{2(k+1)^2}}\to 0$$

as $|z|\to\infty$, uniformly in k. Lemma 6 implies that the composition operator C_{φ_k} is compact on $\mathscr{F}^\infty_\alpha(\mathbb{C}^N)$. Hence the product operator $V_gC_{\varphi_k}$ is also compact from $\mathscr{F}^\infty_\alpha(\mathbb{C}^N)$ into $\mathscr{F}^p_\beta(\mathbb{C}^N)$. By the definition of $\|V_g\|_e$ we have

$$\|V_g\|_e^p\leqslant\sup_{\|f\|_{\infty,\alpha}\leqslant 1}\|V_g(I-C_{\varphi_k})f\|^p_{p,\beta},\tag{6}$$

where I denotes the identity operator on $\mathscr{F}^\infty_\alpha(\mathbb{C}^N)$. It follows from Lemma 4 that the right term in (6) is dominated by

$$\sup_{\|f\|_{\infty,\alpha}\leqslant 1}\int_{\mathbb{C}^N}\frac{|\mathscr{R}[V_g(I-C_{\varphi_k})f](z)|^p}{(1+|z|)^{2p}}e^{-\frac{p\beta}{2}|z|^2}dV(z).\tag{7}$$

Fix $\varepsilon>0$ arbitrarily. The assumption (c) indicates that we can choose $R>0$ such that

$$\int_{|z|>R}\frac{|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}}e^{\frac{p(\alpha-\beta)}{2}|z|^2}dV(z)<\varepsilon.\tag{8}$$

Take a positive integer k and $f\in\mathscr{F}^\infty_\alpha(\mathbb{C}^N)$ with $\|f\|_{\infty,\alpha}\leqslant 1$. Note that it holds

$$\begin{aligned}\mathscr{R}[V_g(I-C_{\varphi_k})f](z)&=\mathscr{R}[V_gf](z)-\mathscr{R}[V_g(f\circ\varphi_k)](z)\\&=f(z)\mathscr{R}g(z)-f(\varphi_k(z))\mathscr{R}g(z).\end{aligned}$$

Thus (8) gives

$$\begin{aligned}&\int_{|z|>R}\frac{|f(z)|^p|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}}e^{-\frac{p\beta}{2}|z|^2}dV(z)\\&\leqslant\|f\|^p_{\infty,\alpha}\int_{|z|>R}\frac{|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}}e^{\frac{p(\alpha-\beta)}{2}|z|^2}dV(z)<\varepsilon.\end{aligned}\tag{9}$$

Furthermore the relation $|\varphi_k(z)| < |z|$ implies

$$\int_{|z|>R} \frac{|f(\varphi_k(z))|^p |\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{-\frac{p\beta}{2}|z|^2} dV(z)$$
$$\leqslant \int_{|z|>R} |f(\varphi_k(z))|^p e^{-\frac{p\alpha}{2}|\varphi_k(z)|^2} \frac{|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|z|^2} dV(z) < \varepsilon. \tag{10}$$

On the other hand, we have

$$\int_{|z|\leqslant R} \frac{|\mathscr{R}[V_g(I-C_{\varphi_k})f](z)|^p}{(1+|z|)^{2p}} e^{-\frac{p\beta}{2}|z|^2} dV(z)$$
$$\leqslant \left(\sup_{|z|\leqslant R} |f(z)-f(\varphi_k(z))| e^{-\frac{\alpha}{2}|z|^2}\right)^p \int_{|z|\leqslant R} \frac{|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|z|^2} dV(z)$$
$$\leqslant \left(\sup_{|z|\leqslant R} |f(z)-f(\varphi_k(z))|\right)^p \int_{\mathbb{C}^N} \frac{|\mathscr{R}g(z)|^p}{(1+|z|)^{2p}} e^{\frac{p(\alpha-\beta)}{2}|z|^2} dV(z). \tag{11}$$

By using the mean value theorem in $|f(z)-f(\varphi_k(z))|$, we see

$$\sup_{|z|\leqslant R} |f(z)-f(\varphi_k(z))| \leqslant \sup_{|z|\leqslant R} \frac{|z|}{k+1} \sup_{|w|\leqslant R} |\nabla f(w)|.$$

Since $|\nabla f(w)| \leqslant \sqrt{N} \max_{1\leqslant j\leqslant N} |\frac{\partial f}{\partial w_j}(w)|$, Lemma 5 gives

$$\begin{aligned}
\sup_{|z|\leqslant R} |f(z)-f(\varphi_k(z))| &\leqslant \sup_{|z|\leqslant R} \frac{|z|}{k+1} \sup_{|w|\leqslant R} |\nabla f(w)| \\
&\leqslant \frac{R(R+1)e^{\frac{\alpha}{2}R^2}}{k+1} \sup_{|w|\leqslant R} \frac{|\nabla f(w)|}{(1+|w|)} e^{-\frac{\alpha}{2}|w|^2} \\
&\leqslant \sqrt{N} \frac{R(R+1)e^{\frac{\alpha}{2}R^2}}{k+1} \sup_{|w|\leqslant R} \max_{1\leqslant j\leqslant N} \frac{|\frac{\partial f}{\partial w_j}(w)|}{(1+|w|)} e^{-\frac{\alpha}{2}|w|^2} \\
&\lesssim \sqrt{N} \frac{R(R+1)e^{\frac{\alpha}{2}R^2}}{k+1} \|f\|_{\infty,\alpha} \to 0
\end{aligned}$$

as $k\to\infty$. Combining this with (11) and the assumption (c), we obtain

$$\lim_{k\to\infty} \sup_{\|f\|_{\infty,\alpha}\leqslant 1} \int_{|z|\leqslant R} \frac{|\mathscr{R}[V_g(I-C_{\varphi_k})f](z)|^p}{(1+|z|)^{2p}} e^{-\frac{p\beta}{2}|z|^2} dV(z) = 0. \tag{12}$$

Hence estimates (9), (10) and (12) show that

$$\sup_{\|f\|_{\infty,\alpha}\leqslant 1} \int_{\mathbb{C}^N} \frac{|\mathscr{R}[V_g(I-C_{\varphi_k})f](z)|^p}{(1+|z|)^{2p}} e^{-\frac{p\beta}{2}|z|^2} dV(z) \lesssim \varepsilon$$

if letting $k\to\infty$ in (7), and so $\|V_g\|_e^p \lesssim \varepsilon$. Since $\varepsilon>0$ was arbitrarily, this implies $\|V_g\|_e = 0$, namely V_g is compact from $\mathscr{F}_\alpha^\infty(\mathbb{C}^N)$ into $\mathscr{F}_\beta^p(\mathbb{C}^N)$. We accomplish the proof. □

4. Examples

Now we describe examples of g which induces the bounded (and also compact) operator $V_g : \mathscr{F}^\infty_\alpha(\mathbb{C}) \to \mathscr{F}^p_\beta(\mathbb{C})$. In order to explain the examples briefly, we deal with the case $N = 1$ only.

The case $\alpha > \beta$. First we observe

$$\left(\frac{|a_j| r^j}{1+r} e^{\frac{\alpha-\beta}{2} r^2}\right)^p \lesssim \int_{\mathbb{C}} \frac{|F(w)|^p}{(1+|w|)^p} e^{\frac{p(\alpha-\beta)}{2}|w|^2} dV(w) \tag{13}$$

for $F \in H(\mathbb{C})$ with $F(z) = \sum_{j \geqslant 0} a_j z^j$ and any $r > 0$. We consider the entire function

$$f(w) = F(w+z) e^{(\alpha-\beta) w\bar{z} + \frac{\alpha-\beta}{2}|z|^2}.$$

As in the proof of Lemma 1, the subharmonic property of $|f|^p$ and the relation

$$|f(w)|^p e^{\frac{p(\alpha-\beta)}{2}|w|^2} = |F(w+z)|^p e^{\frac{p(\alpha-\beta)}{2}|w+z|^2}$$

give

$$\left(\frac{|F(z)|}{1+|z|} e^{\frac{\alpha-\beta}{2}|z|^2}\right)^p \lesssim \int_{B(z,1)} \frac{|F(w)|^p}{(1+|w|)^p} e^{\frac{p(\alpha-\beta)}{2}|w|^2} dV(w).$$

Since this estimate is uniform in z, we also have

$$\left(\frac{\sup_{|z|=r}|F(z)|}{1+r} e^{\frac{\alpha-\beta}{2} r^2}\right)^p \lesssim \int_{\mathbb{C}} \frac{|F(w)|^p}{(1+|w|)^p} e^{\frac{p(\alpha-\beta)}{2}|w|^2} dV(w).$$

Combining this with the fact $|a_j| r^j \lesssim \sup_{|z|=r}|F(z)|$, we obtain the desired estimate (13). Hence if $\int_{\mathbb{C}} \frac{|F(w)|^p}{(1+|w|)^p} e^{\frac{p(\alpha-\beta)}{2}|w|^2} dV(w) < \infty$, then (13) together with the assumption $\alpha > \beta$ shows for any integer $j \geqslant 0$

$$|a_j| \lesssim r^{1-j} e^{-\frac{\alpha-\beta}{2} r^2} \to 0$$

as $r \to \infty$, that is $F \equiv 0$. By applying this argument to g', we see that g must be the constant function. However $V_g \equiv 0$ for a constant function g.

The case $\alpha = \beta$. Since the arguments used to derive inequality (13) are applicable to this case as well, we obtain

$$\frac{|a_j| r^j}{1+r} \lesssim \left(\int_{\mathbb{C}} \frac{|g'(w)|^p}{(1+|w|)^p} dV(w)\right)^{\frac{1}{p}}$$

for $g'(z) = \sum_{j \geqslant 0} a_j z^j$ and a nonnegative integer j. This inequality and (c) in Corollary 1 imply that $a_j = 0$ if $j \geqslant 2$, and so $g(z) = az^2 + bz + c$. Since $|g'(z)| \approx 1 + |z|$ if $a \neq 0$, we see that the above integral is not finite, so a polynomial g with $\deg(g) = 2$ does not induce the bounded operator V_g from $\mathscr{F}_\alpha^\infty(\mathbb{C})$ into $\mathscr{F}_\alpha^p(\mathbb{C})$. Thus we put $g(z)=bz+c$. If $p>2$, then

$$\int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+|z|)^p} dV(z) \approx \int_0^\infty \frac{r}{(1+r)^p} dr < \infty,$$

and so $g(z)=bz+c$ induces the bounded $V_g : \mathscr{F}_\alpha^\infty(\mathbb{C}) \to \mathscr{F}_\alpha^p(\mathbb{C})$.

The case $\alpha<\beta$. We put $g(z)=\int_0^z e^{-\frac{\alpha-\beta}{2}\zeta^2} d\zeta$. Since

$$|g'(z)|^p = e^{-\frac{p(\alpha-\beta)}{2}\mathrm{Re}(z^2)} \leqslant e^{-\frac{p(\alpha-\beta)}{2}|z|^2},$$

we also see

$$\int_{\mathbb{C}} \frac{|g'(z)|^p}{(1+|z|)^p} e^{\frac{p(\alpha-\beta)}{2}|z|^2} dV(z) \lesssim \int_0^\infty \frac{r}{(1+r)^p} dr < \infty$$

if $p>2$. Hence this function g induces the bounded operator $V_g : \mathscr{F}_\alpha^\infty(\mathbb{C}) \to \mathscr{F}_\beta^p(\mathbb{C})$ when $p>2$.

Acknowledgement. The author would like to thank the anonymous referee for careful review of our paper. The referee has made valuable comments and suggestions for improving the earlier version of the paper. This research is partly supported by JSPS KAKENHI Grants-in-Aid for Scientific Research (C), Grant Number 21K03301.

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(Received August 10, 2021)

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