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HARDY AND SOBOLEV INEQUALITIES FOR DOUBLE PHASE FUNCTIONALS ON THE UNIT BALL

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Abstract. We prove Hardy and Sobolev inequalities for double phase functionals $\Phi(x,t) = t^p + (b(x)t)^q$ on the unit ball $\mathbf{B}$, as a continuation of our paper [26], where $1 \leqslant p < q$, $b(\cdot)$ is non-negative and (radially) Hölder continuous of order $\theta \in (0,1]$. The Sobolev conjugate for Φ is given by $\Phi^*(x,t) = t^{p^*} + (b(x)t)^{q^*}$, where p^* and q^* denote the Sobolev exponent of p and q, respectively, that is, $1/p^* = 1/p - 1/n$ and $1/q^* = 1/q - 1/n$.

1. Introduction

The Hardy-Sobolev inequality says that for $f \geqslant 0$

$$\left(\int_0^1 \left(\int_0^x f(y)dy \right)^q (1-x)^\alpha dx \right)^{1/q} \leqslant C \left(\int_0^1 f(y)^p (1-y)^\beta dy \right)^{1/p}, \tag{1}$$

where $1 \leqslant p \leqslant q$, $\beta > p-1$ and $\alpha = \beta q/p - q/p' - 1$ $(1/p + 1/p' = 1)$ and

$$\left(\int_0^1 \left(\int_x^1 f(y)dy \right)^q (1-x)^\alpha dx \right)^{1/q} \leqslant C \left(\int_0^1 f(y)^p (1-y)^\beta dy \right)^{1/p}, \tag{2}$$

where $1 \leqslant p \leqslant q$, $\beta < p-1$ and $\alpha = \beta q/p - q/p' - 1$ (see e.g. [10, 15, 16, 21, 22, 28]).

The double phase functional introduced by Zhikov ([31]) in the 1980s has been studied intensively by many researchers. Regarding regularity theory of differential equations, Baroni, Colombo and Mingione [1, 2, 5, 6] studied a double phase functional

$$\tilde{\Phi}(x,t) = t^p + a(x)t^q,\ x \in \mathbb{R}^n,\ t \geqslant 0,$$

where $1 \leqslant p < q$, $a(\cdot)$ is non-negative, bounded and Hölder continuous of order $\theta \in (0,1]$. We refer to [17, 24, 25] for Sobolev inequality for double phase functionals and to [19, 20] for variational problems with nonstandard growth. For other recent works, see e.g. [3, 4, 7, 8, 9, 11, 12, 13, 14, 23, 27, 29].

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Let $\mathbf{B}$ be the unit ball $B(0,1)$ of $\mathbb{R}^n$. In the present paper, relaxing the continuity of $a(\cdot)$, we consider the case $\Phi(x,t)$ is a double phase functional given by

$$\Phi(x,t) = t^p + (b(x)t)^q,$$

where $1 \leqslant p < q$, $b(\cdot)$ is non-negative and (radially) Hölder continuous of order $\theta \in (0,1]$, that is,

$$|b(x) - b(y)| \leqslant C||x| - |y||^\theta \quad \text{for all } x, y \in \mathbf{B}.$$

Note that if we write

$$\Phi(x,t) = t^p + a(x)t^q$$

with

$$a(x) = b(x)^q,$$

then a is not always Hölder continuous of order θq.

In the previous paper [26], Hardy-Sobolev inequalities were established when

$$\int_{\mathbf{B}} \Phi(y, (1-|y|)^{\beta/q}|f(y)|)dy \leqslant 1 \tag{3}$$

with $\beta > q-1$ and $1/q = 1/p - \theta > 0$ and

$$\int_{\mathbf{B}} \Phi(y, (1-|y|)^{\beta/p}|f(y)|)dy \leqslant 1 \tag{4}$$

with $\beta < p-1$.

The Sobolev conjugate Φ^* for Φ will be given in Section 3 by

$$\Phi^*(x,t) = t^{p^*} + (b(x)t)^{q^*},$$

where p^* and q^* denote the Sobolev exponent of p and q, respectively, that is, $1/p^* = 1/p - 1/n$ and $1/q^* = 1/q - 1/n$. Our aim in this paper is to give a continuation of our paper [26] by the use of $\Phi^*(x,t)$, which was not used in [26], when (3) holds with $\beta > q-1$ and $1/q = 1/p - \theta > 0$ in Theorem 3.1 and (4) holds with $\beta < p-1$ in Theorem 3.3. Our strategy is to apply Theorems 2.1 and 2.4, which are extensions of the classical Hardy-Sobolev inequalities (1) and (2) to the unit ball. which are the Hardy-Sobolev inequality in $\mathbf{B}$. Both Theorems 3.1 and 3.3 are strongly affected by double phase as will be seen in Remark 4.1 below.

As an application of Theorems 2.1, 2.4, 3.1 and 3.3, we discuss behaviors near the sphere for functions in $C^1(\mathbb{R}^n)$ (see Propositions given after the theorems).

Our final aim is to treat the borderline case in Theorems 3.1 and 3.3. For this purpose, we prepare Theorems 5.1 and 5.4, which are the critical case of Theorems 2.1 and 2.4.

Throughout this paper, let C denote various positive constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \leqslant g \leqslant Ch$ for some constant $C > 0$.

2. Hardy-Sobolev inequality in B

Let $B(x,r)$ denote the open ball centered at $x \in \mathbf{B}$ with radius r.
In $\mathbf{B}$, Hardy-Sobolev inequality can be stated in the following (cf. [28]):

THEOREM 2.1. ([26, Theorem 2.1]) *Let* $1 \leqslant p \leqslant q$, $\beta > p-1$ *and* $\alpha = \beta q/p - q/p' - 1$, *where* $1/p + 1/p' = 1$. *Then there exists a constant* $C > 0$ *such that*

$$\left(\int_{\mathbf{B}} \left(\int_{B(0,|x|)} |f(y)| dy\right)^q (1-|x|)^\alpha dx\right)^{1/q} \leqslant C \left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^\beta dy\right)^{1/p}.$$

By Theorem 2.1 we discuss a behavior near the sphere.

PROPOSITION 2.2. (cf. [18, Theorem 6.5], [22, Theorem 4.3.1], [30]) *Let* $1 \leqslant p \leqslant q$ *and* $0 < \varepsilon < 1/q$. *Let* u *be a function in* $C^1(\mathbb{R}^n)$ *such that*

$$\int_{\mathbf{B}} (|\nabla u(y)|(1-|y|)^\varepsilon)^p (1-|y|)^{p-1} dy < \infty.$$

Then

$$\lim_{r \to 1} (1-r)^\varepsilon \frac{1}{|\mathbf{B} \setminus B(0,r)|} \int_{\mathbf{B} \setminus B(0,r)} |u(x)| dx = 0$$

and

$$\lim_{r \to 1} (1-r)^\varepsilon \left(\frac{1}{|S(0,r)|} \int_{S(0,r)} |u(r\xi)| dS(\xi)\right) = 0,$$

where $|S(0,r)|$ *denotes the surface area of the spherical surface* $S(0,r)$.

Proof. First note

$$\begin{aligned} U_0(r\xi) &:= |u(r\xi) - u(0)| \\ &= \left|\int_0^r \frac{d}{dt} u(t\xi) dt\right| \\ &\leqslant \int_0^r |\nabla u(t\xi)| dt \end{aligned}$$

for $0 < r < 1$ and $|\xi| = 1$. Then

$$\frac{1}{|S(0,1)|} \int_{S(0,1)} U_0(r\xi) dS(\xi) \leqslant \frac{1}{|S(0,1)|} \int_{B(0,r)} |\nabla u(y)| |y|^{1-n} dy. \tag{5}$$

Thus for $r > 0$ Theorem 2.1 gives

$$\begin{aligned}
&(1-r)^{\varepsilon}\frac{1}{1-r}\int_r^1\left(\frac{1}{|S(0,1)|}\int_{S(0,1)}U_0(t\xi)dS(\xi)\right)dt\\
&\leqslant (1-r)^{\varepsilon}\left(\frac{1}{1-r}\int_r^1\left(\frac{1}{|S(0,1)|}\int_{B(0,t)}|\nabla u(y)|\,|y|^{1-n}dy\right)^q dt\right)^{1/q}\\
&\leqslant C\left(\int_{\mathbf{B}\setminus B(0,r)}\left(\int_{B(0,t)}|\nabla u(y)|\,|y|^{1-n}dy\right)^q(1-|x|)^{\alpha}|x|^{1-n}dx\right)^{1/q}\\
&\leqslant Cr^{(1-n)/q}\left(\int_{\mathbf{B}\setminus B(0,r)}\left(|\nabla u(y)|\,|y|^{1-n}\right)^p(1-|y|)^{\beta}dy\right)^{1/p}\\
&\leqslant C\int_{\mathbf{B}\setminus B(0,r)}\left(|\nabla u(y)|(1-|y|)^{\varepsilon}\right)^p(1-|y|)^{p-1}dy
\end{aligned}$$

since $(1-n)/q+1-n<0$, where $\beta=(p-1)+\varepsilon p>p-1$ and $\alpha=\beta q/p-q/p'-1=\varepsilon q-1<0$. Here note

$$\begin{aligned}
&(1-r)^{\varepsilon}\frac{1}{1-r}\int_r^1\left(\frac{1}{|S(0,1)|}\int_{S(0,1)}U_0(t\xi)dS(\xi)\right)dt\\
&\sim (1-r)^{\varepsilon}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)}U_0(x)dx \quad \text{for } 0<r<1,
\end{aligned}$$

which yields

$$\lim_{r\to 1}(1-r)^{\varepsilon}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)}U_0(x)dx=0.$$

Moreover, it is seen from (5) that

$$\lim_{r\to 1}(1-r)^{\varepsilon}\left(\frac{1}{|S(0,r)|}\int_{S(0,r)}U_0(r\xi)dS(\xi)\right)=0.$$

Noting that $|u(x)|\leqslant|U_0(x)|+|u(0)|$, we establish the required result. $\square$

REMARK 2.3. Consider the function

$$u(x)=(1-|x|)^{-a}-1$$

for $a>0$. Then

(1) $|\nabla u(x)|=a(1-|x|)^{-a-1}$ and

$$\int_{\mathbf{B}}(|\nabla u(x)|(1-|x|)^{\varepsilon})^p(1-|x|)^{p-1}dx<\infty$$

if and only if $(-a-1+\varepsilon)p+p>0$;

(2)

$$\lim_{r\to 1}(1-r)^{\varepsilon}\left(\frac{1}{|S(0,r)|}\int_{S(0,r)} u(r\xi)dS(\xi)\right)=0$$

if and only if $-a+\varepsilon>0$.

This implies the best possibility as to the power ε in Proposition 2.2.

THEOREM 2.4. ([26, Theorem 2.2]) *Let* $1\leqslant p\leqslant q$, $\beta<p-1$ *and* $\alpha=\beta q/p-q/p'-1$. *Then there exists a constant* $C>0$ *such that*

$$\left(\int_{\mathbf{B}}\left(\int_{\mathbf{B}\setminus B(0,|x|)}|f(y)|dy\right)^{q}(1-|x|)^{\alpha}dx\right)^{1/q}\leqslant C\left(\int_{\mathbf{B}}|f(y)|^{p}(1-|y|)^{\beta}dy\right)^{1/p}.$$

By Theorem 2.4 we discuss a behavior near the sphere, as in Proposition 2.2.

PROPOSITION 2.5. *Let* $p\geqslant 1$ *and* $\varepsilon>0$. *Let* u *be a function in* $C^1(\mathbb{R}^n)$ *such that*

$$\int_{\mathbf{B}}\left(|\nabla u(y)|(1-|y|)^{-\varepsilon}\right)^{p}(1-|y|)^{p-1}dy<\infty.$$

Set

$$U_1(x)=U_1(r\xi)=\limsup_{t\to 1}\ |u(r\xi)-u(t\xi)|.$$

Then

$$\lim_{r\to 1}(1-r)^{-\varepsilon}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)}U_1(x)dx=0$$

and

$$\lim_{r\to 1}(1-r)^{-\varepsilon}\left(\frac{1}{|S(0,r)|}\int_{S(0,r)}U_1(r\xi)dS(\xi)\right)=0.$$

Proof. To show this, note

$$\begin{aligned}|u(r\xi)-u(t\xi)|&=\left|\int_r^t\frac{d}{dt}u(t\xi)dt\right|\\&\leqslant\int_r^t|\nabla u(t\xi)|dt\end{aligned}$$

for $0<r<t<1$ and $|\xi|=1$. Then

$$\frac{1}{|S(0,1)|}\int_{S(0,1)}U_1(r\xi)dS(\xi)\leqslant\frac{1}{|S(0,1)|}\int_{\mathbf{B}\setminus B(0,r)}|\nabla u(y)|\,|y|^{1-n}dy. \tag{6}$$

Thus for $r > 0$ Theorem 2.4 gives

$$
\begin{aligned}
&(1-r)^{-\varepsilon}\frac{1}{1-r}\int_r^1\left(\frac{1}{|S(0,1)|}\int_{S(0,1)}U_1(t\xi)dS(\xi)\right)dt\\
&\leqslant (1-r)^{-\varepsilon}\left(\frac{1}{1-r}\int_r^1\left(\frac{1}{|S(0,1)|}\int_{\mathbf{B}\setminus B(0,t)}|\nabla u(y)|\,|y|^{1-n}dy\right)^q dt\right)^{1/q}\\
&\leqslant C\left(\int_{\mathbf{B}\setminus B(0,r)}\left(\int_{\mathbf{B}\setminus B(0,t)}|\nabla u(y)|\,|y|^{1-n}dy\right)^q(1-|x|)^{\alpha}|x|^{1-n}dx\right)^{1/q}\\
&\leqslant Cr^{(1-n)/q}\left(\int_{\mathbf{B}\setminus B(0,r)}\left(|\nabla u(y)|\,|y|^{1-n}\right)^p(1-|y|)^{\beta}dy\right)^{1/p}\\
&\leqslant C\int_{\mathbf{B}\setminus B(0,r)}\left(|\nabla u(y)|(1-|y|)^{-\varepsilon}\right)^p(1-|y|)^{p-1}dy,
\end{aligned}
$$

where $1 \leqslant p \leqslant q$, $\beta = (p-1)-\varepsilon p < p-1$ and $\alpha = \beta q/p - q/p' - 1 = -\varepsilon q - 1$. Here note

$$
\begin{aligned}
&(1-r)^{-\varepsilon}\frac{1}{1-r}\int_r^1\left(\frac{1}{|S(0,1)|}\int_{S(0,1)}U_1(t\xi)dS(\xi)\right)dt\\
&\sim (1-r)^{-\varepsilon}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)}U_1(x)dx \quad \text{for } 0<r<1,
\end{aligned}
$$

which yields

$$
\lim_{r\to 1}(1-r)^{-\varepsilon}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)}U_1(x)dx = 0.
$$

Moreover, it is seen from (6) that

$$
\lim_{r\to 1}(1-r)^{-\varepsilon}\left(\frac{1}{|S(0,r)|}\int_{S(0,r)}U_1(r\xi)dS(\xi)\right) = 0. \quad \square
$$

REMARK 2.6. Consider the function

$$
u(x) = (1-|x|)^a
$$

for $a > 0$. Then

(1) $|\nabla u(x)| = a(1-|x|)^{a-1}$ and

$$
\int_{\mathbf{B}}(|\nabla u(x)|(1-|x|)^{-\varepsilon})^p(1-|x|)^{p-1}dx < \infty
$$

if and only if $(a-1-\varepsilon)p + p > 0$;

(2)

$$
\lim_{r\to 1}(1-r)^{-\varepsilon}\left(\frac{1}{|S(0,r)|}\int_{S(0,r)}u(r\xi)dS(\xi)\right) = 0
$$

if and only if $a - \varepsilon > 0$.

This implies the best possibility as to the power ε in Proposition 2.5.

3. Hardy-Sobolev inequality in $\mathbf{B}$ for double phase functionals

In this section, we give Hardy-Sobolev inequality in $\mathbf{B}$ when Φ is a double phase functional.

Let p^* and q^* denote the Sobolev exponent of p and q, respectively, that is, $1/p^* = 1/p - 1/n$ and $1/q^* = 1/q - 1/n$. The Sobolev conjugate for Φ is given by

$$\Phi^*(x,t) = t^{p^*} + (b(x)t)^{q^*}.$$

THEOREM 3.1. *Let $1 \leqslant p < q < n$, $\beta > q-1$ and $1/q = 1/p - \theta > 0$. Set $F(x) = \int_{B(0,|x|)} |f(y)|dy$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbf{B}} \Phi^*\left(x, (1-|x|)^{\beta/q - 1/n'} F(x)\right) dx \leqslant C$$

when $\int_{\mathbf{B}} \Phi(y, (1-|y|)^{\beta/q}|f(y)|)dy \leqslant 1$.

Proof. First note from Theorem 2.1

$$\left(\int_{\mathbf{B}} \left((1-|x|)^{\beta/q-1/n'} \int_{B(0,|x|)} |f(y)|dy\right)^{p^*} dx\right)^{1/p^*}$$
$$\leqslant C\left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{\beta p/q} dy\right)^{1/p}$$

when $(\beta/q - 1/n')p^* = (\beta p/q)p^*/p - p^*/p' - 1$ and $\beta p/q > p-1$ since $\beta > q-1$. In this case,

$$\left(\int_{\mathbf{B}} ((1-|x|)^{\beta/q-1/n'} F(x))^{p^*} dx\right)^{1/p^*} \leqslant C\left(\int_{\mathbf{B}} ((1-|y|)^{\beta/q}|f(y)|)^p dy\right)^{1/p}.$$

Next note

$$\begin{aligned} b(x)F(x) &= \int_{B(0,|x|)} |f(y)|\{b(x)-b(y)\}dy + \int_{B(0,|x|)} |f(y)|b(y)dy \\ &\leqslant C\int_{B(0,|x|)} |f(y)|(1-|y|)^{\theta} dy + \int_{B(0,|x|)} |f(y)|b(y)dy. \end{aligned}$$

Theorem 2.1 gives

$$\left(\int_{\mathbf{B}} \left((1-|x|)^{\beta/q-1/n'} \int_{B(0,|x|)} |f(y)|(1-|y|)^{\theta} dy\right)^{q^*} dx\right)^{1/q^*}$$
$$\leqslant C\left(\int_{\mathbf{B}} |f(y)(1-|y|)^{\theta}|^p (1-|y|)^{(\beta/q-\theta)p} dy\right)^{1/p}$$
$$\leqslant C\left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{\beta p/q} dy\right)^{1/p}$$

when $(\beta/q-1/n')q^* = (\beta p/q-\theta p)q^*/p - q^*/p' - 1$ and $\beta p/q-\theta p > p-1$ since $\beta > q-1$. Moreover,

$$\left(\int_{\mathbf{B}}\left((1-|x|)^{\beta/q-1/n'}\int_{B(0,|x|)}|f(y)|b(y)dy\right)^{q^*}dx\right)^{1/q^*}$$
$$\leqslant C\left(\int_{\mathbf{B}}|f(y)b(y)|^q(1-|y|)^\beta dy\right)^{1/q}$$

since $(\beta/q-1/n')q^* = \beta q^*/q - q^*/q' - 1$ and $\beta > q-1$. Thus

$$\int_{\mathbf{B}}\left((1-|x|)^{\beta/q-1/n'}b(x)F(x)\right)^{q^*}dx \leqslant C$$

when $\int_{\mathbf{B}}\Phi(y,(1-|y|)^{\beta/q}|f(y)|)dy \leqslant 1$. This completes the proof. $\square$

In the same way as Proposition 2.2, we have the following result.

PROPOSITION 3.2. *Let* $1 \leqslant p < q < n$, $\beta > q-1$ *and* $1/q = 1/p-\theta > 0$. *Let* u *be a function in* $C^1(\mathbb{R}^n)$ *such that*

$$\int_{\mathbf{B}}\Phi(y,(1-|y|)^{\beta/q}|\nabla u(y)|)dy < \infty.$$

Then

$$\lim_{r\to 1}\left((1-r)^{\beta/q-1/p'}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)}|u(x)|dx\right.$$
$$\left.+(1-r)^{\beta/q-1/q'}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)}b(x)|u(x)|dx\right)=0$$

and

$$\lim_{r\to 1}\left((1-r)^{\beta/q-1/p'}\frac{1}{|S(0,r)|}\int_{S(0,r)}|u(r\xi)|dS(\xi)\right.$$
$$\left.+(1-r)^{\beta/q-1/q'}\frac{1}{|S(0,r)|}\int_{S(0,r)}b(r\xi)|u(r\xi)|dS(\xi)\right)=0.$$

For this, note that

$$(\beta/q-1/n')p^* = (\beta p/q)p^*/p - p^*/p' - 1 \quad \text{and} \quad \beta p/q = (\beta/q-1/p')p + p - 1$$

and

$$(\beta/q-1/n')q^* = \beta q^*/q - q^*/q' - 1 \quad \text{and} \quad \beta = (\beta/q-1/q')q + q - 1.$$

Next we consider the dual Hardy operator.

THEOREM 3.3. *Let* $1 \leqslant p < q < n$, $\beta < p-1$ *and* $1/q = 1/p - \theta > 0$. *Set* $G(x) = \int_{\mathbf{B}\setminus B(0,|x|)} |f(y)|dy$. *Then there exists a constant* $C > 0$ *such that*

$$\int_{\mathbf{B}} \Phi^* \left(x, (1-|x|)^{\beta/p-1/n'} G(x)\right) dx \leqslant C$$

when $\int_{\mathbf{B}} \Phi(y, (1-|y|)^{\beta/p}|f(y)|)dy \leqslant 1$.

Proof. First note from Theorem 2.4

$$\left(\int_{\mathbf{B}} \left((1-|x|)^{\beta/p-1/n'} \int_{\mathbf{B}\setminus B(0,|x|)} |f(y)|dy\right)^{p^*} dx\right)^{1/p^*} \leqslant C\left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{\beta} dy\right)^{1/p}$$

when $(\beta/p - 1/n')p^* = \beta p^*/p - p^*/p' - 1$ and $\beta < p-1$. In this case,

$$\left(\int_{\mathbf{B}} ((1-|x|)^{\beta/p-1/n'} G(x))^{p^*} dx\right)^{1/p^*} \leqslant C\left(\int_{\mathbf{B}} ((1-|y|)^{\beta/p}|f(y)|)^p dy\right)^{1/p}.$$

Next note

$$\begin{aligned} b(x)G(x) &= \int_{\mathbf{B}\setminus B(0,|x|)} \{b(x)-b(y)\}|f(y)|dy + \int_{\mathbf{B}\setminus B(0,|x|)} b(y)|f(y)|dy \\ &\leqslant C(1-|x|)^{\theta} \int_{\mathbf{B}\setminus B(0,|x|)} |f(y)|dy + \int_{\mathbf{B}\setminus B(0,|x|)} |f(y)|b(y)dy. \end{aligned}$$

Theorem 2.4 gives

$$\left(\int_{\mathbf{B}} \left((1-|x|)^{\beta/p-1/n'+\theta} \int_{\mathbf{B}\setminus B(0,|x|)} |f(y)|dy\right)^{q^*} dx\right)^{1/q^*} \leqslant C\left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{\beta} dy\right)^{1/p}$$

when $(\beta/p - 1/n' + \theta)q^* = \beta q^*/p - q^*/p' - 1$ and $\beta < p-1$. Moreover,

$$\left(\int_{\mathbf{B}} \left((1-|x|)^{\beta/p-1/n'} \int_{\mathbf{B}\setminus B(0,|x|)} |f(y)|b(y)dy\right)^{q^*} dx\right)^{1/q^*} \leqslant C\left(\int_{\mathbf{B}} |f(y)b(y)|^q (1-|y|)^{\beta q/p} dy\right)^{1/q}$$

when $(\beta/p - 1/n')q^* = (\beta q/p)q^*/q - q^*/q' - 1$ and $\beta q/p < q-1$ since $\beta < p-1$. Thus

$$\int_{\mathbf{B}} \left((1-|x|)^{\beta/p-1/n'} b(x)G(x)\right)^{q^*} dx \leqslant C$$

when $\int_{\mathbf{B}} \Phi(y,(1-|y|)^{\beta/p}|f(y)|)dy \leqslant 1$. The proof is now completed. □

In the same way as Proposition 2.5, we have the following result.

PROPOSITION 3.4. *Let $1 \leqslant p < q < n$, $\beta < p-1$ and $1/q = 1/p - \theta > 0$. Let u be a function in $C^1(\mathbb{R}^n)$ such that*

$$\int_{\mathbf{B}} \Phi(y,(1-|y|)^{\beta/p}|\nabla u(y)|)dy < \infty.$$

Then

$$\lim_{r\to 1}\Big((1-r)^{-(\beta/q-1/p')}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)} U_1(x)dx + (1-r)^{-(\beta/q-1/q')}\frac{1}{|\mathbf{B}\setminus B(0,r)|}\int_{\mathbf{B}\setminus B(0,r)} b(x)U_1(x)dx\Big) = 0$$

and

$$\lim_{r\to 1}\Big((1-r)^{-(\beta/q-1/p')}\frac{1}{|S(0,r)|}\int_{S(0,r)} U_1(r\xi)dS(\xi) + (1-r)^{-(\beta/q-1/q')}\frac{1}{|S(0,r)|}\int_{S(0,r)} b(r\xi)U_1(r\xi)dS(\xi)\Big) = 0.$$

4. Sharpness

We discuss the sharpness of Theorem 3.1 in the double phase setting.

REMARK 4.1. In Theorem 3.1, the single condition that

$$\int_{\mathbf{B}} (|f(y)|b(y))^q(1-|y|)^{\beta}dy \leqslant 1$$

may not imply

$$\int_{\mathbf{B}} \Big(b(x)(1-|x|)^{\beta/q-1/n'}F(x)\Big)^{q^*}dx \leqslant C.$$

In fact, for $0 < r < 1$ and $a > 0$ consider

$$b(x) = \begin{cases} |x|^{\theta} - r^{\theta} & \text{on } \mathbf{B}\setminus B(0,r); \\ 0 & \text{on } B(0,r); \end{cases}$$

and

$$f_a(y) = \begin{cases} 0 \text{ on } \mathbf{B}\setminus B(0,r); \\ a \text{ on } B(0,r). \end{cases}$$

Then note

(1) $\int_{\mathbf{B}} (|f_a(y)|b(y))^q(1-|y|)^{\beta}dy = 0$;

(2) $\int_{B(0,|x|)} |f_a(y)|dy \geqslant Cr^n a$ for $x \in \mathbf{B} \setminus B(0,r)$;

(3) $\lim_{a\to\infty} \int_{\mathbf{B}} \left(b(x)(1-|x|)^{\beta/q-1/n'} \int_{B(0,|x|)} |f_a(y)|dy \right)^{q^*} dx = \infty.$

5. The borderline case

The borderline case $\beta = p-1$ in Theorem 2.1 is known.

THEOREM 5.1. ([26, Theorem 5.1]) *Let* $1 \leqslant p \leqslant q$. *Then there exists a constant* $C > 0$ *such that*

$$\left(\int_{\mathbf{B}} \left(\int_{B(0,|x|)} |f(y)|dy \right)^q (1-|x|)^{-1} (\log(e/(1-|x|)))^{-a} dx \right)^{1/q}$$
$$\leqslant C \left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{p-1} (\log(e/(1-|y|)))^{p/p'+(1-a)p/q} dy \right)^{1/p}$$

when $a > 1$.

REMARK 5.2. Let p,q be as in Theorem 5.1. Suppose $a_1 > a > 1$ and

$$1 + (1-a_1)/q < \gamma < 1 + (1-a)/q.$$

Consider

$$f(y) = |y|^{-1} (\log(e/(1-|y|)))^{-\gamma} \quad \text{on } \mathbf{B}.$$

Then

(1) $\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{p-1} (\log(e/(1-|y|)))^{p/p'+(1-a_1)p/q} dy < \infty$;

(2) $\int_{\mathbf{B}} \left(\int_{B(0,|x|)} |f(y)|dy \right)^q (1-|x|)^{-1} (\log(e/(1-|x|)))^{-a} dx = \infty.$

This implies that in Theorem 5.1, the exponent $p/p' + (1-a)p/q$ could not be replaced by a smaller $p/p' + (1-a_1)p/q$.

The borderline case $\beta = q-1$ in Theorem 3.1 is treated in the following.

THEOREM 5.3. *Let* $1 \leqslant p < q < n$ *and* $1/q = 1/p - \theta > 0$. *Set*

$$F(x) = \int_{B(0,|x|)} |f(y)|dy$$

as before and

$$L(y) = 1 + (\log(e/(1-|y|)))^{q/q'+(1-a)q/q^*} + (\log(e/(1-|y|)))^{p/p'+(1-a)p/q^*}.$$

If $a > 1$, *then there exists a positive constant* $C > 0$ *such that*

$$\int_{\mathbf{B}} \Phi^* \left(x, (1-|x|)^{-1/q+1/n} F(x) \right) (\log(e/(1-|x|)))^{-a} dx \leqslant C$$

when $\int_{\mathbf{B}} \Phi(y, (1-|y|)^{(q-1)/q} |f(y)|) \tilde{L}(y) dy \leqslant 1$.

Proof. In view of Theorem 2.1 we have

$$\begin{aligned}
&\left(\int_{\mathbf{B}}\left((1-|x|)^{-1/q+1/n}F(x)\right)^{p^*}dx\right)^{1/p^*}\\
&=\left(\int_{\mathbf{B}}F(x)^{p^*}(1-|x|)^{(-1/q+1/n)p^*}dx\right)^{1/p^*}\\
&\leqslant C\left(\int_{\mathbf{B}}|f(y)|^p(1-|y|)^{p(q-1)/q}dy\right)^{1/p}\\
&\leqslant C\left(\int_{\mathbf{B}}|f(y)|^p(1-|y|)^{p(q-1)/q}L(y)\,dy\right)^{1/p}
\end{aligned}$$

when $(-1/q+1/n)p^* = (p(q-1)/q)p^*/p - p^*/p' - 1$ and $p(q-1)/q > p-1$.

As in the proof of Theorem 3.1, note

$$b(x)F(x) \leqslant C\int_{B(0,|x|)}|f(y)|(1-|y|)^{\theta}dy + \int_{B(0,|x|)}|f(y)|b(y)dy.$$

Theorem 5.1 gives

$$\begin{aligned}
&\left(\int_{\mathbf{B}}\left((1-|x|)^{-1/q+1/n}\int_{B(0,|x|)}|f(y)|(1-|y|)^{\theta}dy\right)^{q^*}(\log(e/(1-|x|)))^{-a}dx\right)^{1/q^*}\\
&\leqslant C\left(\int_{\mathbf{B}}\left(|f(y)|(1-|y|)^{\theta}\right)^{p}(1-|y|)^{p-1}(\log(e/(1-|y|)))^{p/p'+(1-a)p/q^*}dy\right)^{1/p}\\
&= C\left(\int_{\mathbf{B}}|f(y)|^p(1-|y|)^{p-1+\theta p}(\log(e/(1-|y|)))^{p/p'+(1-a)p/q^*}dy\right)^{1/p}\\
&\leqslant C\left(\int_{\mathbf{B}}|f(y)|^p(1-|y|)^{p(q-1)/q}L(y)\,dy\right)^{1/p}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left(\int_{\mathbf{B}}\left((1-|x|)^{-1/q+1/n}\int_{B(0,|x|)}|f(y)|b(y)dy\right)^{q^*}(\log(e/(1-|x|)))^{-a}dx\right)^{1/q^*}\\
&\leqslant C\left(\int_{\mathbf{B}}(|f(y)|b(y))^q(1-|y|)^{q-1}(\log(e/(1-|y|)))^{q/q'+(1-a)q/q^*}dy\right)^{1/q}\\
&\leqslant C\left(\int_{\mathbf{B}}(|f(y)|b(y))^q(1-|y|)^{q-1}L(y)\,dy\right)^{1/q}
\end{aligned}$$

since $a>1$.

Hence we find

$$\int_{\mathbf{B}}((1-|x|)^{-1/q+1/n}b(x)F(x))^{q^*}(\log(e/(1-|x|)))^{-a}dx \leqslant C$$

when $a>1$ and $\int_{\mathbf{B}}\Phi(y,(1-|y|)^{(q-1)/q}|f(y)|)L(y)\,dy \leqslant 1$.

Thus we obtain the required inequality. $\square$

Finally we treat the dual Hardy operator. The borderline case $\beta = p-1$ in Theorem 2.4 is known.

THEOREM 5.4. ([26, Theorem 5.5]) *Let* $1 \leqslant p \leqslant q$ *and* $a<1$. *Then there exists a constant* $C>0$ *such that*

$$\left(\int_{\mathbf{B}} \left(\int_{\mathbf{B} \setminus B(0,|x|)} |f(y)| dy \right)^q (1-|x|)^{-1} (\log(e/(1-|x|)))^{-a} dx \right)^{1/q}$$

$$\leqslant C \left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{p-1} (\log(e/(1-|y|)))^{p/p'+(1-a)p/q} dy \right)^{1/p}.$$

With the aid of Theorem 5.4, we obtain the following result as in Theorem 3.3.

THEOREM 5.5. *Let* $1 \leqslant p < q < n$ *and* $1/q = 1/p - \theta > 0$. *Set*

$$G(x) = \int_{\mathbf{B} \setminus B(0,|x|)} |f(y)| dy$$

as before. If $a<1$, *then*

$$\int_{\mathbf{B}} \Phi^* \left(x, (1-|x|)^{-1/p+1/n} G(x) \right) (\log(e/(1-|x|)))^{-a} dx \leqslant C$$

when $\displaystyle\int_{\mathbf{B}} \Phi(y,(1-|y|)^{(p-1)/p} |f(y)|) (\log(e/(1-|y|)))^{p/p'+(1-a)p/p^*} dy \leqslant 1.$

Proof. In view of Theorem 5.4 we have

$$\left(\int_{\mathbf{B}} \left((1-|x|)^{-1/p+1/n} G(x) \right)^{p^*} (\log(e/(1-|x|)))^{-a} dx \right)^{1/p^*}$$

$$\leqslant C \left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{p-1} (\log(e/(1-|y|)))^{p/p'+(1-a)p/p^*} dy \right)^{1/p}.$$

As in the proof of Theorem 3.3 note

$$b(x)G(x) \leqslant C(1-|x|)^{\theta} \int_{\mathbf{B} \setminus B(0,|x|)} |f(y)| dy + \int_{\mathbf{B} \setminus B(0,|x|)} |f(y)| b(y) dy.$$

By Theorem 5.4 we obtain

$$\left(\int_{\mathbf{B}} \left((1-|x|)^{-1/p+1/n} (1-|x|)^{\theta} \int_{\mathbf{B} \setminus B(0,|x|)} |f(y)| dy \right)^{q^*} (\log(e/(1-|x|)))^{-a} dx \right)^{1/q^*}$$

$$= \left(\int_{\mathbf{B}} \left((1-|x|)^{-1/q+1/n} \int_{\mathbf{B} \setminus B(0,|x|)} |f(y)| dy \right)^{q^*} (\log(e/(1-|x|)))^{-a} dx \right)^{1/q^*}$$

$$\leqslant C \left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{p-1} (\log(e/(1-|y|)))^{p/p'+(1-a)p/q^*} dy \right)^{1/p}$$

$$\leqslant C \left(\int_{\mathbf{B}} |f(y)|^p (1-|y|)^{p-1} (\log(e/(1-|y|)))^{p/p'+(1-a)p/p^*} dy \right)^{1/p}$$

when $a<1$. Moreover, Theorem 2.4 gives

$$\left(\int_{\mathbf{B}} \left((1-|x|)^{-1/p+1/n} \int_{\mathbf{B} \setminus B(0,|x|)} |f(y)| b(y) dy \right)^{q^*} dx \right)^{1/q^*}$$

$$\leqslant C \left(\int_{\mathbf{B}} (|f(y)| b(y))^q (1-|y|)^{q(p-1)/p} dy \right)^{1/q}$$

when $(-1/p+1/n)q^* = (q(p-1)/p)q^*/q - q^*/q' - 1$ and $q(p-1)/p < q-1$.

Hence we find

$$\int_{\mathbf{B}} ((1-|x|)^{-1/p+1/n} b(x)G(x))^{q^*} (\log(e/(1-|y|)))^{-a} \, dx \leqslant C$$

when $\int_{\mathbf{B}} \Phi(y,(1-|y|)^{(p-1)/p}|f(y)|)(\log(e/(1-|y|)))^{p/p'+(1-a)p/p^*} \, dy \leqslant 1$.

Thus Theorem 5.5 is proved. □

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