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COVERING CONVEX HULLS OF COMPACT CONVEX SETS WITH SMALLER HOMOTHETIC COPIES

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Abstract. Estimations of the covering functional of a convex body that can be expressed as the convex hull of a finite number of compact convex sets are presented. In particular, an upper bound of the covering functional of zonotopes that can be written as the sum of $n+k$ segments is given.

1. Introduction

Estimating covering functionals of *convex bodies* (i.e., compact convex sets having interior points) in $\mathbb{R}^n$ is an important part of Chuanming Zong's computer-based program (cf. [17]) to attack Hadwiger's covering conjecture, which asserts that each convex body in $\mathbb{R}^n$ can be covered by at most 2^n of its smaller homothetic copies (see [5], [10], [7], [1], and [2] for more information about this conjecture). In contrast to the traditional way to study Hadwiger's covering conjecture, which is based on estimating the least number $c(K)$ of smaller homothetic copies of K needed to cover K, Zong's program is based on good estimations of *covering functionals* $\Gamma_m(K)$ defined by

$$\Gamma_m(K) := \min\left\{\gamma > 0 \;\middle|\; \exists \{x_i \mid i \in [m]\} \subseteq \mathbb{R}^n \text{ s.t. } K \subseteq \bigcup_{i=1}^{m}(x_i + \gamma K)\right\}, \; \forall m \in \mathbb{Z}^+.$$

When K is an n-dimensional convex body, i.e., when K is affinely equivalent to a convex body in $\mathbb{R}^n$, $\Gamma_{2^n}(K)$ is called *the* covering functional of K.

Compared with estimating $c(K)$, it is more difficult to obtain tight estimations of $\Gamma_m(K)$. For example, it is clear that $c(K) = n+1$ when K is an n-dimensional convex body with smooth boundary, while no nontrivial upper bound of $\Gamma_{2^n}(K)$, when K is in this class, is known. Another example is that, for n-dimensional zonotopes and zonoids that are not parallelotopes, we have $c(K) \leqslant 3 \cdot 2^{n-2}$ (cf. [9], [3], [5, Theorem 42.1], and [4]), but no good estimations of the covering functionals of these two classes of convex bodies is known yet.

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We continue our recent works [8], [15], [14], [12], [13], and [16] by providing estimations of covering functionals of convex hulls of compact convex sets. We also present an upper bound for the covering functional of zonotopes in terms of the number of segments needed to generate it.

Let K be a compact convex subset of $\mathbb{R}^n$ that contains distinct points. We denote by $\operatorname{relint} K$, $\operatorname{relbd} K$, $\operatorname{int} K$, $\operatorname{bd} K$, and $\operatorname{ext} K$ the *relative interior*, *relative boundary*, *interior*, *boundary*, and the set of *extreme points* of K, respectively. For each $x \in \mathbb{R}^n$ and $\lambda \in (0,1)$, the set $x+\lambda K := \{x+\lambda y \mid y \in K\}$ is called a *smaller homothetic copy* of K. We denote by $\mathscr{K}^n$ the set of convex bodies in $\mathbb{R}^n$. A convex body is called a *zonotope* if it is the Minkowski sum of a finite number of segments, and is called a *zonoid* if it is the limit (with respect to the Hausdorff metric) of a converging sequence of zonotopes.

For each $m \in \mathbb{Z}^+$, we use the short-hand notation $[m] = \{t \in \mathbb{Z}^+ \mid 1 \leqslant t \leqslant m\}$.

In Section 2, we estimate covering functionals of convex hulls of compact convex sets via the knowledge of covering $\operatorname{ext} K$ by smaller homothetic copies of K. In Section 3, we apply results from Section 2 to estimate the covering functional of n-dimensional zonotopes.

2. Covering convex hulls of compact convex sets

LEMMA 1. ([13]) *For each $K \in \mathscr{K}^n$, $c(K)$ is the least number of smaller homothetic copies of K needed to cover* $\operatorname{ext} K$.

Let K be a compact convex set whose affine dimension is not less than 1 and $x \in K$. Put

$$\lambda(x,K) = \sup\{\lambda \in [0,1] \mid \exists\, v \in \operatorname{ext} K \text{ and } y \in K \text{ s.t. } x = \lambda v + (1-\lambda)y\}.$$

Set

$$\lambda(K) = \inf\{\lambda(x,K) \mid x \in K\}.$$

We note that, when K is a convex polytope, the definitions of $\lambda(x,K)$ and $\lambda(K)$ here coincide with that of $\lambda(x,K)$ and $\lambda(K)$ defined in [14], respectively.

LEMMA 2. *If K is a nonempty compact convex set and* $\operatorname{ext} K$ *is closed, then "*sup*" in the definition of $\lambda(x,K)$ can be replaced with "*max*".*

Proof. Let x be an arbitrary point in K. Put $\lambda = \lambda(x,K)$. By Caratheódory's Theorem and the Krein–Milman Theorem we have that $\lambda \geqslant 1/(m+1)$, where m is the affine dimenson of K. For each $k \in \mathbb{Z}^+$ satisfying $1/k \in (0,\lambda)$, there exist $\lambda_k \in (\lambda - 1/k, \lambda]$, $v_k \in \operatorname{ext} K$ and $y_k \in K$ such that

$$x = \lambda_k v_k + (1-\lambda_k) y_k.$$

Since $[0,1]$, $\operatorname{ext}K$, and K are all compact, without loss of generality, we may assume that $\{\lambda_k\}_{k=1}^{\infty}$ converges to a number $\lambda_0 \in [0,1]$, $\{v_k\}_{k=1}^{\infty}$ converges to a point $v_0 \in \operatorname{ext}K$, and that $\{y_k\}_{k=1}^{\infty}$ converges to a point $y_0 \in K$. It follows that $\lambda_0 = \lambda$ and

$$x = \lambda_0 v_0 + (1-\lambda_0)y_0 = \lambda(x,K)v_0 + (1-\lambda(x,K))y_0. \qquad \square$$

Let $C \subseteq \operatorname{ext}K$. Put

$$\gamma(C,K) = \inf\{\gamma \geqslant 0 \mid \exists c \in \mathbb{R}^n \text{ s.t. } C \subseteq c + \gamma K\}.$$

LEMMA 3. ([13]) *Let $K \in \mathscr{K}^n$ be a convex body. Then*

$$\lambda(K) \geqslant \frac{1}{n+1},$$

and equality holds for n-dimensional simplices.

LEMMA 4. *Suppose that $K, K_1, \ldots, K_m$ are nonempty compact convex sets and K is the convex hull of $\cup_{i\in[m]} K_i$. Then*

$$\operatorname{ext}K \subseteq \bigcup_{i\in[m]} \operatorname{ext}K_i.$$

Proof. By Theorem 7.17 in [11], we have $K_i = \operatorname{conv}(\operatorname{ext}K_i)$, $\forall i \in [m]$. It is clear that

$$\bigcup_{i\in[m]} \operatorname{ext}K_i \subseteq \bigcup_{i\in[m]} K_i \subseteq K.$$

By Theorem 7.18 in [11], it suffices to show that $K = \operatorname{conv}\left(\bigcup_{i\in[m]} \operatorname{ext}K_i\right)$.

By Theorem 3.13 and Corollary 3.14 in [11], we have

$$\begin{aligned}
&\operatorname{conv}\left(\bigcup_{i\in[m]} \operatorname{ext}K_i\right)\\
&=\left\{\sum_{i\in[m]} \lambda_i x_i \;\middle|\; x_i \in \operatorname{conv}(\operatorname{ext}K_i),\ \lambda_i \in [0,1],\ \forall i \in [m],\ \sum_{i\in[m]} \lambda_i = 1\right\}\\
&=\left\{\sum_{i\in[m]} \lambda_i x_i \;\middle|\; x_i \in K_i,\ \lambda_i \in [0,1],\ \forall i \in [m],\ \sum_{i\in[m]} \lambda_i = 1\right\}\\
&=\operatorname{conv}\left(\bigcup_{i\in[m]} K_i\right) = K.
\end{aligned}$$

This completes the proof. $\square$

For each $m \in \mathbb{Z}^+$ and each compact convex set K, put

$$\gamma_m^1(K) = \inf\left\{\gamma > 0 \,\middle|\, \exists\{x_i \mid i \in [m]\} \subseteq \mathbb{R}^n \text{ s.t. } \operatorname{ext} K \subseteq \bigcup_{i=1}^{m}(x_i + \gamma K)\right\}.$$

PROPOSITION 5. *The "*inf*" in the definition of $\gamma_m^1(\cdot)$ can be replaced with "*min*".*

Proof. Let K be a compact convex set and $m \in \mathbb{Z}^+$. Without loss of generality, we may assume that o lies in the relative interior of K. By the definition of $\gamma_m^1(K)$, for each $k \in \mathbb{Z}^+$, there exists a set $\left\{x_i^k \mid i \in [m]\right\}$ such that

$$\operatorname{ext} K \subseteq \bigcup_{i \in [m]} \left(x_i^k + \left(\gamma_m^1(K) + \frac{1}{k}\right) K\right).$$

We may also require that, for each $i \in [m]$ and each $k \in \mathbb{Z}^+$,

$$\left(x_i^k + \left(\gamma_m^1(K) + \frac{1}{k}\right) K\right) \cap \operatorname{ext} K \neq \emptyset.$$

It follows that, for each $i \in [m]$, the sequence $\{x_i^k\}_{k \in \mathbb{Z}^+}$ is bounded. By choosing subsequences if necessary, we may assume that

$$\lim_{k \to \infty} x_i^k = x_i, \ \forall i \in [m].$$

Put $C = \{x_i \mid i \in [m]\}$. In the rest, we show that $\operatorname{ext} K \subseteq C + \gamma_m^1(K)K$. Let v be an arbitrary point in $\operatorname{ext} K$. Then there exist an $i_0 \in [m]$ and a subsequence $\{x_{i_0}^{k_l}\}_{l \in \mathbb{Z}^+}$ such that

$$v \in x_{i_0}^{k_l} + \left(\gamma_m^1(K) + \frac{1}{k_l}\right) K, \ \forall l \in \mathbb{Z}^+.$$

Put $K_l = \left(\gamma_m^1(K) + \frac{1}{k_l}\right) K$. Since o is a relative interior point of K, $\{K_l\}_{l \in \mathbb{Z}^+}$ is a decreasing sequence of compact convex sets. Moreover, for each $l_0 \in \mathbb{Z}^+$, $\{v - x_{i_0}^{k_l}\}_{l \geqslant l_0}$ is a converging sequence in K_{l_0}, which implies that

$$v - x_{i_0} \in K_l, \ \forall l \in \mathbb{Z}^+.$$

Thus $v - x_{i_0} \in \gamma_m^1(K)K$ (see the proof of Theorem 2.37 in [11] for more details), or, equivalently, $v \in x_{i_0} + \gamma_m^1(K)K$. □

THEOREM 6. *Let $K \in \mathscr{K}^n$ be a convex body, $c_1, \dots, c_m$ be m points, and $\gamma_1, \dots, \gamma_m \in (0,1)$ be m numbers such that $\operatorname{ext} K \subseteq \cup_{i \in [m]}(c_i + \gamma_i K)$. Then*

$$\Gamma_m(K) \leqslant 1 + \lambda(K) \cdot \max\{\gamma_i \mid i \in [m]\} - \lambda(K).$$

Proof. Without loss of generality, we may assume that o is the centroid of K. In this situation we have $-K \subseteq nK$. Put

$$\gamma_0 = \max\{\gamma_i \mid i \in [m]\} \quad \text{and} \quad \lambda = \lambda(K).$$

Let x be an arbitrary point in K. For each $\varepsilon \in (0, \lambda(x,K))$, there exist a point $v \in \operatorname{ext} K$, a number $\lambda_0 \in (\lambda(x,K) - \frac{\varepsilon}{2}, \lambda(x,K)]$ and a point $y \in K$ such that

$$x = \lambda_0 v + (1-\lambda_0)y.$$

If $\lambda_0 \geqslant \lambda$, then

$$x = \lambda v + (\lambda_0 - \lambda)v + (1-\lambda_0)y \in \lambda v + (1-\lambda)K.$$

Otherwise, $\lambda(x,K) - \frac{\varepsilon}{2} < \lambda_0 < \lambda \leqslant \lambda(x,K)$, which implies that $0 < \lambda - \lambda_0 < \frac{\varepsilon}{2}$. Thus,

$$\begin{aligned} x &= \lambda v + (\lambda_0 - \lambda)v + (1-\lambda_0)y \\ &\in \lambda v + n(\lambda - \lambda_0)K + \left(1 - \lambda + \frac{\varepsilon}{2}\right)K \\ &\subseteq \lambda v + \left(1 - \lambda + \frac{(n+1)\varepsilon}{2}\right)K. \end{aligned}$$

Thus, in both cases, we have

$$x \in \lambda v + \left(1 - \lambda + \frac{(n+1)\varepsilon}{2}\right)K.$$

Without loss of generality we may assume that $v \in c_1 + \gamma_1 K$. Therefore,

$$\begin{aligned} x &\in \lambda v + \left(1 - \lambda + \frac{(n+1)\varepsilon}{2}\right)K \\ &= \lambda c_1 + \lambda(v - c_1) + \left(1 - \lambda + \frac{(n+1)\varepsilon}{2}\right)K \\ &\subseteq \lambda c_1 + \lambda\gamma_1 K + \left(1 - \lambda + \frac{(n+1)\varepsilon}{2}\right)K \\ &\subseteq \lambda c_1 + \left(1 + \lambda\gamma_0 - \lambda + \frac{(n+1)\varepsilon}{2}\right)K. \end{aligned}$$

Thus, $K \subseteq \{\lambda c_i \mid i \in [m]\} + (1 + \lambda\gamma_0 - \lambda)K$. This completes the proof. □

COROLLARY 7. *Let $K \in \mathscr{K}^n$ be the convex hull of compact convex sets $K_1, \ldots, K_p$, and $m_1, \ldots, m_p$ be p positive integers. Then*

$$\Gamma_{m_1+\cdots+m_p}(K) \leqslant 1 + \lambda(K) \cdot \max\left\{\gamma^1_{m_i}(K_i) \mid i \in [p]\right\} - \lambda(K).$$

Proof. For each $i \in [p]$, there exists a set $\left\{c^i_j \mid j \in [m_i]\right\}$ such that

$$\operatorname{ext} K_i \subseteq \bigcup_{j\in[m_i]} (c^i_j + \gamma^1_{m_i}(K_i)K_i) \subseteq \bigcup_{j\in[m_i]} (c^i_j + \gamma^1_{m_i}(K_i)K).$$

By Lemma 4,

$$\operatorname{ext} K \subseteq \bigcup_{i\in[p]} \operatorname{ext} K_i \subseteq \bigcup_{i\in[p]} \bigcup_{j\in[m_i]} (c^i_j + \gamma^1_{m_i}(K_i)K).$$

From Theorem 6 the desired inequality follows. □

EXAMPLE 1. Let L be the segment having $(0,0,1)$ and $(0,0,-1)$ as vertices, and M be the square with vertices

$$(1,0,0),\ (-1,0,0),\ (0,1,0), \quad \text{and} \quad (0,-1,0).$$

Then $K = \operatorname{conv}(M \cup L)$ is the unit ball B^3_1 of the Banach space l^3_1. Since all the facets of B^3_1 are triangles, Theorem 12 and Theorem 16 in [14] show that $\lambda(K) = \frac{1}{3}$. Since

$$\gamma^1_2(L) = \gamma^1_4(M) = 0,$$

Corollary 7 shows that $\Gamma_6(B^3_1) \leqslant \frac{2}{3}$. In fact, we have $\Gamma_6(B^3_1) = \frac{2}{3}$.

REMARK 8. The sum of a compact convex set K and a segment $[a,b]$ can also be viewed as the convex hull of $K+a$ and $K+b$. On the one hand, $K+[a,b]$ is a convex set containing $K+a$ and $K+b$. Thus $\operatorname{conv}((K+a)\cup(K+b)) \subseteq K+[a,b]$. Conversely, let x be an arbitrary point in $K+[a,b]$. There exist a point $y \in K$ and a number $\lambda \in [0,1]$ such that

$$x = y + \lambda a + (1-\lambda)b = \lambda(y+a) + (1-\lambda)(y+b) \in \operatorname{conv}((K+a)\cup(K+b)).$$

By Lemma 3, we have the following:

COROLLARY 9. *Suppose that $K \subseteq \mathbb{R}^n$ is the convex hull of two non-empty compact convex sets L and M, $m_1, m_2 \in \mathbb{Z}^+$. Then*

$$\begin{aligned}\Gamma_{m_1+m_2}(K) &\leqslant 1 + \frac{1}{n+1}(\max\{\gamma^1_{m_1}(L), \gamma^1_{m_2}(M)\} - 1)\\ &\leqslant 1 + \frac{1}{n+1}(\max\{\Gamma_{m_1}(L), \Gamma_{m_2}(M)\} - 1).\end{aligned}$$

3. Covering zonotopes

Clearly $[0,1]^n$ is a zonotope and it is well known that

$$\Gamma_{2^n}([0,1]^n) = \frac{1}{2},\ \forall n \geqslant 2.$$

In the rest of this paper we present an upper bound of the covering functional of n-dimensional zonotopes.

LEMMA 10. *If K is a zonotope then $\lambda(K) = \frac{1}{2}$.*

Proof. The desired equality follows from Theorem 15 and Theorem 16 in [14], and the fact that all faces of a zonotope are centrally symmetric. □

LEMMA 11. *Suppose that $k \in \mathbb{Z}^+$ and $K = \sum_{i\in[n+k]} [-u_i, u_i]$ is an n-dimensional zonotope. If $C = \sum_{i\in[n]} [-u_i, u_i]$ is a parallelotope and there exists $\gamma \geqslant 0$ such that*

$$[-u_j, u_j] \subseteq \gamma C, \ \forall j \in [n+k] \setminus [n],$$

then

$$\Gamma_{2^n}(K) \leqslant 1 - \frac{1}{2+2\gamma k}.$$

Proof. Let v be an arbitrary extreme point of K. Then v can be written in the form

$$v = \sum_{i\in[n+k]} \sigma_i u_i,$$

where $\sigma_i \in \{-1, 1\}$, for each $i \in [n+k]$. We have

$$\begin{aligned} v - \sum_{i\in[n]} \sigma_i u_i &= \sum_{i=n+1}^{n+k} \sigma_i u_i \\ &= \sum_{i=n+1}^{n+k} \frac{1}{1+\gamma k} \sigma_i u_i + \sum_{i=n+1}^{n+k} \left(1 - \frac{1}{1+\gamma k}\right) \sigma_i u_i \\ &\in \sum_{i=n+1}^{n+k} \frac{\gamma}{1+\gamma k} C + \sum_{i=n+1}^{n+k} \left(1 - \frac{1}{1+\gamma k}\right) \sigma_i u_i \\ &= \left(1 - \frac{1}{1+\gamma k}\right) \left(C + \sum_{i=n+1}^{n+k} \sigma_i u_i\right) \\ &\subseteq \left(1 - \frac{1}{1+\gamma k}\right) K. \end{aligned}$$

It follows that

$$\operatorname{ext} K \subseteq \left\{ \sum_{i\in[n]} \sigma_i u_i \,\middle|\, \sigma_i \in \{-1, 1\}, \ \forall i \in [n] \right\} + \left(1 - \frac{1}{1+\gamma k}\right) K.$$

By Theorem 6, we have

$$\Gamma_{2^n}(K) \leqslant 1 + \frac{1}{2}\left(1 - \frac{1}{1+\gamma k}\right) - \frac{1}{2} = 1 - \frac{1}{2+2\gamma k}. \quad \square$$

THEOREM 12. *Let $k \in \mathbb{Z}^+$ and K be an n-dimensional zonotope that can be written as the sum of $n+k$ segments. Then*

$$\Gamma_{2^n}(K) \leqslant 1 - \frac{1}{2+2k}.$$

Proof. By a suitable translation if necessary, we may assume that K is the sum of $n+k$ segments $[-u_i, u_i]$, $i \in [n+k]$. Since K is n-dimensional, the set $\{u_i \mid i \in [n+k]\}$ contains n linearly independent points. Without loss of generality, we may assume that $u_1, \ldots, u_n$ are linearly independent. Thus $C = \sum_{i \in [n]} [-u_i, u_i]$ is a parallelotope. We also assume that $u_1, \ldots, u_n$ are choosen such that the volume of C is as large as possible. It follows that $\{-u_j, u_j\} \in C$ holds for each $j \in [n+k] \setminus [n]$. By Lemma 11,

$$\Gamma_{2^n}(K) \leqslant 1 - \frac{1}{2+2k},$$

as claimed. □

REMARK 13. Theorem 12 is also valid when $k = 0$. It can be used to estimate the covering functional of an n-dimensional zonoid that is sufficiently close to a sum of $n+k$ segments. See, e.g., [6] for results on approximation of zonoids by zonotopes.

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