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ON BACKWARD ALUTHGE ITERATES OF COMPLEX SYMMETRIC OPERATORS

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Abstract. For a nonnegative integer k, an operator $T \in \mathscr{L}(\mathscr{H})$ is called a *backward Aluthge iterate of a complex symmetric operator of order* k if the kth Aluthge iterate $\widetilde{T}^{(k)}$ of T is a complex symmetric operator, denoted by $T \in BAIC(k)$. In this paper, we study several properties of the backward Aluthge iterate of a complex symmetric operator. We show that every nilpotent operator of order $k+2$ belongs to $BAIC(k)$. Moreover, we prove that if T belongs to $BAIC(k)$, then T has the property (β) if and only if T is decomposable. Finally, we show that, under some conditions, operators in $BAIC(k)$ have nontrivial hyperinvariant subspaces and we consider Weyl type theorems for such operators.

1. Introduction and preliminaries

Let $\mathscr{H}$ be a separable complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be a p-*hyponormal* operator if $(T^*T)^p \geqslant (TT^*)^p$, where $0 < p < \infty$. If $p = 1$, T is called *hyponormal* and if $p = \frac{1}{2}$, T is called *semi-hyponormal*. ([3]) It is well known that

$$\text{hyponormal} \Rightarrow p\text{-hyponormal } (0 < p < 1).$$

An operator $T \in \mathscr{L}(\mathscr{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $ker(U) = ker(|T|) = ker(T)$ and $ker(U^*) = ker(T^*)$. We call the Aluthge transform of $T \in \mathscr{L}(\mathscr{H})$ given by $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ ([15]). For an arbitrary $T \in \mathscr{L}(\mathscr{H})$, the sequence $\{\widetilde{T}^{(n)}\}$ of the Aluthge iterates of T is defined by $\widetilde{T}^{(0)} = T$ and $\widetilde{T}^{(n)} = \widetilde{\widetilde{T}^{(n-1)}}$ for $n \in \mathbb{N}$ where $\mathbb{N}$ denotes the set of positive integers. A. Aluthge [3] showed that if T is p-hyponormal with

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$0 < p < \frac{1}{2}$, then $\widetilde{T}^{(2)}$ is hyponormal. In [17], I.B. Jung, E. Ko, and C. Pearcy proved that if T is a quasiaffinity, then $\mathrm{Lat}(T)$ is nontrivial if and only if $\mathrm{Lat}(\widetilde{T})$ is nontrivial, and the same it true of the hyperinvariant subspace lattices HLat(T) and HLat($\widetilde{T}$).

A *conjugation* C on $\mathscr{H}$ is an antilinear operator $C : \mathscr{H} \to \mathscr{H}$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathscr{H}$ and $C^2 = I$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathscr{H}$ such that $T = CT^*C$. In this case, we say that T is a complex symmetric operator with a conjugation C. Complex symmetric operators can be considered as a generalization of complex symmetric matrices; in fact, if $T \in \mathscr{L}(\mathscr{H})$ and if C is a given conjugation on $\mathscr{H}$, then the operator CT^*C comes to be the transpose of the matrix for T with respect to an orthonormal basis which is fixed by C (see [13]). In 2006, S.R. Garcia and M. Putinar provide a lot of useful properties of complex symmetric operators [13]–[14]. There are many authors studying complex symmetric operators (see [10]–[14], [27], and [28], etc.).

In 2000, I. B. Jung, E. Ko and C. Pearcy [15] firstly considered the backward Aluthge iterate of a hyponormal operator. In 2007, Ko [24] proved that the backward Aluthge iterates of a hyponormal operator have scalar extensions. In 2015, Ko and Lee [25] examined various properties of the backward Aluthge iterates of a hyponormal operator. In view of these results, we also study the backward Aluthge iterate of a complex symmetric operator.

DEFINITION 1. For a nonnegative integer k, an operator $T \in \mathscr{L}(\mathscr{H})$ is called a *backward Aluthge iterate of a complex symmetric operator of order* k if $\widetilde{T}^{(k)}$ is a complex symmetric operator.

We denote by $BAIC(k)$ the class of all backward Aluthge iterate of a complex symmetric operator of order k. In particular, $BAIC(0)$ is the set of complex symmetric operators which contains 2×2 matrices, normal operators, nilpotent operator of order 2, algebraic operators of order 2, Aluthge transform of complex symmetric operators, Hankel operators, truncated Toeplitz operators, and Volterra integration operators (see [10], [12] and [22]). In general, even if $T \in BAIC(1)$, then T may not be complex symmetric (see Example 1). In addition, it is clear that $BAIC(1)$ contains complex symmetric operators.

We next state some elementary properties for $BAIC(k)$ without proof.

PROPOSITION 1. Let $T \in BAIC(k)$ for some $k \in \mathbb{N}$. Then the following statements hold.

(i) $\lambda T \in BAIC(k)$ for any $\lambda \in \mathbb{C}$.
(ii) $U^*TU \in BAIC(k)$ where U is unitary.
(iii) If T is invertible, then $T^{-1} \in BAIC(k)$.

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to \mathscr{H}$ such that $(T - z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For an operator $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, the *resolvent set* $\rho_T(x)$ of T at x is defined to consist of z_0

in $\mathbb{C}$ such that there exists an analytic function $f(z)$ on a neighborhood of z_0, with values in $\mathscr{H}$, which verifies $(T-z)f(z) \equiv x$. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. Using this local spectra, we define the *local spectral subspace* of T by $\mathscr{H}_T(F) = \{x \in \mathscr{H} : \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Dunford's property* (C) if $\mathscr{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathscr{H}$ of $\mathscr{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well known from [26] that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

An operator $T \in \mathscr{L}(\mathscr{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathscr{M}$ and $\mathscr{N}$ such that $\mathscr{H} = \mathscr{M} + \mathscr{N}$, $\sigma(T|_{\mathscr{M}}) \subset U$, and $\sigma(T|_{\mathscr{N}}) \subset V$. In [26], it is shown that both T and T^* have the property (β) if and only if T is decomposable. For an operator $T \in \mathscr{L}(\mathscr{H})$, we define *a spectral maximal space* of T to be a closed T-invariant subspace $\mathscr{M}$ of $\mathscr{H}$ with the property that $\mathscr{M}$ contains any closed T-invariant subspace $\mathscr{N}$ of $\mathscr{H}$ such that $\sigma(T|_{\mathscr{N}}) \subset \sigma(T|_{\mathscr{M}})$, where $T|_{\mathscr{M}}$ denotes the restriction of T to $\mathscr{M}$.

In this paper, we focus on several properties of the backward Aluthge iterate of a complex symmetric operator. We prove that every nilpotent operator of order $k+2$ belongs to $BAIC(k)$. Moreover, we prove that if T belongs to $BAIC(k)$, then T has the property (β) if and only if T is decomposable. Finally, we show that, under some conditions, operators in $BAIC(k)$ have nontrivial hyperinvariant subspaces and we consider Weyl type theorems for such operators.

2. Main results

In this section, we study several properties of the backward Aluthge iterates of a complex symmetric operator of order k. It is known from [10] that if T is a complex symmetric operator, then $\widetilde{T}$ is also a complex symmetric operator. However, its converse does not hold. The following example shows that T is not complex symmetric, but $\widetilde{T}$ is complex symmetric.

EXAMPLE 1. Let $T \in \mathscr{L}(\mathbb{C}^3)$ be defined as

$$T = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}$ and hence $\widetilde{T}$ is complex symmetric since $\widetilde{T}$ is nilpotent of order 2. But, T is not complex symmetric from [12, Example 1, p 6068]. Hence $T \in BAIC(1)$.

In general, if T is nilpotent operator of order 2, then it is complex symmetric from [10]. But, if T is nilpotent operator of order $k > 2$, then T is not complex symmetric. Note that some Volterra integral operator is complex symmetric and it belongs to $BAIC(0)$, but it is not nilpotent. So, the second statement of Theorem 1 is a bit trivial for $n = 0$. In the following theorem, we prove that every nilpotent operator of order $n+2$ belongs to $BAIC(n)$.

THEOREM 1. *Let n be a nonnegative integer. Every bounded linear nilpotent operator of order $n+2$ belongs to $BAIC(n)$. Moreover, the class of all nilpotent operators of order $n+2$ forms a proper subclass of $BAIC(n)$.*

Proof. If $T \in \mathscr{L}(\mathscr{H})$ is a nilpotent operator of order $n+2$, then $\widetilde{T}$ is a nilpotent operator of order $n+1$ and then $\widetilde{T}^{(2)}$ is a nilpotent operator of order n by [16, Proposition 4.6]. By repeated applications of [16], $\widetilde{T}^{(n)}$ is a nilpotent operator of order 2. Therefore $\widetilde{T}^{(n)}$ is complex symmetric by [12, Corollary 5]. Thus T belongs to $BAIC(n)$ (cf. [6]).

On the other hand, let

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} \oplus I_n$$

where I_n is the identity matrix. Then T is not a nilpotent operator. Since

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \oplus I_n \text{ and } |T| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \oplus I_n,$$

it follows that

$$\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \oplus I_n.$$

Then $\widetilde{T}$ is complex symmetric since it is normal. Thus $T \in BAIC(1)$ and hence $T \in BAIC(n)$. Hence there exists a nonnilpotent operator T in $BAIC(n)$. $\square$

COROLLARY 1. *If N is a nilpotent operator of order $n+2$ and S is a complex symmetric operator, then $N \oplus S \in BAIC(n)$.*

Proof. Since $N \in BAIC(n)$ by Theorem 1 and S is a complex symmetric operator, we have $\widetilde{N \oplus S}^{(n)} = \widetilde{N}^{(n)} \oplus \widetilde{S}^{(n)}$. Moreover, since $\widetilde{N}^{(n)}$ and $\widetilde{S}^{(n)}$ are complex symmetric operators, $\widetilde{N \oplus S}^{(n)}$ is a complex symmetric operator. Hence we have $N \oplus S \in BAIC(n)$. $\square$

EXAMPLE 2. Let

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then T is nilpotent of order 4. By Theorem 1, we know that $T \in BAIC(2)$ which is not complex symmetric.

EXAMPLE 3. Let

$$T = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $|a| = |f|$ and $|b| = |e|$. Then T is nilpotent of order 4. By Theorem 1, we know that $T \in BAIC(2)$. Moreover, since T is unitarily equivalent to a complex symmetric operator by [11, Theorem 2], it follows that T is a complex symmetric operator by Proposition 1.

LEMMA 1. Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{L}(\mathcal{H})$. If U is unitary, then $(\widetilde{T})^*$ and $\widetilde{T^*}$ are unitarily equivalent.

Proof. Since $TT^* = U|T|^2U^*$, it follows that $|T^*| = U|T|U^*$. If $T^* = V|T^*|$ is the polar decomposition of T^*, then $V = U^*$ and $|T^*| = U|T|U^*$. Hence we have

$$\begin{aligned} \widetilde{T^*} &= |T^*|^{\frac{1}{2}}V|T^*|^{\frac{1}{2}} \\ &= U|T|^{\frac{1}{2}}U^*U^*U|T|^{\frac{1}{2}}U^* \\ &= U|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}}U^* \\ &= U(\widetilde{T})^*U^*. \end{aligned}$$

Thus $(\widetilde{T})^*$ and $\widetilde{T^*}$ are unitarily equivalent. □

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is a *quasiaffinity* if it has trivial kernel and dense range. We now investigate the decomposability of an operator T which belongs to $BAIC(k)$.

THEOREM 2. Let $T \in BAIC(k)$. If T is a quasiaffinity, then the following statements are equivalent.

(i) T has the property (β).

(ii) T is decomposable.

(iii) T^* is decomposable.

Proof. Since (ii)⇔(iii) and (ii)⇒(i) are well-known from [26], it suffices to show that (i)⇒(ii). Assume that T has the property (β). Then $\widetilde{T}^{(k)}$ has the property (β)

by [23, Theorem 1.14]. Since $\widetilde{T}^{(k)}$ is complex symmetric, it follows from [22, Theorem 2.1] that $\widetilde{T}^{(k)}$ is decomposable. Hence $(\widetilde{T}^{(k)})^*$ has the property (β). Since T is a quasiaffinity, $\widetilde{T}$ is a quasiaffinity. By induction, $\widetilde{T}^{(k-1)}$ is a quasiaffinity. Let $\widetilde{T}^{(k-1)} = V|\widetilde{T}^{(k-1)}|$ be the polar decomposition of $\widetilde{T}^{(k-1)}$. Since $\widetilde{T}^{(k-1)}$ is a quasiaffinity, it follows that V is unitary. By Lemma 1, we have

$$(\widetilde{T}^{(k)})^* = \left(\widetilde{\widetilde{T}^{(k-1)}}\right)^* = V(\widetilde{\widetilde{T}^{(k-1)}})^*V^*.$$

Hence $(\widetilde{\widetilde{T}^{(k-1)}})^*$ has the property (β). Thus $(\widetilde{T}^{(k-1)})^*$ has the property (β) by [23]. So, $\widetilde{T}^{(k-1)}$ is decomposable since $\widetilde{T}^{(k-1)}$ has the property (β). By repeated applications, we know that T^* has the property (β). Hence T is decomposable. □

COROLLARY 2. *Let $T \in BAIC(k)$ where T is a quasiaffinity. Then the following statements hold.*

(i) *If T has the single-valued extension property, then T^* has.*

(ii) *If T has the Dunford's property (C), then T^* has.*

(iii) *If T has the property (β), then for all closed $F \subset \sigma(T)$, $H_T(F)$ is a spectral maximal space of T and $\sigma(T|_{H_T(F)}) \subset F$.*

Proof. (i) Let T have the single-valued extension property. Since $\widetilde{T}^{(k)}$ is complex symmetric and $\widetilde{T}^{(k)}$ has the single-valued extension property by [23, Theorem 1.1], it follows from [19, Lemma 3.5] that $(\widetilde{T}^{(k)})^*$ has the single-valued extension property. By the similar method as in the proof of Theorem 2, T^* has the single-valued extension property.

(ii) Let T have the Dunford's property (C). Since $\widetilde{T}^{(k)}$ is complex symmetric and $\widetilde{T}^{(k)}$ has the Dunford's property (C) by [23, Theorem 1.12], it follows from [22, Theorem 3.2] that $(\widetilde{T}^{(k)})^*$ has the Dunford's property (C). By the similar method as in the proof of Theorem 2, T^* has the Dunford's property (C).

(iii) Since T is decomposable by Theorem 2, the proof follows from [8, Proposition 3.8]. □

PROPOSITION 2. *Assume that $T \in \mathscr{L}(\mathscr{H})$ has the single-valued extension property. Let $T \in BAIC(k)$ with a conjugation C and let $\widetilde{T}^{(j)} = U_j|\widetilde{T}^{(j)}|$ be the polar decomposition of $\widetilde{T}^{(j)}$ for $j = 0,1,2,\cdots,k$ where $\widetilde{T}^{(0)} = T$. Then the following statements hold.*

(i) $\sigma_T((\prod_{i=0}^{k-1} U_i|T_i|^{\frac{1}{2}})Cx) \subset \sigma_{\widetilde{T}^{(k)*}}(x)$.

(ii) $(\Pi_{i=0}^{k-1} U_i|T_i|^{\frac{1}{2}}C)\mathscr{H}_{\widetilde{T}^{(k)*}}(F) \subset \mathscr{H}_T(F)$ *for any subset $F \subset \mathbb{C}$.*

Proof. (i) Let $\widetilde{T}^{(j)} = U_j|\widetilde{T}^{(j)}|$ be the polar decomposition of $\widetilde{T}^{(j)}$ for $j = 0,1,\cdots,k-1$. Assume that $T \in BAIC(k)$. Since $\widetilde{T}^{(k)}$ is complex symmetric, it follows from [22, Lemma 3.1] that

$$\sigma_{\widetilde{T}^{(k)}}(Cx)^* \subset \sigma_{\widetilde{T}^{(k)*}}(x). \tag{1}$$

Since $T \in BAIC(k)$, by [23, Corollary 1.2]

$$\sigma_T((\prod_{i=0}^{k-1} U_i|T_i|^{\frac{1}{2}})Cx)^* \subset \sigma_{\widetilde{T}}((\prod_{i=1}^{k-1} U_i|T_i|^{\frac{1}{2}})Cx)^* \subset \cdots \subset \sigma_{\widetilde{T}^{(k)}}(Cx)^*. \tag{2}$$

Hence by (1) and (2), we have

$$\sigma_T((\prod_{i=0}^{k-1} U_i|T_i|^{\frac{1}{2}})Cx) \subset \sigma_{\widetilde{T}^{(k)*}}(x). \tag{3}$$

(ii) If $x \in \mathscr{H}_{\widetilde{T}^{(k)*}}(F)$ for any subset $F \subset \mathbb{C}$, then $\sigma_{\widetilde{T}^{(k)*}}(x) \subset F$ and so

$$\sigma_T(\Pi_{i=0}^{k-1} U_i|T_i|^{\frac{1}{2}} Cx) \subset F$$

from the inclusion (3). This means that $\Pi_{i=0}^{k-1} U_i|T_i|^{\frac{1}{2}} Cx \in \mathscr{H}_T(F)$ holds. Hence

$$(\Pi_{i=0}^{k-1} U_i|T_i|^{\frac{1}{2}} C)\mathscr{H}_{\widetilde{T}^{(k)*}}(F) \subset \mathscr{H}_T(F)$$

for any subset $F \subset \mathbb{C}$. □

For $T \in \mathscr{L}(\mathscr{H})$, the *algebraic core* $Alg(T)$ is defined as the greatest (not necessarily closed) subspace $\mathscr{M}$ of $\mathscr{H}$ satisfying $T\mathscr{M} = \mathscr{M}$. The *analytical core* of T is the set $Anal(T)$ of all $x \in \mathscr{H}$ such that there exists a sequence $\{u_n\} \subset \mathscr{H}$ and a constant $\delta > 0$ such that $x = u_0$, $Tu_{n+1} = u_n$, and $\|u_n\| \leqslant \delta^n \|x\|$ for every $n \in \mathbb{N}$.

PROPOSITION 3. *Let $T \in BAIC(k)$ be with a conjugation C. Suppose that $\widetilde{T}^{(j)} = U_j|\widetilde{T}^{(j)}|$ is the polar decomposition of $\widetilde{T}^{(j)}$ for $j = 0, 1, \cdots, k$. Then the following statements hold.*

(i) $\begin{cases} Alg(\widetilde{T}^{(k)^*}) = C(\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})Alg(T) \text{ and} \\ Alg(T) = (\prod_{j=k-1}^{0} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})CAlg(\widetilde{T}^{(k)^*}). \end{cases}$

(ii) $\begin{cases} Anal(\widetilde{T}^{(k)^*}) = C(\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})Anal(T) \text{ and} \\ Anal(T) = (\prod_{j=k-1}^{0} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})CAnal(\widetilde{T}^{(k)^*}) \text{ if } T \text{ is invertible.} \end{cases}$

Proof. Assume $\widetilde{T}^{(k)}$ is a complex symmetric operator with a conjugation C.
(i) Since $\widetilde{T}^{(k)} Alg(\widetilde{T}^{(k)}) = Alg(\widetilde{T}^{(k)})$, we get that

$$C\widetilde{T}^{(k)^*} CAlg(\widetilde{T}^{(k)}) = Alg(\widetilde{T}^{(k)}).$$

Hence $\widetilde{T}^{(k)^*} CAlg(\widetilde{T}^{(k)}) = CAlg(\widetilde{T}^{(k)})$. Thus $CAlg(\widetilde{T}^{(k)}) \subseteq Alg(\widetilde{T}^{(k)^*})$.

On the other hand, since $\widetilde{T}^{(k)^*} Alg(\widetilde{T}^{(k)^*}) = Alg(\widetilde{T}^{(k)^*})$,

$$C\widetilde{T}^{(k)} CAlg(\widetilde{T}^{(k)^*}) = Alg(\widetilde{T}^{(k)^*}).$$

Hence $\widetilde{T}^{(k)} CAlg(\widetilde{T}^{(k)^*}) = CAlg(\widetilde{T}^{(k)^*})$. Therefore $CAlg(\widetilde{T}^{(k)^*}) \subseteq Alg(\widetilde{T}^{(k)})$ and thus

$$Alg(\widetilde{T}^{(k)^*}) \subseteq CAlg(\widetilde{T}^{(k)}).$$

So we have $CAlg(\widetilde{T}^{(k)}) = Alg(\widetilde{T}^{(k)^*})$. Since $Alg(\widetilde{T}^{(k)}) = (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})Alg(T)$ by [25, Proposition 2], we get that

$$Alg(\widetilde{T}^{(k)^*}) = C(\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})Alg(T).$$

Since $CAlg(\widetilde{T}^{(k)}) \subseteq Alg(\widetilde{T}^{(k)^*})$ and $Alg(T) = (\prod_{j=0}^{k-1} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})Alg(\widetilde{T}^{(k)})$ by [25, Proposition 2], it follows that

$$Alg(T) = (\prod_{j=0}^{k-1} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})CAlg(\widetilde{T}^{(k)^*}).$$

(ii) Let $x \in Anal(\widetilde{T}^{(k)})$. Then there exists a sequence $\{u_n\} \subset \mathscr{H}$ and a constant $\delta > 0$ such that $x = u_0$, $\widetilde{T}^{(k)}u_{n+1} = u_n$, and $\|u_n\| \leqslant \delta^n\|x\|$ for every $n \in \mathbb{N}$. Since $\widetilde{T}^{(k)^*}Cx = \widetilde{T}^{(k)^*}Cu_0$, $\widetilde{T}^{(k)^*}Cu_{n+1} = C\widetilde{T}^{(k)}u_{n+1} = Cu_n$ and

$$\|Cu_n\| \leqslant \|C\|\|u_n\| \leqslant \delta^n\|x\| = \delta^n\|Cx\|$$

for all $n \in \mathbb{N}$, it holds that $CAnal(\widetilde{T}^{(k)}) \subseteq Anal(\widetilde{T}^{(k)^*})$.

On the other hand, let $y \in Anal(\widetilde{T}^{(k)^*})$. Then there exists a sequence $\{v_n\} \subset \mathscr{H}$ and a constant $\delta > 0$ such that $y = v_0$, $\widetilde{T}^{(k)^*}v_{n+1} = v_n$, and $\|v_n\| \leqslant \delta^n\|y\|$ for every $n \in \mathbb{N}$. Since $\widetilde{T}^{(k)}Cy = \widetilde{T}^{(k)}Cv_0$, $\widetilde{T}^{(k)}Cv_{n+1} = C\widetilde{T}^{(k)^*}v_{n+1} = Cv_n$ and

$$\|Cv_n\| \leqslant \|C\|\|v_n\| \leqslant \delta^n\|y\| = \delta^n\|Cy\|$$

for every $n \in \mathbb{N}$, it holds that $CAnal(\widetilde{T}^{(k)^*}) \subseteq Anal(\widetilde{T}^{(k)})$. Thus $CAnal(\widetilde{T}^{(k)}) = Anal(\widetilde{T}^{(k)^*})$. Since $Anal(\widetilde{T}^{(k)}) = (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})Anal(T)$ by [25, Proposition 2],

$$Anal(\widetilde{T}^{(k)^*}) = C(\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}})Anal(T).$$

Since $CAlg(\widetilde{T}^{(k)}) \subseteq Alg(\widetilde{T}^{(k)^*})$ and $Anal(T) = (\prod_{j=0}^{k-1} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})Anal(\widetilde{T}^{(k)})$ by [25, Proposition 2], we obtain that

$$Anal(T) = (\prod_{j=0}^{k-1} U_j|\widetilde{T}^{(j)}|^{\frac{1}{2}})CAnal(\widetilde{T}^{(k)^*}).$$

So we complete the proof. □

COROLLARY 3. *If $T \in \mathscr{L}(\mathscr{H})$ is invertible, then*

$$Alg(T^*) = (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{-\frac{1}{2}}U_{j+1})C(\prod_{j=0}^{k-1} |\widetilde{T}^{(j)}|^{\frac{1}{2}})Alg(T)$$

and

$$Anal(T^*) = (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{-\frac{1}{2}} U_{j+1}) C (\prod_{j=0}^{k-1} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Anal(T)$$

where $\widetilde{T}^{(j)} = U_j |\widetilde{T}^{(j)}|$ *is the polar decomposition of* $\widetilde{T}^{(j)}$ *for* $j = 0, 1, \cdots, k$.

Proof. By Lemma 1, we can put $\widetilde{\widetilde{T}^{(k-1)*}} = U_k \widetilde{T}^{(k)*} U_k^*$ for some $k \geqslant 1$. Then $U_k^* Alg(\widetilde{T}^{(k-1)*}) = Alg(\widetilde{T}^{(k)*})$. Thus we get that

$$\begin{aligned} Alg(\widetilde{T}^{(k)*}) &= U_k^* Alg(\widetilde{\widetilde{T}^{(k-1)*}}) \\ &= U_k^* |\widetilde{T}^{(k-1)*}|^{\frac{1}{2}} Alg(\widetilde{T}^{(k-1)*}) \\ &= U_k^* |\widetilde{T}^{(k-1)*}|^{\frac{1}{2}} U_{k-1}^* |\widetilde{T}^{(k-2)*}|^{\frac{1}{2}} Alg(\widetilde{T}^{(k-2)*}) \\ &\ \vdots \\ &= \prod_{j=k-1}^{0} U_{j+1}^* |\widetilde{T}^{(j)}|^{\frac{1}{2}} Alg(T^*). \end{aligned}$$

Since T is invertible, it follows from Proposition 3 that

$$Alg(T^*) = (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{-\frac{1}{2}} U_{j+1}) C (\prod_{j=0}^{k-1} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Alg(T)$$

where $\widetilde{T}^{(j)} = U_j |\widetilde{T}^{(j)}|$ is the polar decomposition of $\widetilde{T}^{(j)}$ for $j = 0, 1, \cdots, k$.

For the proof of the second equation, let $\widetilde{\widetilde{T}^{(k-1)*}} = U_k \widetilde{T}^{(k)*} U_k^*$ for some $k \geqslant 1$. If $x \in Anal(\widetilde{\widetilde{T}^{(k-1)*}})$, then $x = u_0$, $\widetilde{\widetilde{T}^{(k-1)*}} u_{n+1} = u_n$, and $\|u_n\| \leqslant \delta^n \|x\|$. Since $\widetilde{T}^{(k)^*} U_k^* x = \widetilde{T}^{(k)^*} U_k^* u_0$, $\widetilde{T}^{(k)^*} U_k^* u_{n+1} = U_k^* \widetilde{\widetilde{T}^{(k-1)^*}} u_{n+1} = U_k^* u_n$, and

$$\|U_k^* u_n\| \leqslant \|U_k^*\| \|u_n\| \leqslant \delta^n \|x\|$$

for all $n \in \mathbb{N}$, it holds that $U_k^* Anal(\widetilde{\widetilde{T}^{(k-1)^*}}) \subseteq Anal(\widetilde{T}^{(k)^*})$. Similarly, we obtain the reverse inclusion. Hence $U_k^* Anal(\widetilde{\widetilde{T}^{(k-1)*}}) = Anal(\widetilde{T}^{(k)*})$. From this, we get that

$$\begin{aligned} Anal(\widetilde{T}^{(k)*}) &= U_k^* Anal(\widetilde{\widetilde{T}^{(k-1)*}}) \\ &= U_k^* |\widetilde{T}^{(k-1)*}|^{\frac{1}{2}} Anal(\widetilde{T}^{(k-1)*}) \\ &= U_k^* |\widetilde{T}^{(k-1)*}|^{\frac{1}{2}} U_{k-1}^* |\widetilde{T}^{(k-2)*}|^{\frac{1}{2}} Anal(\widetilde{T}^{(k-2)*}) \\ &\ \vdots \\ &= \prod_{j=k-1}^{0} U_{j+1}^* |\widetilde{T}^{(j)}|^{\frac{1}{2}} Anal(T^*). \end{aligned}$$

By Proposition 3, we get that

$$Anal(T^*) = (\prod_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{-\frac{1}{2}} U_{j+1}) C (\prod_{j=0}^{k-1} |\widetilde{T}^{(j)}|^{\frac{1}{2}}) Anal(T)$$

where $\widetilde{T}^{(j)} = U_j|\widetilde{T}^{(j)}|$ is the polar decomposition of $\widetilde{T}^{(j)}$ for $j = 0, 1, \cdots, k$. $\square$

Recall that an operator T in $\mathcal{L}(\mathcal{H})$ will be said to have *the property* (PS) if there exist sequences $\{S_n\} \subset \{T\}'$ and $\{K_n\} \subset \mathcal{K}(\mathcal{H})$ such that $\|S_n - K_n\| \to 0$ and $\{K_n\}$ is a nontrivial sequence of compact operators. For $T \in \mathcal{L}(\mathcal{H})$, we write T' for the commutant of T, that is, for the algebra of all $S \in \mathcal{L}(\mathcal{H})$ such that $TS = ST$. A subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for $T \in \mathcal{L}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$, and a subspace $\mathcal{M}$ is *hyperinvariant* for T if it is an invariant subspace for all $S \in \{T\}'$. We next examine a nontrivial hyperinvariant subspace of $T \in BAIC(k)$.

THEOREM 3. *Let $T \in BAIC(k)$ for some $k \in \mathbb{N}$ and let $T \neq 0$, λI for any $\lambda \in \mathbb{C}$. Suppose $\widetilde{T}^{(k)}$ has the property (PS). Then T has a nontrivial hyperinvariant subspace.*

Proof. Let $T = U|T|$ be the polar decomposition of $T \in \mathcal{L}(\mathcal{H})$. If T is not a quasiaffinity, then $0 \in \sigma_p(T) \cup \sigma_p(T^*)$ where $\sigma_p(T)$ denotes the point spectrum of T. Hence T has a nontrivial hyperinvariant subspace. Assume that T is a quasiaffinity. Then $|T|$ is a quasiaffinity and U is unitary. Thus $\widetilde{T}$ is a quasiaffinity. Hence $\widetilde{T}^{(j)}$ is a quasiaffinity for $j = 0, 1, 2, \cdots, k-1$ by the induction. Since $\widetilde{T}^{(k)}$ has the property (PS), there exists a sequence $\{Q_n\} \subset \{\widetilde{T}^{(k)}\}'$ and $\{H_n\}$ such that $\|Q_n - H_n\| \to 0$ and $\{H_n\}$ is a nontrivial sequence of compact operators. Let $\widetilde{T}^{(j)} = U_j|\widetilde{T}^{(j)}|$ be the polar decomposition of $\widetilde{T}^{(j)}$ for $j = 0, 1, 2, \cdots, k-1$. Put

$$S_n := \mathbf{A}Q_n\mathbf{B} \text{ and } K_n := \mathbf{A}H_n\mathbf{B}$$

where $\mathbf{A} = \Pi_{j=0}^{k-1} U_j |\widetilde{T}^{(j)}|^{\frac{1}{2}}$ and $\mathbf{B} = \Pi_{j=k-1}^{0} |\widetilde{T}^{(j)}|^{\frac{1}{2}}$. Since $\widetilde{T}^{(j)}$ is a quasiaffinity, $\{K_n\}$ is a nontrivial sequence of compact operators. Then

$$S_nT = AQ_nBT = AQ_n\widetilde{T}^{(k)}B = A\widetilde{T}^{(k)}Q_nB = TAQ_nB = TS_n.$$

Since

$$\|S_n - K_n\| \leqslant \|\mathbf{A}\|\|Q_n - H_n\|\|\mathbf{B}\| \to 0 \text{ , as } n \to 0,$$

we obtain that T has the property (PS). Hence T has a nontrivial hyperinvariant subspace from [2]. $\square$

COROLLARY 4. *Let $T \in BAIC(k)$ for some $k \in \mathbb{N}$ and let $T \neq 0, \lambda I$ for any $\lambda \in \mathbb{C}$. Suppose that $\widetilde{T}^{(k)}$ has the property (PS) and T is a quasiaffinity. Then both T and T^* have the property (PS) and hence both T and T^* have the property (PS).*

Proof. Since $\widetilde{T}^{(k)}$ has the property (PS) and complex symmetric, it follows from [21] that $(\widetilde{T}^{(k)})^*$ has the property (PS). By similar methods, we know that T^* has a nontrivial hyperinvariant subspace. In this case, both T and T^* have the property (PS). $\square$

PROPOSITION 4. *Let $T \in \mathcal{L}(\mathcal{H})$ be p-hyponormal for $0 < p < \frac{1}{2}$. If $T \in BAIC(2)$, then T is normal and hence Lat (T) is nontrivial.*

Proof. If T is p-hyponormal and $T \in BAIC(2)$, then $\widetilde{T}^{(2)}$ is hyponormal and complex symmetric. Hence $\widetilde{T}^{(2)}$ is normal. By [9, Corollary 2], $\widetilde{T}$ is normal and hence T is normal. Thus Lat (T) is nontrivial. □

If $T \in \mathscr{L}(\mathscr{H})$ and $x \in \mathscr{H}$, then $\{T^n x\}_{n=0}^{\infty}$ is called *the orbit of x under T*, and is denoted by $O(x,T)$. If $O(x,T)$ is dense in $\mathscr{H}$, then x is called *a hypercyclic vector* for T. An operator $T \in \mathscr{L}(\mathscr{H})$ is called *hypercyclic* if there is a nonzero hypercyclic vector $x \in \mathscr{H}$ for T, and T is said to be *hypertransitive* if every nonzero vector in $\mathscr{H}$ is hypercyclic for T. Denote the set of all nonhypertransitive operators in $x \in \mathscr{H}$ by (NHT). The hypertransitive operator problem is the open question whether $(NHT) = \mathscr{L}(\mathscr{H})$.

PROPOSITION 5. *Let $T \in BAIC(k)$ and be invertible. Then the following properties hold.*

(i) *T is hypercyclic if and only if T^* is hypercyclic.*

(ii) *$T^n \in (NHT)$ if and only if $(T^*)^n \in (NHT)$.*

Proof. Let $T = U|T|$ be the polar decomposition of $T \in \mathscr{L}(\mathscr{H})$.

(i) Suppose that T is hypercyclic. Since $|T|^{\frac{1}{2}}T = \widetilde{T}|T|^{\frac{1}{2}}$, there is a hypercyclic vector $x \in \mathscr{H}$ such that

$$\overline{\mathscr{O}(|T|^{\frac{1}{2}}x, \widetilde{T})} = |T|^{\frac{1}{2}}\mathscr{H}.$$

Since T is invertible, it follows that

$$|T|^{-\frac{1}{2}}\overline{\mathscr{O}(|T|^{\frac{1}{2}}x, \widetilde{T})} = \mathscr{H}.$$

Thus $\widetilde{T}$ is hypercyclic. By the induction, $\widetilde{T}^{(k)}$ is hypercyclic. Since $T \in BAIC(k)$, $\widetilde{T}^{(k)}$ is complex symmetric. From [[22], Lemma 3.8], $(\widetilde{T}^{(k)})^*$ is hypercyclic. By the similar method as the above, T^* is hypercyclic.

(ii) Suppose that $T \in (NHT)$. By (i), T is hypercyclic if and only if T^* is hypercyclic. Since $|T|^{\frac{1}{2}}\mathscr{H} = \mathscr{H}$ for invertible T, we have $T^* \in (NHT)$. It is known from [17, Theorem 1.7] that $T \in (NHT)$ if and only if $T^m \in (NHT)$ for $m \in \mathbb{N}$. Hence $T^n \in (NHT)$ if and only if $(T^*)^n \in (NHT)$. □

Finally, we concern Weyl type theorems for operators belong to $BAIC(k)$. We state the definitions of some spectra;

$$\sigma_{ea}(T) := \cap\{\sigma_a(T+K) : K \in \mathscr{K}(\mathscr{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T) := \cap\{\sigma_a(T+K) : TK = KT \text{ and } K \in \mathscr{K}(\mathscr{H})\}$$

is the Browder essential approximate point spectrum. We put

$$\pi_{00}(T) := \{\lambda \in \text{iso}\,\sigma(T) : 0 < dim \ker(T-\lambda) < \infty\}$$

and

$$\pi_{00}^{a}(T) := \{\lambda \in \text{iso } \sigma_a(T) : 0 < dim \ker(T-\lambda) < \infty\}.$$

Let $T \in \mathcal{L}(\mathcal{H})$. We say that
(i) *a-Browder's theorem holds for T* if $\sigma_{ea}(T) = \sigma_{ab}(T)$;
(ii) *a-Weyl's theorem holds for T* if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}^{a}(T)$;
(iii) T has *the property (w)* if $\sigma_a(T) \setminus \sigma_{ea}(T) = \pi_{00}(T)$.
It is known that

$$\begin{array}{ccc} \text{Property } (w) & \Longrightarrow & a\text{-Browder's theorem} \\ \Downarrow & & \Uparrow \\ \text{Weyl's theorem} & \Longleftarrow & a\text{-Weyl's theorem.} \end{array}$$

We refer the reader to [1] for more details.

Let $T_n = T|_{\text{ran}(T^n)}$ for each nonnegative integer n; in particular, $T_0 = T$. If T_n is upper semi-Fredholm for some nonnegative integer n, then T is called a *upper semi-B-Fredholm* operator. In this case, by [5], T_m is a upper semi-Fredholm operator and $ind(T_m) = ind(T_n)$ for each $m \geqslant n$. Therefore, one can consider the *index* of T, denoted by $ind_B(T)$, as the index of the semi-Fredholm operator T_n. Similarly, we define lower semi-B-Fredholm operators. We say that $T \in \mathcal{L}(\mathcal{H})$ is *B-Fredholm* if it is both upper and lower semi-B-Fredholm. In [5], Berkani proved that $T \in \mathcal{L}(\mathcal{H})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$ where T_1 is Fredholm and T_2 is nilpotent. Let $SBF_{+}^{-}(\mathcal{H})$ be the class of all upper semi-B-Fredholm operators such that $ind_B(T) \leqslant 0$, and let

$$\sigma_{SBF_{+}^{-}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_{+}^{-}(\mathcal{H})\}.$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *B-Weyl* if it is B-Fredholm of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator }\}.$$

We say that $\lambda \in \sigma_a(T)$ is a *left pole* of T if it has finite ascent, i.e., $a(T) < \infty$ and $\text{ran}(T^{a(T)+1})$ is closed where $a(T) = dim \ker(T)$. The notation $p_0(T)$ (respectively, $p_0^a(T)$) denotes the set of all poles (respectively, left poles) of T, while $\pi_0(T)$ (respectively, $\pi_0^a(T)$) is the set of all eigenvalues of T which is an isolated point in $\sigma(T)$ (respectively, $\sigma_a(T)$).

Let $T \in \mathcal{L}(\mathcal{H})$. We say that
(i) T satisfies *generalized Browder's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus p_0(T)$;
(ii) T satisfies *generalized a-Browder's theorem* if $\sigma_{SBF_{+}^{-}}(T) = \sigma_a(T) \setminus p_0^a(T)$;
(iii) T satisfies *generalized Weyl's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T)$;
(iv) T satisfies *generalized a-Weyl's theorem* if $\sigma_{SBF_{+}^{-}}(T) = \sigma_a(T) \setminus \pi_0^a(T)$.
It is known that

$$\begin{array}{ccc} \text{generalized } a\text{-Weyl's theorem} & \Longrightarrow & \text{generalized Weyl's theorem} \\ \Downarrow & & \Downarrow \\ \text{generalized } a\text{-Browder's theorem} & \Longrightarrow & \text{generalized Browder's theorem.} \end{array}$$

THEOREM 4. *Let $T \in BAIC(k)$. If T is a quasiaffinity, then the following properties hold.*

(i) *If T satisfies Weyl's theorem, then T^* satisfies Weyl's theorem.*

(ii) *If T satisfies Browder's theorem, then T^* satisfies Browder's theorem.*

Proof. (i) Suppose that T satisfies Weyl's theorem. Then $\widetilde{T}^{(k)}$ satisfies Weyl's theorem by [18, Theorem 1.21]. In this case, since $\widetilde{T}^{(k)}$ is complex symmetric, $(\widetilde{T}^{(k)})^*$ satisfies Weyl's theorem from [22, Theorem 4.4]. Since T is a quasiaffinity, $\widetilde{T}$ is a quasiaffinity. By induction, $\widetilde{T}^{(k-1)}$ is a quasiaffinity. Let $\widetilde{T}^{(k-1)} = V|\widetilde{T}^{(k-1)}|$ be the polar decomposition of $\widetilde{T}^{(k-1)}$. Since $\widetilde{T}^{(k-1)}$ is a quasiaffinity, V is unitary. By Lemma 1,

$$(\widetilde{T}^{(k)})^* = \left(\widetilde{\widetilde{T}^{(k-1)}}\right)^* = V(\widetilde{\widetilde{T}^{(k-1)}})^* V^*.$$

Then $(\widetilde{\widetilde{T}^{(k-1)}})^*$ satisfies Weyl's theorem. Hence $(\widetilde{T}^{(k-1)})^*$ satisfies Weyl's theorem by [18, Theorem 1.21]. By repeated applications, T^* satisfies Weyl's theorem.

(ii) Suppose that T satisfies Browder's theorem. Then $\widetilde{T}^{(n)}$ satisfies Browder's theorem by [18]. Moreover, since $\widetilde{T}^{(n)}$ is complex symmetric, $(\widetilde{T}^{(n)})^*$ satisfies Browder's theorem from [22, Theorem 4.4]. Hence T^* satisfies Browder's theorem by the similar proof with (i). □

COROLLARY 5. *Let $T \in BAIC(k)$. If T and T^* are quasiaffinities, then the following properties hold.*

(i) *If T satisfies Weyl's theorem if and only if T^* satisfies Weyl's theorem.*

(ii) *If T satisfies Browder's theorem if and only if T^* satisfies Browder's theorem.*

Proof. The proof follows from Theorem 4. □

As usual, we write $\sigma(T)$, $\sigma_a(T)$, $\sigma_p(T)$ and $\sigma_s(T)$ for the spectrum, the approximate point spectrum, the point spectrum, and the surjective spectrum of T, respectively.

LEMMA 2. *Let $T \in BAIC(k)$. Then the following properties hold.*

(i) $\sigma(T) = \sigma_a(T)$.

(ii) $\sigma(T) = \sigma_a(T) = \sigma_s(T)$ *if T has the single-valued extension property.*

(iii) $\sigma_p(T^*) \setminus (0) = \sigma_p(T)^* \setminus (0)$.

(iv) $\sigma_a(T^*) \setminus (0) = \sigma_a(T)^* \setminus (0)$.

(v) $\sigma_{le}(T) = \sigma_e(T)$ *and* $\sigma_{le}(T) \setminus (0) = \sigma_{re}(T) \setminus (0) = \sigma_e(T) \setminus (0)$.

(vi) $\sigma_e(T) = \sigma_{ea}(T) = \sigma_w(T)$.

Proof. (i) Since $\widetilde{T}^{(k)}$ is complex symmetric, it follows from Lemma 3.22 in [20] that $\sigma(\widetilde{T}^{(k)}) = \sigma_a(\widetilde{T}^{(k)})$. Moreover, by Theorem 1.3 in [15], we have

$$\sigma(\widetilde{T}^{(k)}) = \sigma(T) \text{ and } \sigma_a(\widetilde{T}^{(k)}) = \sigma_a(T).$$

Hence we obtain that $\sigma(T) = \sigma_a(T)$.

(ii) Since $\widetilde{T}^{(k)}$ is complex symmetric, it follows from Lemma 3.22 in [20] that

$$\sigma(\widetilde{T}^{(k)}) = \sigma_a(\widetilde{T}^{(k)}) = \sigma_s(\widetilde{T}^{(k)}).$$

It is well known from [1] that if T has the single-valued extension property, then we have $\sigma(T) = \sigma_s(T)$. Since $\sigma_a(T) = \sigma_s(T^*)^*$ for $T \in \mathscr{L}(\mathscr{H})$, we have

$$\sigma_s(T) = \sigma(T) = \sigma(\widetilde{T}^{(k)}) = \sigma_a(\widetilde{T}^{(k)}) = \sigma_a(T).$$

(iii) Since $\widetilde{T}^{(k)}$ is complex symmetric, it follows that $\sigma_p([\widetilde{T}^{(k)}]^*) = [\sigma_p(\widetilde{T}^{(k)})]^*$. By [15], we have $\sigma_p(T^*) \setminus (0) = \sigma_p(T)^* \setminus (0)$.

(iv) The proof follows from the proof of (ii).

(v) Since $\widetilde{T}^{(k)}$ is a complex symmetric operator, it follows from [20, Lemma 3.22] that

$$\sigma_{le}(\widetilde{T}^{(k)}) = \sigma_{re}(\widetilde{T}^{(k)}) = \sigma_e(\widetilde{T}^{(k)}).$$

Moreover, since note that for any $T \in \mathscr{L}(\mathscr{H})$,

$$\sigma_e(T) = \sigma_e(\widetilde{T}), \sigma_{le}(T) = \sigma_{le}(\widetilde{T}), \text{ and } \sigma_{re}(T) \setminus (0) = \sigma_{re}(\widetilde{T}) \setminus (0)$$

hold, it follows from [15, Theorem 1.5] that $\sigma_e(T) = \sigma_e(\widetilde{T}^{(k)}), \sigma_{le}(T) = \sigma_{le}(\widetilde{T}^{(k)})$, and $\sigma_{re}(T) \setminus (0) = \sigma_{re}(\widetilde{T}^{(k)}) \setminus (0)$. Hence we obtain that

$$\sigma_{le}(T) = \sigma_e(T) \text{ and } \sigma_{le}(T) \setminus (0) = \sigma_{re}(T) \setminus (0) = \sigma_e(T) \setminus (0).$$

(vi) Since $\widetilde{T}^{(k)}$ is a complex symmetric operator, it follows from [20, Lemma 3.22] that

$$\sigma_e(\widetilde{T}^{(k)}) = \sigma_{ea}(\widetilde{T}^{(k)}) = \sigma_w(\widetilde{T}^{(k)}).$$

Moreover, since for any $T \in \mathscr{L}(\mathscr{H})$, $\sigma_w(T) = \sigma_w(\widetilde{T})$ holds from [18, Theorem 1.21], we know that $\sigma_w(T) = \sigma_w(\widetilde{T}^{(k)})$. On the other hand, it is known that $\lambda \notin \sigma_{ea}(T)$ if and only if $T - \lambda$ is semi-Fredholm with $\text{ind}\,(T - \lambda) \leqslant 0$. From this fact and [18, Theorem 1.10], we know that $\sigma_{ea}(T) = \sigma_{ea}(\widetilde{T})$ and so $\sigma_{ea}(T) = \sigma_{ea}(\widetilde{T}^{(k)})$. Hence we obtain that $\sigma_e(T) = \sigma_{ea}(T) = \sigma_w(T)$. □

THEOREM 5. *Let $T \in BAIC(k)$. Then the following statements are equivalent:*
(i) *a-Weyl's theorem holds for T.*
(ii) *Weyl's theorem holds for T.*
(iii) *T has the property (w).*

Proof. By the definition, it is trivial that (i) $\Rightarrow$ (ii). Assume that T satisfies Weyl's theorem. Since T is complex symmetric, it follows from Lemma 2 that $\sigma_a(T) = \sigma(T)$ and $\sigma_w(T) = \sigma_{ea}(T)$, which gives that

$$\pi_{00}^a(T) = \pi_{00}(T) = \sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{ea}(T).$$

Hence a-Weyl's theorem holds for T. Thus we have (ii) $\Rightarrow$ (i). Similarly, since $\pi_{00}^a(T) = \pi_{00}(T)$, we show that (i) $\Leftrightarrow$ (iii). □

COROLLARY 6. *Let $T \in BAIC(k)$ be a quasiaffinity and let $T^* \in BAIC(k)$. Then the following statements holds.*

(i) *If T satisfies a-Weyl's theorem, then T^* does.*

(ii) *If T has the property (w), then T^* does.*

Proof. (i) If T satisfies a-Weyl's theorem, then Weyl's theorem holds for T. Since T is a quasiaffinity, it follows that Weyl's theorem holds for T^* by Theorem 4. Since $T^* \in BAIC(k)$, it satisfies a-Weyl's theorem by Theorem 5.

(ii) Let T have the property (w). Since $T \in BAIC(k)$ and T is a quasiaffinity, it follows from Theorem 5 that T^* has the property (w). $\square$

THEOREM 6. *Let $T \in BAIC(k)$ have the single-valued extension property. If T is a quasiaffinity, then the following statements are equivalent.*

(i) *T satisfies generalized a-Weyl's theorem.*

(ii) *T satisfies generalized Weyl's theorem.*

Proof. Since (i) $\Rightarrow$ (ii) follows from [7, Theorem 3.7], it suffices to show that (ii) $\Rightarrow$ (i). Suppose that T satisfies generalized Weyl's theorem. Then we have $\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T)$. Since $T \in BAIC(k)$, it follows from Lemma 2 that $\sigma_a(T) = \sigma(T)$ and so

$$\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T) = \sigma_a(T) \setminus \pi_0^a(T).$$

Hence it suffices to show that $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. If $\lambda \notin \sigma_{SBF_+^-}(T)$, then $T - \lambda$ is semi-B-Fredholm and $ind_B(T - \lambda) \leqslant 0$. Since $T \in BAIC(k)$ and T has the single-valued extension property, it follows from Corollary 2 that T^* has the single-valued extension property. Therefore, we obtain from [1] that $ind_B(T - \lambda) \geqslant 0$ for every $\lambda \notin \sigma_{SBF_+^-}(T)$. Thus we have $ind_B(T - \lambda) = 0$ for every $\lambda \notin \sigma_{SBF_+^-}(T)$, which means that $\sigma_{SBF_+^-}(T) \supset \sigma_{BW}(T)$. Since $\sigma_{SBF_+^-}(T) \subset \sigma_{BW}(T)$ always holds, we obtain that

$$\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_a(T) \setminus \pi_{00}^a(T),$$

that is, generalized a-Weyl's theorem holds for T. $\square$

COROLLARY 7. *Let $T \in BAIC(k)$. If T is a quasiaffinity, then the following arguments are equivalent.*

(i) *T satisfies Browder's theorem.*

(ii) *T satisfies a-Browder's theorem.*

(iii) *T satisfies the generalized Browder's theorem.*

(iv) *T satisfies the generalized a-Browder's theorem.*

Proof. It is well known that (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) from [4, Theorems 2.1 and 2.2]. Since $\sigma(T) = \sigma_a(T)$ from Lemma 2, we know that $p_0(T) = p_0^a(T)$. In addition, $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$ as in the proof of Theorem 6. Using these facts, we obtain that (iii) $\Leftrightarrow$ (iv). Hence we complete the proof. $\square$

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REFERENCES

[1] P. AIENA, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Pub., 2004.

[2] C. APOSTOL, H. BERCOVICI, C. FOIAS, AND C. PEARCY, *Quasiaffine transforms of operators*, Michigan Math. J. **29** (1982), 243–255.

[3] A. ALUTHGE, *On p-hyponormal operators for* $0 < p < 1$, Inter. Equ. Oper. Theory **13** (1990), 307–315.

[4] M. AMOUCH AND H. ZGUITTI, *On the equivalence of Browder's and generalized Browder's theorem*, Glasgow Math. J. **48** (2006), 179–185.

[5] M. BERKANI AND J. J. KOLIHA, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. **69** (2003), 359–376.

[6] C. BENHIDA, M. CHŌ, E. KO, AND J. E. LEE, *Characterizations of a symmetric operator matrix and its transforms*, preprint.

[7] M. BERKANI AND J. J. KOLIHA, *Weyl type theorems for bounded linear operators*, Acta Sci. Math. **69** (2003), 359–376.

[8] I. COLOJOARĂ AND C. FOIAŞ, *Theory of generalized spectral operators*, Gordon and Breah, New York, 1968.

[9] B P. DUGGAL, *Quasisimilar p-hyponormal operators*, Integr. Equ. Oper. Theory **26** (1996), 338–345.

[10] S. R. GARCIA, *Aluthge transforms of complex symmetric operators*, Integr. Equ. Oper. Theory **60** (2008), 357–367.

[11] S. R. GARCIA, D. E. POORE AND J. E. TENER, *Unitary equivalence to a complex symmetric matrix: Low dimensions*, Linear Algebra Appl. **437** (2012), 271–284.

[12] S. R. GARCIA AND W. R. WOGEN, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362** (2010), 6065–6077.

[13] S. R. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006) 1285–1315.

[14] S. R. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications II*, Trans. Amer. Math. Soc. **359** (2007) 3913–3931.

[15] I. JUNG, E. KO AND C. PEARCY, *Aluthge transform of operators*, Integr. Equ. Oper. Theory, **37** (2000), 437–448.

[16] I. JUNG, E. KO AND C. PEARCY, *The iterated Aluthge transform of an operator*, Integr. Equ. Oper. Theory **45** (2003), 375–387.

[17] I. JUNG, E. KO AND C. PEARCY, *Some nonhypertransitive operators*, Pacific J. Math., **220** (2005), 329–340.

[18] I. JUNG, E. KO AND C. PEARCY, *Spectral pictures of Aluthge transform of an operators*, Integr. Equ. Oper. Theory **40** (2001), 52–60.

[19] S. JUNG, E. KO, AND J. E. LEE, *On scalar extensions and spectral decompositions of complex symmetric operators*, J. Math. Anal. Appl., 384 (2011), 252–260.

[20] S. JUNG, E. KO, AND J. E. LEE, *On complex symmetric operators matrices*, J. Math. Anal. Appl. **406** (2013), 373–385.

[21] S. JUNG, E. KO, AND J. E. LEE, *Properties of complex symmetric operators*, Operators and Matrices, **8** (4) (2014), 957–974.

[22] S. JUNG, E. KO, M. LEE, AND J. E. LEE, *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl. **379** (2011), 325–333.

[23] M. KIM AND E. KO, *Some connections between an operator and its Aluthge transform*, Glasgow Math. J. **47** (2005), 167–175.

[24] E. KO, *Backward Aluthge iterates of a hyponormal operator have scalar extensions*, Integr. Equ. Oper. Theory **57** (2007), 567–582.

[25] E. KO AND M. LEE, *On backward Aluthge iterates of a hyponormal operator*, Math. Inequal. Appl. **18** (3) (2015), 1121–1133.

[26] K. B. LAURSEN AND M. M. NEUMANN, *Introduction to Local spectral theory*, London Math. Soc. Monograghs New Series, Claredon Press, Oxford, 2000.

[27] X. H. WANG AND Z. S. GAO, *A note on Aluthge transforms of complex symmetric operators and applications*, Integr. Equ. Oper. Theory, **65** (2009), no. 4, 573–580.
[28] S. ZHU AND C. G. LI, *Complex symmetric weighted shifts*, Trans. Am. Math. Soc. **365** (1), 511–530 (2013).

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