Mathematical
Inequalities
& Applications
Volume 25, Number 2 (2022), 397–405 doi:10.7153/mia-2022-25-24

# DIFFERENTIAL HARNACK ESTIMATE OF SOLUTIONS TO A CLASS OF SEMILINEAR PARABOLIC EQUATION

HUI WU* AND CUIXIAN KONG

(*Communicated by I. Perić*)

*Abstract.* In this paper, we obtain a differential Harnack estimate for a semilinear parabolic equation using parabolic maxinum principle. As applications of this estimate, we derive blow-up of solutions for this equation and a classical Harnack inequality by integrating along space-time paths.

## 1. Introduction

In this paper, we study the following Cauchy problem :

$$\begin{cases} \frac{\partial}{\partial t}w = \triangle w + e^{\gamma t}w^q, & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ w(x,0) = w_0(x), & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1}$$

where $q > 1$, $\gamma \geqslant 0$ and $w_0(x) \geqslant 0$.

In [2], H.Fujita studied the solutions of problem (1) when $\gamma = 0$. He proved that there exists a critical value $q^* := 1 + \frac{2}{n}$, which has the following properties: for $1 < q < q^*$, every non-negative solution blow up in finite time and for $q > q^*$ global solutions exist, if $w_0(x)$ is small enough. For $q = q^*$, the solution of problem (1) blows up in finite time (see [7]). When $\gamma > 0$, Meier [6] found that there are properties similar to those in [2], and that the critical value changes into $q^* = 1 + \frac{\gamma}{\lambda}$ where $\lambda$ denotes the first eigenvalue of the operator -$\triangle$ with homogeneous Dirichlet boundary conditions. The same phenomenon has been observed for many other differential problems (see [4, 8, 10]).

Harnack estimates for elliptic and parabolic partial differential equations have a long history. The study of differential Harnack inequalities was first initiated by P. Li and S.-T. Yau in [5]. This method was later brought into the study of geometric flows by Hamilton (see [3]) and played an important role in the field, especially for the study of the Ricci flow. Differential Harnack inequalities are important in the study of parabolic problems. Some applications include deriving Hölder continuity, obtaining estimates on the heat kernel, classifying ancient and eternal solutions, and so on. For $\gamma = 0$, X.D.

*Mathematics subject classification* (2020): 53C44, 58J35.
*Keywords and phrases*: Differential Harnack inequality, semilinear parabolic equations, maximum principle.
* Corresponding author.


Paper MIA-25-24

Cao [1] gave differential Harnack estimates of solution for (1). On hyperbolic space, [9] described a differential Harnack estimate for (1). Our purpose is to obtain results in Euclidean space, which is different from the case in [9].

Now we address our main result:

THEOREM 1. (Main) *Let $w(x,t)$ be a positive classical solution to* (1)*, and $l(x,t)$ $:= \log w$. There exists $\rho$, $\sigma$, $a$, $c$ $d$ and $\gamma$ satisfing*

$$\rho > \sigma \geqslant 0,\ \gamma \geqslant 0,\ d > 0,\ \frac{\rho(q-1)+2\sigma}{q} \geqslant c \geqslant \frac{(q-1)n\rho^2}{4(\rho-\sigma)}, \tag{2}$$

*and*

$$a \geqslant \frac{nd\rho^2}{2(\rho-\sigma)} > 0. \tag{3}$$

*We have*

$$H_0 \equiv \rho \triangle l + \sigma|\nabla l|^2 + ce^{\gamma t + l(q-1)} + \frac{a}{1-e^{-dt}} \geqslant 0, \tag{4}$$

*for all $t$.*

REMARK 1. The case of $n = 1$ and $q = 2$ was studied by Hamilton in [3]. In particular, we apply Theorem 1.1 with $n = 1$ and $q = 2$ and by picking $\rho = 1$, $\sigma = 0$, $\gamma = 0$, $a = \frac{d}{2}$ and $c = \frac{1}{4}$, to conclude

$$l_{xx} + \frac{1}{4}e^l + \frac{d}{2(1-e^{-dt})} \geqslant 0,$$

yielding

$$w_t + \frac{d}{2(1-e^{-dt})}w \geqslant \frac{w_x^2}{w} + \frac{3}{4}w^2.$$

If $d$ is small enough, the estimate in [3] will be improved.

The organization of this paper is as follows. In Section 2 we derive a differential Harnack estimate. There are applications of our differential Harnack inequality in Section 3. One of our applications is a reproof of the classical result in [6], which states that any positive solution of problem (1) blows up in the finite time provided $1 < q < 1 + \frac{\gamma}{\lambda}$.

## 2. Differential Harnack estimate

In this section, we shall first derive our differential Harnack estimate, relying on the parabolic maximum principle.

LEMMA 1. *Assume that $w(x,t)$ is a positive solution to* (1) *and $l = \log w$. Suppose that $H$ is defined as follows:*

$$H := \rho \triangle l + \sigma|\nabla l|^2 + ce^{\gamma t + l(q-1)} + \psi(x,t), \tag{5}$$

*Where $\rho,\sigma,c\in\mathbb{R},\gamma\geqslant 0$ and $\psi$ is a test function to be chosen later. Then we have*

$$\begin{aligned} H_t =&\triangle H+2\nabla H\cdot\nabla l+(q-1)e^{\gamma t+l(q-1)}H+2(\rho-\sigma)|\nabla\nabla l|^2\\ &+(\rho(q-1)+\sigma-cq)(q-1)e^{\gamma t+l(q-1)}|\nabla l|^2+c\gamma e^{\gamma t+l(q-1)}\\ &-(q-1)e^{\gamma t+l(q-1)}\psi+\psi_t-\triangle\psi-2\nabla\psi\cdot\nabla l. \end{aligned} \tag{6}$$

*Proof.* Substituting $w=\mathrm{e}^l$ into (1), we have

$$l_t=\triangle l+|\nabla l|^2+e^{\gamma t+l(q-1)}.$$

Recall the formula:

$$\Delta|\nabla l|^2=2|\nabla\nabla l|^2+2\nabla l\cdot\nabla\Delta l.$$

Then we can compute the following evolution equations:

$$\begin{aligned} H_t&=\rho(\triangle l)_t+\sigma(|\nabla l|^2)_t+c(e^{\gamma t+l(q-1)})_t+\psi_t,\\ \partial_t(\triangle l)&=\triangle(\triangle l)+\triangle|\nabla l|^2+(q-1)^2e^{\gamma t+l(q-1)}|\nabla l|^2+(q-1)e^{\gamma t+l(q-1)}\triangle l,\\ \partial_t(|\nabla l|^2)&=\triangle|\nabla l|^2-2|\nabla\nabla l|^2+2\nabla l\cdot\nabla|\nabla l|^2+2(q-1)e^{\gamma t+l(q-1)}|\nabla l|^2, \end{aligned}$$

and

$$\begin{aligned} \partial_t(e^{\gamma t+l(q-1)})=&\gamma e^{\gamma t+l(q-1)}+(q-1)e^{\gamma t+l(q-1)}(\triangle l)\\ &+(q-1)e^{\gamma t+l(q-1)}|\nabla l|^2+(q-1)e^{2(\gamma t+l(q-1))}. \end{aligned}$$

Hence we get

$$\begin{aligned} H_t =&\rho\left[\triangle(\triangle l)+\triangle|\nabla l|^2+(q-1)e^{\gamma t+l(q-1)}\triangle l+(q-1)^2e^{\gamma t+l(q-1)}|\nabla l|^2\right]\\ &+\sigma\left[\triangle|\nabla l|^2-2|\nabla\nabla l|^2+2\nabla l\cdot\nabla|\nabla l|^2+2(q-1)e^{\gamma t+l(q-1)}|\nabla l|^2\right]\\ &+c(q-1)e^{\gamma t+l(q-1)}\left[\frac{\gamma}{q-1}+\triangle l+|\nabla l|^2+e^{\gamma t+l(q-1)}\right]+\psi_t. \end{aligned} \tag{7}$$

A direct calculation gives

$$\triangle H=\rho\triangle(\triangle l)+\sigma\triangle(|\nabla l|^2)+c(q-1)e^{\gamma t+l(q-1)}\Big((q-1)|\nabla l|^2+\triangle l\Big)+\triangle\psi,$$

and

$$\nabla H=\rho\nabla(\triangle l)+\sigma\nabla(|\nabla l|^2)+c(q-1)e^{\gamma t+l(q-1)}\nabla l+\nabla\psi.$$

Hence reordering (7), we get (6). □

*Proof of Theorem* 1. Difine the $n$-rectangle $D:=\prod_{i=1}^{n}[p_i,q_i]\subset\mathbb{R}^n$. Set

$$\psi_D(x,t)=\frac{a}{1-e^{-dt}}+\sum_{k=1}^{n}\left(\frac{b}{(x_k-p_k)^2}+\frac{b}{(q_k-x_k)^2}\right) \tag{8}$$

for $t>0,\ d>0,\ b>0$ and $x=(x_1,\cdots,x_n)\in D$, while $\psi_D\to+\infty$ as $x_i\to p_i,q_i$ or $t\to 0$.

The corresponding (5) is

$$H_D := \rho\triangle l+\sigma|\nabla l|^2+ce^{\gamma t+l(q-1)}+\psi_D(x,t).$$

Note that $H_D\to H_0$ as $D\to\mathbb{R}^n$, and $H_D>0$ for small $t$.

To get a contradiction, assume that there exists a first time $t_0$ and point $x_0\in D$ such that $H_D(x_0,t_0)=0$. At $(x_0,t_0)$, we have

$$(H_D)_t\leqslant 0,\ \nabla H_D=0,\ \triangle H_D\geqslant 0,$$

and

$$\triangle l=-\frac{1}{\rho}(\sigma|\nabla l|^2+ce^{\gamma t+l(q-1)}+\psi_D). \tag{9}$$

Using (6) and Cauchy-Schwarz inequality $|\nabla\nabla l|^2\geqslant\frac{1}{n}(\triangle l)^2$, we can get

$$\begin{aligned}0\geqslant (H_D)_t =&\triangle H_D+2\nabla H_D\cdot\nabla l+(q-1)e^{\gamma t+l(q-1)}H_D+2(\rho-\sigma)|\nabla\nabla l|^2\\&+[\rho(q-1)+\sigma-cq](q-1)e^{\gamma t+l(q-1)}|\nabla l|^2+c\gamma e^{\gamma t+l(q-1)}\\&-(q-1)e^{\gamma t+l(q-1)}\psi_D+(\psi_D)_t-\triangle\psi_D-2\nabla\psi_D\cdot\nabla l\\\geqslant&\frac{2(\rho-\sigma)}{n}(\triangle l)^2+[\rho(q-1)+\sigma-cq](q-1)e^{\gamma t+l(q-1)}|\nabla l|^2\\&+c\gamma e^{\gamma t+l(q-1)}-(q-1)e^{\gamma t+l(q-1)}\psi_D+(\psi_D)_t-\triangle\psi_D-2\nabla\psi_D\cdot\nabla l.\end{aligned} \tag{10}$$

Set $X=e^{\gamma t+l(q-1)}$ and $Y=|\nabla l|^2$. Applying (9) to (10) and combining terms gives

$$\begin{aligned}0\geqslant&\frac{2(\rho-\sigma)}{n\rho^2}(\sigma|\nabla l|^2+ce^{\gamma t+l(q-1)}+(\psi_D)^2+[\rho(q-1)+\sigma-cq](q-1)e^{\gamma t+l(q-1)}|\nabla l|^2\\&+c\gamma e^{\gamma t+l(q-1)}-(q-1)e^{\gamma t+l(q-1)}\psi_D+(\psi_D)_t-\triangle\psi_D-2\nabla\psi_D\cdot\nabla l\\=&\frac{2(\rho-\sigma)}{n\alpha^2}[\sigma Y+cX+\psi_D]^2+[\rho(q-1)+\sigma-cq](q-1)XY\\&+c\gamma X-(q-1)X\psi_D+(\psi_D)_t-\triangle\psi_D-2\nabla\psi_D\cdot\nabla l\\=&\frac{2(\rho-\sigma)}{n\rho^2}(c^2X^2+\sigma^2Y^2)+\left[\rho(q-1)-cq+\sigma+\frac{4(\rho-\sigma)\sigma c}{n\rho^2(q-1)}\right](q-1)XY+c\gamma X\\&+\left[\frac{4(\rho-\sigma)c}{n\rho^2}-(q-1)\right]\psi_D X+\frac{4(\rho-\sigma)\sigma}{n\rho^2}\psi_D Y\\&+(\psi_D)_t-\triangle\psi_D-2\nabla\psi_D\cdot\nabla l+\frac{2(\rho-\sigma)}{n\rho^2}(\psi_D)^2.\end{aligned} \tag{11}$$

According to (2), we get

$$\rho(q-1)-cq+\sigma+\frac{4(\rho-\sigma)\sigma c}{n\rho^2(q-1)}\geqslant 0,\quad \frac{4(\rho-\sigma)c}{n\rho^2}-(q-1)\geqslant 0. \tag{12}$$

Completing the square, we have

$$\frac{4(\rho-\sigma)\sigma}{n\rho^2}\psi_D Y - 2\nabla\psi_D\cdot\nabla l \geqslant \frac{-n\rho^2|\nabla\psi_D|^2}{4(\rho-\sigma)\sigma\psi_D}. \tag{13}$$

Applying (12) and (13) to (11) gives

$$0 \geqslant (\psi_D)_t - \triangle\psi_D - \frac{n\rho^2|\nabla\psi_D|^2}{4(\rho-\sigma)\sigma\psi_D} + \frac{2(\rho-\sigma)}{n\rho^2}(\psi_D)^2. \tag{14}$$

To arrived at a contradiction, it suffices to have

$$(\psi_D)_t - \triangle\psi_D - \frac{n\rho^2|\nabla\psi_D|^2}{4(\rho-\sigma)\sigma\psi_D} + \frac{2(\rho-\sigma)}{n\rho^2}(\psi_D)^2 > 0. \tag{15}$$

We can compute

$$\triangle\psi_D = \sum_{k=1}^{n}\Big(\frac{6b}{(x_k-p_k)^4} + \frac{6b}{(q_k-x_k)^4}\Big), \tag{16}$$

$$|\nabla\psi_D|^2 = \sum_{k=1}^{n}\Big(-\frac{2b}{(x_k-p_k)^3} + \frac{2b}{(q_k-x_k)^3}\Big)^2,$$

and

$$\begin{aligned}\frac{|\nabla\psi_D|^2}{\psi_D} &= \sum_{k=1}^{n}\Big(-\frac{2b}{(x_k-p_k)^3\sqrt{\psi_D}} + \frac{2b}{(q_k-x_k)^3\sqrt{\psi_D}}\Big)^2\\ &\leqslant \sum_{k=1}^{n}\Big(\frac{4b}{(x_k-p_k)^4} + \frac{4b}{(q_k-x_k)^4}\Big).\end{aligned} \tag{17}$$

For the sake of simplicity, we set

$$A := \frac{2(\rho-\sigma)}{n\rho^2} > 0,\ B := \frac{n\rho^2}{4(\rho-\sigma)\sigma} > 0.$$

Next, plugging (8), (16) and (17) into (15), we get

$$\begin{aligned}&A\Big[\frac{a}{(1-e^{-dt})} + \sum_{k=1}^{n}\Big(\frac{b}{(x_k-p_k)^2} + \frac{b}{(q_k-x_k)^2}\Big)\Big]^2 - \Big[\sum_{k=1}^{n}\Big(\frac{6b}{(x_k-p_k)^4} + \frac{6b}{(q_k-x_k)^4}\Big)\Big]\\ &\quad - B\Big[\sum_{k=1}^{n}\Big(-\frac{2b}{(x_k-p_k)^3\sqrt{\psi_D}} + \frac{2b}{(q_k-x_k)^3\sqrt{\psi_D}}\Big)^2\Big] - \frac{da}{(1-e^{-dt})^2e^{dt}}\\ &\geqslant \frac{Aa^2e^{dt}-da}{(1-e^{-dt})^2e^{dt}} + (Ab^2-6b-4bB)\Big[\sum_{k=1}^{n}\Big(\frac{1}{(x_k-p_k)^4} + \frac{1}{(q_k-x_k)^4}\Big)\Big].\end{aligned}$$

According (3), $Aa^2e^{dt}-da \geqslant 0$.

If we prove (15), it needs

$$Ab^2 - b(6+4B) > 0.$$

In summary, $a$ and $b$ satisfy

$$a \geqslant \frac{nd\rho^2}{2(\rho-\sigma)}, \quad b > \frac{n\rho^2}{2(\rho-\sigma)}\Big[6+\frac{n\rho^2}{(\rho-\sigma)\sigma}\Big].$$

According to the range of $\rho$, $\sigma$, $a$, $c$ in Theorem1, the right hand side of (11) is positive, which is a contradiction. This is because $b$ is independent of $x_k$, $k=1,\cdots,n$, so when the solution exists in $\mathbb{R}^n$ we can let $D \to \mathbb{R}^n$ and these terms drop out. Theorem 1 is thus proved. $\square$

## 3. Applications

In this section, we shall give a few applications of Theorem 1. Firstly, we use it to obtain blow-up of solutions of (1) in finite time, then we integrate along space-time paths to derive a classical Harnack inequality.

### 3.1. Finite time blow-Up

PROPOSITION 1. *Assume that $w$ is a positive solution to* (1)*, $c$ is a constant satisfies that $0 < \lambda(q-1) \leqslant c < \gamma$. Then $w$ blows up in finite time provided that*

$$w(x_0,t_0) \geqslant \left(\frac{d\gamma n}{(\gamma-c)(1-e^{-d})e^{\gamma}}\right)^{\frac{1}{(q-1)}}. \tag{18}$$

*at some point $(x_0,t_0)$.*

*Proof.* Picking $\rho=\gamma$, $\sigma=\frac{\gamma}{2}$, $a=d\gamma n$ and $\max\{\frac{(q-1)n\rho^2}{4(\rho-\sigma)},\lambda(q-1)\} \leqslant c < \gamma$ in (4) yields

$$\gamma\triangle w - \frac{\gamma}{2w}|\nabla w|^2 + ce^{\gamma t}w^q + \frac{d\gamma n}{(1-e^{-dt})}w \geqslant 0.$$

Since $w_t = \triangle w + e^{\gamma t}w^q$, we have

$$\gamma w_t + \frac{d\gamma n w}{(1-e^{-dt})} \geqslant (\gamma-c)e^{\gamma t}w^q. \tag{19}$$

Hence

$$\gamma\frac{\partial}{\partial t}\Big(\frac{1}{w}\Big) \leqslant \frac{1}{w}\Big(\frac{d\gamma n}{(1-e^{-dt})} - (\gamma-c)e^{\gamma t}w^{q-1}\Big) = \frac{1}{w^{2-q}}\Big(\frac{d\gamma n}{(1-e^{-dt})w^{q-1}} - (\gamma-c)e^{\gamma t}\Big). \tag{20}$$

Without loss of generality, we may assume that $w \geqslant \left(\frac{d\gamma n}{(\gamma-c)(1-e^{-d})e^{\gamma}}\right)^{\frac{1}{(q-1)}}$ at the origin $x_0 = 0$ for $t_0 = 1$. This assumption together with (20) gives

$$\gamma\frac{\partial}{\partial t}\left(\frac{1}{w}\right)(0,t) \leqslant \frac{\gamma-c}{w^{2-q}(0,t)}\left(e^{\gamma}-e^{\gamma t}\right) < 0,$$

so that $w(0,t)$ is strictly increasing for $t \geqslant 1$ provided that $w(0,t)$ is finite.

Using (20) and monotonicity of $w(0,t)$, if $q > 2$, then $w^{q-2}(0,t) \geqslant w^{q-2}(0,1)$ for $t \geqslant 1$ and

$$\gamma\frac{\partial}{\partial t}\left(\frac{1}{w}\right)(0,t) \leqslant \frac{d\gamma n}{(1-e^{-dt})w(0,1)} - (\gamma-c)e^{\gamma t}w^{q-2}(0,1),$$

and if $1 \leqslant q \leqslant 2$, then

$$\frac{\gamma}{q-1}\frac{\partial}{\partial t}\left[\left(\frac{1}{w}\right)^{q-1}\right](0,t) = \gamma w^{2-q}\frac{\partial}{\partial t}\left(\frac{1}{w}\right)(0,t) \leqslant \frac{d\gamma n}{(1-e^{-dt})w^{q-1}(0,t)} - (\gamma-c)e^{\gamma t}.$$

In both cases, when $t$ is large enough, the right hand sides are less than zero, so that $\frac{1}{w} \to 0$ in finite time. This completes the proof. $\square$

### 3.2. Classical Harnack inequality

In this subsection, we integrate our differential Harnack inequality of (4) along space-time curve to derive a classical Harnack inequality.

COROLLARY 1. *Let $w(x,t)$ be a positive classical solution to (1) and $l(x,t) := \log w$. Suppose that $x_1$, $x_2 \in \mathbb{R}^n$ and $t_2 > t_1 > 0$. Assume further that $\rho \geqslant 2\sigma$, $\rho \geqslant c$ and $a = \frac{nd\rho^2}{2(\rho-\sigma)} \leqslant nd\rho$. Then we have*

$$w(x_1,t_1) \leqslant w(x_2,t_2)\left(\frac{e^{dt_2}-1}{e^{dt_1}-1}\right)^n exp\left[\frac{|x_2-x_1|^2}{2(t_2-t_1)}\right]. \tag{21}$$

*Proof.* Let $\mu(t) = (x(t),t)$, $t \in [t_1,t_2]$ be a space-time curve joining two given points $(x_1,t_1)$, $(x_2,t_2) \in \mathbb{R}^n \times [0,+\infty)$ with $0 < t_1 < t_2$.

Applying (4), we have

$$\triangle l \geqslant \frac{1}{\rho}\left(-\sigma|\nabla l|^2 - ce^{\gamma t + l(q-1)} - \frac{a}{(1-e^{-dt})}\right).$$

It yields that

$$\begin{aligned}\frac{d}{dt}[l(x(t),t)] &= \nabla l \cdot \dot{x} + l_t \\ &= \nabla l \cdot \dot{x} + \triangle l + |\nabla l|^2 + ce^{\gamma t + l(q-1)} \\ &\geqslant |\nabla l|^2\left(1 - \frac{\sigma}{\rho}\right) + \nabla l \cdot \dot{x} - \frac{a}{\rho(1-e^{-dt})} + e^{\gamma t + l(q-1)}\left(1 - \frac{c}{\rho}\right) \\ &\geqslant |\nabla l|^2\left(\frac{1}{2} - \frac{\sigma}{\rho}\right) - \frac{1}{2}|\dot{x}|^2 - \frac{a}{\rho(1-e^{-dt})} + e^{\gamma t + l(q-1)}\left(1 - \frac{c}{\rho}\right) \\ &\geqslant -\frac{1}{2}|\dot{x}|^2 - \frac{a}{\rho(1-e^{-dt})},\end{aligned}$$

where $\rho \geqslant 2\sigma$ and $\rho \geqslant c$. Hence, we get

$$\frac{d}{dt}[-l(x(t),t)] \leqslant \frac{1}{2}|\dot{x}|^2 + \frac{nd}{(1-e^{-dt})}.$$

Integrating the above inequality along $\mu$, we have

$$l(x_1,t_1) - l(x_2,t_2) \leqslant \inf_{\mu(t)=(x(t),t)} \int_{t_1}^{t_2} \left[\frac{1}{2}|\dot{x}|^2 + \frac{nd}{(1-e^{-dt})}\right] dt.$$

Recalling that $l = \log w$, we arrive at (21). □

*Acknowledgements.* The authors would like to express their deep gratitude to the anonymous referee for a careful reading and valuable suggestions. This work is partially supported by National Natural Science Foundation of China (No. 11801306), and China Postdoctoral Science Foundation (No. 2020M672023) and the Natural Science Foundation of Shandong Province (No. ZR2020MA018).

## REFERENCES

[1] X. D. CAO, M. CERENZIA AND D. KAZARAS, *Harnack estimate for the Endangered species equation*, Proc. Amer. Math. Soc. **143** (10) (2015) 4537–4545.

[2] H. FUJITA, *On the blowing up of solutions of the Cauchy problem for* $u_t = \triangle u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect.I, **13** (1966) 109–124.

[3] R. S. HAMILTON, *Li-Yau estimates and their Harnack inequalities*, In: Geometry and analysis, No. 1, in: Adv. Lect. Math. (ALM), vol. **17**, Int. Press, Somerville, MA, 2011, 329–362.

[4] H. A. LEVINE, *The role of critical exponents in blowup theorems*, SIAM Rev. **32** (1990) 262–288, 1990.

[5] P. LI AND S.-T. YAU, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (3–4) (1986) 153–201.

[6] P. MEIER, *On the critical exponent for reaction-diffusion equations*, Arch. Ration. Mech. Anal. **109** (1990) 63–71.

[7] F. B. WEISSLER, *Existence and nonexsitence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981) 29–40.

[8] H. WU, *A Fujita-type result for a semilinear equation in hyperbolic space*, J. Aust. Math. Soc. **103** (3) (2017) 420–429.

[9] H. WU AND L. H. MIN, *Differential Harnack estimate for a semilinear parabolic equation on hyperbolic space*, Appl. Math. Lett. **50** (2015) 69–77.
[10] H. WU AND X. P. YANG, *Global existence and finite time blow-up for a parabolic system on hyperbolic space*, J. Math. Phy. **59** (2018) Article ID: 011505, 11 pages.

(Received August 28, 2020)

*Hui Wu*
*School of Mathematical Science*
*Qufu Normal University*
*Qufu 273165, P. R. China*
*e-mail:* huiwu@qfnu.edu.cn

*Cuixian Kong*
*School of Mathematical Science*
*Qufu Normal University*
*Qufu 273165, P. R. China*
*e-mail:* 17862930545@163.com