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ON THE ERDŐS–LAX AND TURÁN INEQUALITIES CONCERNING POLYNOMIALS

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Abstract. In this paper, we prove certain sharp inequalities that relate the uniform norm of the derivative and the polynomial itself, in case when the zeros are outside or inside some closed disk. We further extend the obtained results to the polar derivative of a polynomial. The obtained results strengthen some recently proved Erdős-Lax and Turán-type inequalities contained in a paper published recently by Kumar [*Complex Anal. Oper. Theory* **14**, 65 (2020)], as well as other related inequalities.

1. Introduction

Let $P(z)$ be a polynomial of degree n and $P'(z)$ its derivative. Turán's inequality [23] that relates the norm of a polynomial to that of its derivative on the unit circle states that if $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1}$$

On the other hand, if $P(z)$ has no zeros in $|z| < 1$, then Erdős conjectured and later Lax [10] proved that

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

Thus in (1) as well as in (2) equality holds for those polynomials of degree n which have all their zeros on $|z| = 1$. Various versions of these inequalities are a classical topic in analysis. As a generalization of (1), Govil [7] proved that if $P(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{3}$$

where as, for the class of polynomials not vanishing in $|z| < k$, $k \leqslant 1$, the precise estimate of maximum of $|P'(z)|$ on $|z| = 1$ does not seem to be known in general. In

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1980, it was again Govil [6], who generalized (2) by proving that if $P(z)$ does not vanish in $|z| < k$, $k \leqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|, \tag{4}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P(1/\overline{z})}$. Both the inequalities (3) and (4) are best possible and hold with equality for $P(z) = z^n + k^n$. Although the inequality (3) is sharp but it has a drawback. The bound in this inequality depends on the zero of the largest modulus and not on other zeros even if some of them are very close to the origin. This was taken into consideration by Aziz [1], who proved that if $P(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree n with $|z_\nu| \leqslant k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{2}{1+k^n} \sum_{\nu=1}^{n} \frac{k}{k+|z_\nu|} \max_{|z|=1} |P(z)|. \tag{5}$$

A similar type of modification to (4) was given by Aziz and Ahmad [2], who proved that if $P(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leqslant 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{1}{1+k^n} \left\{ n - k^n \sum_{\nu=1}^{n} \frac{|z_\nu| - k}{|z_\nu| + k} \right\} \max_{|z|=1} |P(z)|, \tag{6}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. Equality in (6) holds for $P(z) = z^n + k^n$.

Very recently, Kumar [9] strengthened the bound in (5) by involving the modulus of each zero and some of the coefficients of the underlying polynomial. In fact, Kumar proved that if

$$P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n} (z - z_\nu)$$

is a polynomial of degree n having all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \left\{ \frac{2}{1+k^n} + \frac{(|a_n|k^n - |a_0|)(k-1)}{(1+k^n)(|a_n|k^n + k|a_0|)} \right\} \sum_{\nu=1}^{n} \frac{k}{|z_\nu| + k} \max_{|z|=1} |P(z)|. \tag{7}$$

In the same paper, Kumar proved an extension of (4) by showing that if $P(z)$ is a polynomial of degree n having no zeros in $|z| < k$, $k \leqslant 1$, and $Q(z) = z^n\overline{P(1/\overline{z})}$, then

$$\max_{|z|=1} |P'(z)| \leqslant \left\{ n - \left[\frac{2k^n}{1+k^n} + \frac{k^n(|a_0| - |a_n|k^n)(1-k)}{(1+k^n)(|a_0|k + |a_n|k^n)} \right] \sum_{\nu=1}^{n} \frac{|z_\nu|}{|z_\nu| + k} \right\} \max_{|z|=1} |P(z)|, \tag{8}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$. Equality in (7) and (8) holds for $P(z) = z^n + k^n$.

For a polynomial $P(z)$ of degree n, we define

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z),$$

the polar derivative of $P(z)$ with respect to the point α. The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left\{\frac{D_\alpha P(z)}{\alpha}\right\}=P'(z),$$

uniformly with respect to z for $|z|\leqslant R$, $R>0$.

Various results of majorization on the polar derivative of a polynomial can be found in the comprehensive books of Milovanović et al. [14], Marden [11] and Rahman and Schmeisser [22], where some approaches to obtaining polynomial inequalities are developed on applying the methods and results of the geometric function theory. For the latest research and development in this direction, one can see some of the papers ([9], [12], [13], [15]–[20]). By using the new version of the Schwarz lemma, Kumar [9] in the same paper also proved the polar derivative generalizations of (7) and (8) in the form of the following results.

THEOREM A. *Let $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu=a_n\prod_{\nu=1}^{n}(z-z_\nu)$ be a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\geqslant 1$. Then for any complex number α with $|\alpha|\geqslant k$,*

$$\max_{|z|=1}|D_\alpha P(z)|\geqslant\frac{2(|\alpha|-k)}{1+k^n}\left\{1+\frac{(|a_n|k^n-|a_0|)(k-1)}{2(|a_n|k^n+k|a_0|)}\right\}\times\sum_{\nu=1}^{n}\frac{k}{|z_\nu|+k}\max_{|z|=1}|P(z)|. \tag{9}$$

THEOREM B. *Let $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu=a_n\prod_{\nu=1}^{n}(z-z_\nu)$ be a polynomial of degree n having no zeros in $|z|<k$, $k\leqslant 1$, and $Q(z)=z^n\overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for any complex numbers α with $|\alpha|\geqslant 1$,*

$$\max_{|z|=1}|D_\alpha P(z)|\leqslant\left\{n|\alpha|-(|\alpha|-1)\left[\frac{2k^n}{1+k^n}+\frac{k^n(|a_0|-|a_n|k^n)(1-k)}{(1+k^n)(|a_0|k+|a_n|k^n)}\right]\right\}\times\sum_{\nu=1}^{n}\frac{|z_\nu|}{|z_\nu|+k}\max_{|z|=1}|P(z)|. \tag{10}$$

REMARK 1. All the previous results can be formulated for monic polynomials, without loss of generality, so that we can put $a_n=1$ in the inequalities (7)–(10), obtaining simpler expressions.

If we divide both sides of (9) and (10) by $|\alpha|$ and let $|\alpha|\to\infty$, we get respectively (7) and (8). It is important to mention here that Kumar [9] has mentioned in the last part of his paper that (8) and (10) are possibly the best available bounds so far towards the problem of generalizing the Erdős-Lax inequality for the class of polynomials having no zeros in $|z|<k$, $k\leqslant 1$. Motivated by this, the authors are curious to establish some improved bounds of Erdős-Lax and Turán-type for the derivative and polar derivative of a polynomial. The obtained results produce refinements of (7)–(10) and related inequalities.

2. Main results

As we mentioned in Remark 1, without loss of generality, in the sequel we consider only the class of monic polynomials $\widehat{\mathcal{P}}_n$ of degree n, with the complex zeros z_ν, $\nu = 1,\dots,n$, i.e.,

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{\nu=1}^{n}(z - z_\nu), \tag{11}$$

for which we introduce the following quantities

$$A_n(k) = \sum_{\nu=1}^{n} \frac{k}{|z_\nu| + k}, \quad B_n(k) = \sum_{\nu=1}^{n} \frac{|z_\nu|}{|z_\nu| + k}, \quad m = \min_{|z|=k} |P(z)|. \tag{12}$$

We note that $A_n(k) + B_n(k) = n$.

We begin now by presenting the following strengthening of (7).

THEOREM 1. *Let $P(z) \in \widehat{\mathcal{P}}_n$ having all its zeros in $|z| \leqslant k$, $k \geqslant 1$. Then for $0 \leqslant t \leqslant 1$, we have*

$$\max_{|z|=1} |P'(z)| \geqslant \frac{2A_n(k)}{1+k^n} \left\{ \left[1 + \frac{(k^n - |a_0| - tm)(k-1)}{2(k^n + k|a_0| - tm)} \right] \max_{|z|=1} |P(z)| \right.$$
$$\left. + \frac{1}{2k^n} \left[k^n - 1 - \frac{(k^n - |a_0| - tm)(k-1)}{k^n + k|a_0| - tm} \right] tm \right\}, \tag{13}$$

Equality in (13) holds for $P(z) = z^n + k^n$.

REMARK 2. For $t = 0$, (13) reduces to (7) $(a_n = 1)$. In fact excepting the case when some or all the zeros of $P(z)$ lie on $|z| = k$, the bound obtained in (13) is always sharper than the bound obtained in (7). As an illustration we consider two polynomials of degree four.

Case (a): Let $P(z) = z^4 - 2z^3 + 4z - 4$, with all zeros $\{-\sqrt{2}, \sqrt{2}, 1-\mathrm{i}, 1+\mathrm{i}\}$ on the circle $|z| = \sqrt{2}$, so that Theorem 1 holds for $k \geqslant \sqrt{2}$. Since

$$M = \max_{|z|=1} |P(z)| = \max_{0 \leqslant \theta < 2\pi} \sqrt{37 - 36\cos\theta - 16\cos 2\theta + 24\cos 3\theta - 8\cos 4\theta}$$

and

$$M_1 = \max_{|z|=1} |P'(z)| = \max_{0 \leqslant \theta < 2\pi} \sqrt{68 - 48\cos\theta - 48\cos 2\theta + 32\cos 3\theta},$$

i.e., $M = 9.6142743738\ldots$ and $M_1 = 12.250756577\ldots$, as well as

$$m = \min_{|z|=k} |P(z)| = P(k) = (k^2 - 2)\big((k-1)^2 + 1\big) \quad (k \geqslant \sqrt{2}),$$

the right hand side of the inequality (13) is presented in Fig. 1 for $t = 0$ and $t = 1$. The curve for $t = 0$ is in fact, the right hand side of (7) $(a_n = 1)$.

Case (b): Let $P(z) = z^4 + 4$, with all zeros $\{1-\mathrm{i}, 1+\mathrm{i}, -1-\mathrm{i}, -1+\mathrm{i}\}$ on the circle $|z| = \sqrt{2}$. Again Theorem 1 holds for $k \geqslant \sqrt{2}$, here with $M = 5$, $M_1 = 4$, and $m = \min_{|z|=k} |P(z)| = k^4 - 4$. The corresponding graphics for $t = 0$, $t = 1/2$ and $t = 1$ are presented in Fig. 2. As we can see the sharpest inequality is obtained for $t = 1$. Evidently, equality in (13) holds for $k = \sqrt{2}$ for the polynomial $z^4 + 4$.

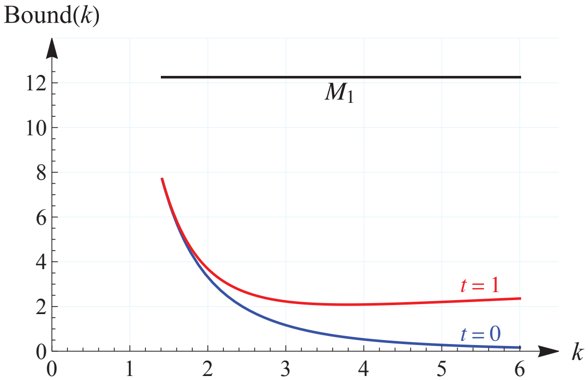

Figure 1: *Bounds obtained by Theorem 1 when $\sqrt{2} \leqslant k \leqslant 6$: Case (a) for $t=0$ and $t=1$*

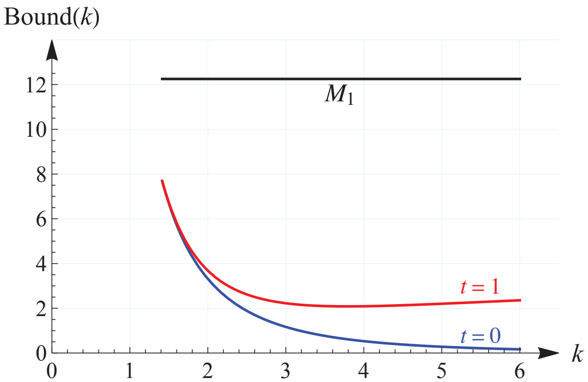

Figure 2: *Bounds obtained by Theorem 1 when $\sqrt{2} \leqslant k \leqslant 6$: Case (b) for $t=0,1/2,1$*

It is also easy to see that (13) includes the following inequality due to Dewan and Upadhye [4]:

$$\max_{|z|=1}|P'(z)| \geqslant \sum_{\nu=1}^{n}\frac{k}{|z_\nu|+k}\left\{\frac{2}{1+k^n}\max_{|z|=1}|P(z)|+\frac{k^n-1}{k^n(k^n+1)}\min_{|z|=k}|P(z)|\right\}.$$

In the sequel we prove the following refinement of (8), which in turn strengthens the bound in (6).

THEOREM 2. *Let $P(z)\in\widehat{\mathcal{P}}_n$ having no zeros in $|z|<k$, $k\leqslant 1$, and let $Q(z)=z^n\overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then for every $0\leqslant t\leqslant 1$, we have*

$$\max_{|z|=1}|P'(z)| \leqslant \left\{n-\left[1+\frac{(|a_0|-k^n-tm)(1-k)}{2k(|a_0|+k^{n-1}-tm)}\right]\frac{2k^nB_n(k)}{1+k^n}\right\}\max_{|z|=1}|P(z)|$$
$$-tm\left(1-k^n-\frac{k^n(|a_0|-k^n-tm)(1-k)}{k(|a_0|+k^{n-1}-tm)}\right)\frac{B_n(k)}{1+k^n}. \tag{14}$$

Equality in (14) *holds for* $P(z)=z^n+k^n$.

REMARK 3. For $t=0$, (14) reduces to (8) $(a_n=1)$. The following result which is a refinement of (6) immediately follows from Theorem 2 for $t=1$.

COROLLARY 1. *Let* $P(z)\in\widehat{\mathcal{P}}_n$ *having no zeros in* $|z|<k$, $k\leqslant 1$, *and let* $Q(z)=z^n\overline{P(1/\overline{z})}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on* $|z|=1$, *then*

$$\max_{|z|=1}|P'(z)| \leqslant \frac{1}{1+k^n}\left\{\left(n-k^n\sum_{\nu=1}^{n}\frac{|z_\nu|-k}{|z_\nu|+k}\right)\max_{|z|=1}|P(z)|\right.$$
$$\left.-(1-k^n)\sum_{\nu=1}^{n}\frac{|z_\nu|}{|z_\nu|+k}\min_{|z|=k}|P(z)|\right\}. \tag{15}$$

Equality in (15) *holds for* $P(z)=z^n+k^n$.

In the literature, we can see a series of papers where inequalities involving the ordinary derivatives have been extended to the polar derivatives. Here, we are interested to extend Theorems 1 and 2 to the polar derivative of a polynomial and to get compact generalizations of these results. These generalizations involve the proofs of Theorems 1 and 2 as well, so it is obligatory to establish Theorems 1 and 2 before their respective generalizations in the form of Theorems 3 and 4. In this context, our next result is a polar derivative generalization of Theorem 1 which also provides a refinement of Theorem A.

THEOREM 3. *Let* $P(z)\in\widehat{\mathcal{P}}_n$ *having all its zeros in* $|z|\leqslant k$, $k\geqslant 1$. *Then for every complex number* α *with* $|\alpha|\geqslant k$ *and* $0\leqslant t\leqslant 1$, *we have*

$$\max_{|z|=1}|D_\alpha P(z)| \geqslant \frac{2(|\alpha|-k)}{1+k^n}\left\{\left(1+\frac{(k^n-|a_0|-tm)(k-1)}{2(k^n+k|a_0|-tm)}\right)\max_{|z|=1}|P(z)|\right.$$
$$\left.+\frac{1}{2k^n}\left(k^n-1-\frac{(k^n-|a_0|-tm)(k-1)}{k^n+k|a_0|-tm}\right)tm\right\}A_n(k). \tag{16}$$

REMARK 4. For $t=0$, Theorem 3 reduces to Theorem A $(a_n=1)$. If we divide both sides of (16) by $|\alpha|$ and let $|\alpha|\to\infty$, we get the inequality (13), and thus Theorem 3 is an extension of Theorem 1.

Finally, we prove the following polar derivative generalization of Theorem 2. The obtained inequality gives a refinement of the inequality (10) as well.

THEOREM 4. *Let* $P(z) \in \widehat{\mathcal{P}}_n$ *having no zeros in* $|z| < k$, $k \leqslant 1$, *and let* $Q(z) = z^n\overline{P(1/\overline{z})}$. *If* $|P'(z)|$ *and* $|Q'(z)|$ *attain maximum at the same point on* $|z| = 1$, *then for every complex number* α *with* $|\alpha| \geqslant 1$ *and* $0 \leqslant t \leqslant 1$, *we have*

$$\max_{|z|=1}|D_\alpha P(z)| \leqslant n|\alpha|\max_{|z|=1}|P(z)| - \frac{|\alpha|-1}{1+k^n}B_n(k)\left\{2k^n\left(1+\frac{(|a_0|-k^n-tm)(1-k)}{2k(|a_0|+k^{n-1}-tm)}\right)\max_{|z|=1}|P(z)| + \left(1-k^n-\frac{k^{n-1}(|a_0|-k^n-tm)(1-k)}{|a_0|+k^{n-1}-tm}\right)tm\right\}. \tag{17}$$

Equality in (17) *holds for* $P(z) = z^n + k^n$.

REMARK 5. For $t = 0$, Theorem 4 reduces to Theorem B $(a_n = 1)$. If we divide both sides of (17) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the inequality (14), and thus Theorem 4 is an extension of Theorem 2.

3. Auxiliary results

We need the following lemmas to prove our theorems. The following lemma is due to Mir et al. [20].

LEMMA 1. *If* $P(z) \in \widehat{\mathcal{P}}_n$ *having no zeros in* $|z| < 1$, *then for* $R \geqslant 1$ *and* $0 \leqslant t \leqslant 1$, *we have*

$$\max_{|z|=R}|P(z)| \leqslant \left(\frac{(1+R^n)(|a_0|+R-tm_1)}{(1+R)(|a_0|+1-tm_1)}\right)\max_{|z|=1}|P(z)| - \left(\frac{(1+R^n)(|a_0|+R-tm_1)}{(1+R)(|a_0|+1-tm_1)}-1\right)tm_1, \tag{18}$$

where $m_1 = \min_{|z|=1}|P(z)|$. *Equality in* (18) *holds for* $P(z) = (\alpha+\beta z^n)/2$, $|\alpha| = |\beta| = 1$.

LEMMA 2. *If* $P(z) \in \widehat{\mathcal{P}}_n$ *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then for* $0 \leqslant t \leqslant 1$, *we have*

$$\max_{|z|=k}|P(z)| \geqslant \left(\frac{2k^n}{1+k^n}+\frac{k^n(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)}\right)\max_{|z|=1}|P(z)| + \left(\frac{k^n-1}{k^n+1}-\frac{(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)}\right)tm, \tag{19}$$

where $m = \min_{|z|=k}|P(z)|$. *Equality in* (19) *holds for* $P(z) = z^n + k^n$.

Proof. Let $T(z) = P(kz)$. Since $P(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, the polynomial $T(z)$ has all its zeros in $|z| \leqslant 1$. Let $H(z) = z^nT(1/z)$ be the reciprocal polynomial of $T(z)$, then $H(z)$ is a polynomial of degree at most n having no zeros in $|z|<1$. Hence applying (18) of Lemma 1 to the polynomial $H(z)$, we get for $k \geqslant 1$ and $0 \leqslant t \leqslant 1$,

$$\max_{|z|=k}|H(z)| \leqslant \left(\frac{(1+k^n)(k^n+k|a_0|-tm^*)}{(1+k)(k^n+|a_0|-tm^*)}\right)\max_{|z|=1}|H(z)| \\ -\left(\frac{(1+k^n)(k^n+k|a_0|-tm^*)}{(1+k)(k^n+|a_0|-tm^*)}-1\right)tm^*, \tag{20}$$

where $m^* = \min_{|z|=1}|H(z)|$.

Since $|H(z)| = |T(z)|$ on $|z|=1$, hence,

$$m^* = \min_{|z|=1}|H(z)| = \min_{|z|=1}\left|z^n P\left(\frac{k}{z}\right)\right| = \min_{|z|=k}|P(z)| = m,$$
$$\max_{|z|=1}|H(z)| = \max_{|z|=1}|T(z)| = \max_{|z|=k}|P(z)|,$$

and

$$\max_{|z|=k}|H(z)| = \max_{|z|=k}\left|z^n P\left(\frac{k}{z}\right)\right| = k^n\max_{|z|=1}|P(z)|,$$

which when substituted in (20) gives

$$\max_{|z|=k}|P(z)| \geqslant \left(\frac{(1+k)(k^n+|a_0|-tm)}{(1+k^n)(k^n+k|a_0|-tm)}\right)k^n\max_{|z|=1}|P(z)| \\ +\left(1-\frac{(1+k)(k^n+|a_0|-tm)}{(1+k^n)(k^n+k|a_0|-tm)}\right)tm. \tag{21}$$

Using the fact that

$$\frac{(1+k)(k^n+|a_0|-tm)}{(1+k^n)(k^n+k|a_0|-tm)} = \frac{2}{1+k^n}+\frac{(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)},$$

in (21), we get

$$\max_{|z|=k}|P(z)| \geqslant \left(\frac{2k^n}{1+k^n}+\frac{k^n(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)}\right)\max_{|z|=1}|P(z)| \\ +\left(\frac{k^n-1}{k^n+1}-\frac{(k^n-|a_0|-tm)(k-1)}{(1+k^n)(k^n+k|a_0|-tm)}\right)tm,$$

which is (19) and this completes the proof of Lemma 2. □

LEMMA 3. *If $P(z) \in \widehat{\mathcal{P}}_n$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then on $|z|=1$,*

$$|P'(z)|+|Q'(z)| \leqslant n\max_{|z|=1}|P(z)|.$$

The above lemma is due to Govil and Rahman [8].

LEMMA 4. *If $P(z)$ is a polynomial of degree n, then for $R \geqslant 1$,*

$$\max_{|z|=R}|P(z)| \leqslant R^n\max_{|z|=1}|P(z)|.$$

The above lemma is a simple consequence of the Maximum Modulus Principle [21]. The following lemma is due to Giroux, Rahman and Schmeisser [5].

LEMMA 5. *If* $P(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ *is a polynomial of degree* n *having all its zeros in* $|z|\leqslant 1$*, then*

$$\max_{|z|=1}|P'(z)|\geqslant \sum_{\nu=1}^{n}\frac{1}{1+|z_\nu|}\max_{|z|=1}|P(z)|.$$

4. Proofs of main results

Proof of Theorem 1. Let $G(z)=P(kz)$. Recall that a monic polynomial $P(z)$ has all its zeros in $|z|\leqslant k$, $k\geqslant 1$, the polynomial

$$G(z)=k^n\prod_{\nu=1}^{n}\left(z-\frac{z_\nu}{k}\right)$$

has all its zeros in $|z|\leqslant 1$, therefore, applying Lemma 5 to $G(z)$, we get

$$\max_{|z|=1}|G'(z)|\geqslant \sum_{\nu=1}^{n}\frac{1}{1+\frac{|z_\nu|}{k}}\max_{|z|=1}|G(z)|, \tag{22}$$

which is equivalent to

$$k\max_{|z|=1}|P'(kz)|\geqslant \sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=k}|P(z)|. \tag{23}$$

Since $P'(z)$ is a polynomial of degree $n-1$ and $k\geqslant 1$, therefore by Lemma 4, we have

$$\max_{|z|=1}|P'(kz)|=\max_{|z|=k}|P'(z)|\leqslant k^{n-1}\max_{|z|=1}|P'(z)|.$$

Using this and Lemma 2 in (23), as well as the notation from (12), we get

$$k^n\max_{|z|=1}|P'(z)|\geqslant \frac{A_n(k)}{1+k^n}\left\{\left(2k^n+\frac{k^n(k^n-|a_0|-tm)(k-1)}{k^n+k|a_0|-tm}\right)\max_{|z|=1}|P(z)|\right.$$
$$\left.+\left(k^n-1-\frac{(k^n-|a_0|-tm)(k-1)}{k^n+k|a_0|-tm}\right)tm\right\},$$

which is equivalent to (13). This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $Q(z)=z^n\overline{P(1/\overline{z})}$. Since $P(z)\neq 0$ in $|z|<k$, $k\leqslant 1$, the polynomial $Q(z)$ of degree n has all its zeros in $|z|\leqslant 1/k$, $1/k\geqslant 1$. On applying

Theorem 1 to $Q(z)$, we get for $0 \leqslant t \leqslant 1$,

$$\max_{|z|=1}|Q'(z)| \geqslant \sum_{\nu=1}^{n} \frac{\frac{1}{k}}{\frac{1}{k}+\frac{1}{|z_\nu|}}\left[\left(\frac{2}{1+\frac{1}{k^n}}+\frac{\left(\frac{|a_0|}{k^n}-1-\frac{tm}{k^n}\right)\left(\frac{1}{k}-1\right)}{\left(1+\frac{1}{k^n}\right)\left(\frac{|a_0|}{k^n}+\frac{1}{k}-\frac{tm}{k^n}\right)}\right)\max_{|z|=1}|P(z)|\right.$$
$$\left.+\left(\frac{\frac{1}{k^n}-1}{\frac{1}{k^n}\left(\frac{1}{k^n}+1\right)}-\frac{\left(\frac{|a_0|}{k^n}-1-\frac{tm}{k^n}\right)\left(\frac{1}{k}-1\right)}{\frac{1}{k^n}\left(\frac{1}{k^n}+1\right)\left(\frac{|a_0|}{k^n}+\frac{1}{k}-\frac{tm}{k^n}\right)}\right)\frac{tm}{k^n}\right], \tag{24}$$

because

$$\min_{|z|=1/k}|Q(z)| = \min_{|z|=1/k}\left|z^n\overline{P(1/\overline{z})}\right| = \frac{1}{k^n}\min_{|z|=k}|P(z)| = \frac{m}{k^n}$$

and

$$\max_{|z|=1}|Q(z)| = \max_{|z|=1}|P(z)|.$$

The above inequality (24) is equivalent to

$$\max_{|z|=1}|Q'(z)| \geqslant \frac{B_n(k)}{1+k^n}\left\{\left(2k^n+\frac{k^n(|a_0|-k^n-tm)(1-k)}{k(|a_0|+k^{n-1}-tm)}\right)\max_{|z|=1}|P(z)|\right.$$
$$\left.+\left(1-k^n-\frac{k^n(|a_0|-k^n-tm)(1-k)}{k(|a_0|+k^{n-1}-tm)}\right)tm\right\}, \tag{25}$$

where $B_n(k)$ is defined in (12).

Also $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, we have

$$\max_{|z|=1}(|P'(z)|+|Q'(z)|) = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{26}$$

On combining (25), (26) and Lemma 3, we get

$$n\max_{|z|=1}|P(z)| \geqslant \max_{|z|=1}|P'(z)|$$
$$+\frac{B_n(k)}{1+k^n}\left\{\left(2k^n+\frac{k^n(|a_0|-k^n-tm)(1-k)}{k(|a_0|+k^{n-1}-tm)}\right)\max_{|z|=1}|P(z)|\right.$$
$$\left.+\left(1-k^n-\frac{k^n(|a_0|-k^n-tm)(1-k)}{k(|a_0|+k^{n-1}-tm)}\right)tm\right\},$$

which after simplification yields (14).

This proves Theorem 2 completely. □

Proof of Theorem 3. Let $G(z)=P(kz)$. Since $P(z)$ has all its zeros in $|z|\leqslant k$, $k\geqslant 1$, therefore, all the zeros of $G(z)$ lie in $|z|\leqslant 1$, hence by (22), we get

$$\max_{|z|=1}|G'(z)| \geqslant \sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|G(z)|. \tag{27}$$

Let $H(z)=z^n\overline{G(1/\overline{z})}$. Then it can be easily verified that

$$|H'(z)| = |nG(z)-zG'(z)| \quad \text{for } |z|=1. \tag{28}$$

The polynomial $H(z)$ has all its zeros in $|z| \geqslant 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$, therefore, by the result of de-Bruijn [3],

$$|H'(z)| \leqslant |G'(z)| \text{ for } |z| = 1. \tag{29}$$

Now, noting that by hypothesis, we have $|\alpha|/k \geqslant 1$, hence on using definition of polar derivative of a polynomial, we get

$$\begin{aligned}\left|D_{\alpha/k}G(z)\right| &= \left|nG(z) + \frac{\alpha}{k}G'(z) - zG'(z)\right| \\ &\geqslant \left|\frac{\alpha}{k}\right| |G'(z)| - |nG(z) - zG'(z)|,\end{aligned}$$

which on using (28) and (29), gives

$$\left|D_{\frac{\alpha}{k}}G(z)\right| \geqslant \frac{|\alpha| - k}{k} \max_{|z|=1} |G'(z)|. \tag{30}$$

Using (27) in (30) and on replacing $G(z)$ by $P(kz)$, we get

$$\max_{|z|=1} \left|D_{\alpha/k}P(kz)\right| \geqslant \frac{|\alpha| - k}{k} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=1} |P(kz)|,$$

which implies

$$\max_{|z|=1} \left|nP(kz) + \left(\frac{\alpha}{k} - z\right) kP'(kz)\right| \geqslant \frac{|\alpha| - k}{k} \sum_{\nu=1}^{n} \frac{k}{k + |z_\nu|} \max_{|z|=k} |P(z)|,$$

which gives by using Lemma 2 on the right hand side that

$$\begin{aligned}\max_{|z|=k} |D_\alpha P(z)| \geqslant \frac{|\alpha| - k}{k} \frac{A_n(k)}{1 + k^n} \Bigg\{ &\left(2k^n + \frac{k^n(k^n - |a_0| - tm)(k-1)}{k^n + k|a_0| - tm}\right) \max_{|z|=1} |P(z)| \\ &+ \left(k^n - 1 - \frac{(k^n - |a_0| - tm)(k-1)}{k^n + k|a_0| - tm}\right) tm \Bigg\},\end{aligned} \tag{31}$$

where $A_n(k)$ is defined in (12). Since $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and $k \geqslant 1$, applying Lemma 4 to the polynomial $D_\alpha P(z)$, we get

$$\max_{|z|=k} |D_\alpha P(z)| \leqslant k^{n-1} \max_{|z|=1} |D_\alpha P(z)|,$$

which on using in (31) gives (16).

This completes the proof of Theorem 3. □

Proof of Theorem 4. Since $Q(z) = z^n \overline{P(1/\overline{z})}$, it is easy to verify that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)|. \tag{32}$$

Also, for any complex number α with $|\alpha| \geqslant 1$, the polar derivative of $P(z)$ with respect to α is

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

This implies by (32) for $|z|=1$, that

$$\begin{aligned}|D_\alpha P(z)| &\leqslant |nP(z)-zP'(z)|+|\alpha||P'(z)| \\ &= |Q'(z)|+|\alpha||P'(z)| \\ &= |Q'(z)|+|P'(z)|-|P'(z)|+|\alpha||P'(z)| \\ &\leqslant n\max_{|z|=1}|P(z)|+(|\alpha|-1)|P'(z)| \quad \text{(by Lemma 3)}.\end{aligned}$$

This gives by using Theorem 2, that

$$\begin{aligned}\max_{|z|=1}|D_\alpha P(z)| \leqslant n\max_{|z|=1}|P(z)| \\ +(|\alpha|-1)\Bigg[\Bigg\{n-\left(2k^n+\frac{k^n(|a_0|-k^n-tm)(1-k)}{k(|a_0|+k^{n-1}-tm)}\right)\frac{B_n(k)}{1+k^n}\Bigg\}\max_{|z|=1}|P(z)| \\ -tm\left(1-k^n-\frac{k^n(|a_0|-k^n-tm)(1-k)}{k(|a_0|+k^{n-1}-tm)}\right)\frac{B_n(k)}{1+k^n}\Bigg],\end{aligned}$$

which after simplification gives (17).

This completes the proof of Theorem 4.

5. Conclusion

We proved some sharp inequalities that relate the uniform norm of the derivative and the polynomial itself, in case when the zeros are outside or inside some closed disk. Also, we extended these results to the polar derivative of a polynomial. The obtained results sharpen and generalize some already known estimates of Erdős-Lax and Turán-type.

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