Mathematical
Inequalities
& Applications
Volume 25, Number 2 (2022), 433–446 doi:10.7153/mia-2022-25-27

STABILITY ESTIMATES FOR A RADICAL FUNCTIONAL EQUATION WITH FIXED–POINT APPROACHES

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(*Communicated by J. Pečarić*)

Abstract. We investigate the stability problem of a radical functional equation using both Brzdęk fixed point theorem and fixed point alternative method.

1. Introduction

In 1940, Ulam [24] proposed a problem called the Hyers-Ulam stability in the field of functional equations: "When is it true that a function which approximately satisfies a certain functional equation must be close to an exact solution for the functional equation?"

In the next year, Hyers [13] gave the first, affirmative, and partial solution to Ulam's question in Banach spaces. Afterwards, the result obtained by Hyers was generalized by Aoki [3] with the condition for the bound of the norm of Cauchy difference. In 1978, Rassias [19] provided the same results as Aoki's in terms of the additive property and the condition for the linearity and later improved them with weaker conditions as follows.

THEOREM 1. [20, 21] *Let E_1 be a normed space, E_2 be a Banach space, and $f : E_1 \to E_2$ be a function. If f satisfies the inequality*

$$||f(x+y)-f(x)-f(y)|| \leqslant \theta(||x||^p+||y||^p) \tag{1}$$

for some $\theta \geqslant 0$, for some $p \in \mathbb{R}$ with $p \neq 1$, and for all $x,y \in E_1-\{0_{E_1}\}$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$||f(x)-A(x)|| \leqslant \frac{2\theta}{|2-2^p|}||x||^p \tag{2}$$

for each $x \in E_1-\{0_{E_1}\}$.

Mathematics subject classification (2020): Primary 39B52; Secondary 46A99.
Keywords and phrases: Stability, radical functional equation, alternative fixed point, Brzdęk fixed point.
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For the recent decades many researchers have studied this problem in a variety of fields (see, e.g., [4] and [15]). Among the approaches to the stability problems the fixed point methods are shown to be very efficient and suitable. In 2011, for example, Brzdęk and Ciepliński [5] introduced the existence theorem of the fixed point for nonlinear operator in metric spaces and also obtained the fixed point results in arbitrary metric space. Moreover, they used the results of the fixed point to investigate the stability problem of functional equations in non-Archimedean metric spaces. Recently, Brzdęk's fixed point method applied to some additive and radical functional equations; see [1], [2] and [8].

Before the Brzdęk's fixed point method, there was another fixed point method for the Hyers-Ulam stability, called Fixed Point Alternative method. In 1996, Isac and Rassias [14] were first to provide applications of new fixed point theorems for the proof of stability theory of functional equations. By using fixed point methods the stability problems of several functional equations have been extensively investigated by a number of authors; see [11], [12], [17], [18] and [22] including Brzdęk's fixed point method as aforementioned.

During the past several years many researchers have studied a variety of generalizations, extensions, and applications of the Hyers-Ulam stability problems for a number of functional equations, in particular, the ones involving a radical expression such as the radical quadratic functional equation of the form $f(\sqrt{x^2+y^2}) = f(x)+f(y)$. Now we continue those investigations with a new form of functional equation, called the radical functional equation

$$f(x+y+2\sqrt{xy}) = f(x)+f(y). \tag{3}$$

For the existence of a solution to (3), there are very general results in [9] where the following conditional equation for functions $f: S \longrightarrow W$ was considered:

$$f(p(\Pi(x)\star\Pi(y))) = f(x)*f(y), \qquad x,y\in S, \Pi(x)\star\Pi(y)\in P_0, \tag{4}$$

where S is a nonempty set, $(Y,\star)$ and $(W,*)$ are groupoids (i.e., Y and W are nonempty sets endowed with binary operations $\star: Y^2 \longrightarrow Y$ and $*: W^2 \longrightarrow W$), $\Pi: S \longrightarrow Y$, and $P_0 := \Pi(S)$ with a section of Π, $p: P_0 \longrightarrow S$, (i.e., $\Pi(p(x)) = x$, $x\in P_0$.) Then the radical functional equation (3) can be thought of as a special case of the generalized radical functional equation (4) simply taking $\Pi(z) = \sqrt{z}$ and $p(z) = z^2$ if $S = \mathbb{R}$ (the set of reals), $(Y,\star)$ is the additive group of real numbers. According to Theorem 2.1 in [9] there exists the unique solution $A: P_0 \longrightarrow W$ for the conditional equation (4) $A(u\star v) = A(u)*A(v)$, $u,v\in P_0$, $u\star v\in P$ such that $f = A\circ\Pi$ or $A = f\circ p$ and hence it follows that every solution $f: \mathbb{R}_+ \longrightarrow \mathbb{R}$ of the radical equation (3) has the form $f(x) = A(\sqrt{x})$, where $A: \mathbb{R} \longrightarrow \mathbb{R}$ is an additive function, i.e., $A(x+y) = A(x)+A(y)$ for all $x,y\in\mathbb{R}$.

Now let us introduce the definition of stability due to Brzdęk [8] here for our main results.

DEFINITION 1. Let $(X,+)$ and $(Y,+)$ be semigroups, d be a metric in Y, $\mathscr{E} \subseteq \mathscr{C} \subseteq \mathbb{R}_+^{X^2}$ be nonempty, and $\mathscr{T}$ be an operator mapping $\mathscr{C}$ into $\mathbb{R}_+^X$ ($\mathbb{R}_+$ denotes the set of nonnegative reals). We say that a radical equation is $(\mathscr{E},\mathscr{T})$-stable provided for every $\varepsilon\in\mathscr{E}$ and $f\in Y^X$ with

$$d(f(x+y+2\sqrt{xy}),f(x)+f(y))\leqslant\varepsilon(x,y) \tag{5}$$

for all $x,y\in X$, there exists a solution $R\in Y^X$ of the equation (3) such that

$$d(f(x),R(x))\leqslant\mathscr{T}\varepsilon(x) \tag{6}$$

for all $x\in X$ (As usual, B^A denotes the family of all functions mapping a set $A\neq\emptyset$ into a set $B\neq\emptyset$).

We note that $(\mathscr{E},\mathscr{T})$-stability of a radical equation (3) means that every approximate (in the sense of (5)) solution of (3) is always close (in the sense of (6)) to an exact solution to (3).

In 2014, Brzdęk [8] used the Brzdęk fixed point method (see Theorem 4 in [8] or Theorem 2 in this article) to prove a general result on Ulam's type stability of the additive functional equation $f(x+y)=f(x)+f(y)$, mapping a commutative group into a commutative group and improve some earlier stability estimations on the additive functional equation. Recently, Brzdęk, El-hady, and Schwaiger [10] analyzed the stability results for the generalized radical functional equation (4) with a very general and uniform approach. We will discuss all the analogies and connections between our main results and stability results from two papers [8] and [10] just after Theorem 3 and Theorem 5, respectively on Remarks in the following sections.

The purpose of this paper is to investigate new stability results for the radical functional equation (3) by using both Brzdęk's fixed point method (for further references on related results see [6]) and fixed point alternative method in the subsequent sections. In addition, we consider the distinctive properties for each method and we raise an open problem in Remarks 1 and 2, not only discussion of correspondences and connections between the main results and previous work.

2. Brzdęk fixed point method

In this section, we will investigate the stability problem for the radical functional equation (3) by using Brzdęk fixed point method; see Theorem 2. As usual, $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{R}_+$ denote the set of non-negative integers, the set of positive integers and the set of non-negative real numbers, respectively. The fixed point method can be based on the following theorem; see [8].

THEOREM 2. ([8]) *Let X be a non-empty set, (Y,d) be a complete metric space and $f_1,f_2:X\to X$ be given mappings. Suppose that $\mathscr{T}:Y^X\to Y^X$ and $\Lambda:\mathbb{R}_+^X\to\mathbb{R}_+^X$ are two operators satisfying the following conditions*

$$d(\mathscr{T}\xi(x),\,\mathscr{T}\mu(x))\leqslant\sum_{j=1}^{2}d(\xi(f_j(x)),\,\mu(f_j(x))) \tag{7}$$

and

$$\Lambda\delta(x) := \sum_{j=1}^{2} \delta(f_j(x)) \tag{8}$$

for all $\xi, \mu \in Y^X$, $\delta \in \mathbb{R}_+^X$ *and* $x \in X$. *If there exist* $\varepsilon : X \to \mathbb{R}_+$ *and* $\phi : X \to Y$ *such that*

$$d(\mathscr{T}\phi(x), \phi(x)) \leqslant \varepsilon(x) \text{ and } \varepsilon^*(x) := \sum_{n=0}^{\infty} (\Lambda^n \varepsilon)(x) < \infty \tag{9}$$

for all $x \in X$, *then the limit* $\lim_{n\to\infty}(\mathscr{T}^n\phi)(x)$ *exists for each* $x \in X$. *Moreover, the function* $\psi(x) := \lim_{n\to\infty}(\mathscr{T}^n\phi)(x)$ *is a fixed point of* $\mathscr{T}$ *with*

$$d(\phi(x), \psi(x)) \leqslant \varepsilon^*(x)$$

for all $x \in X$.

We call the fixed point theorem as in Theorem 2 Brzdęk fixed point method.

THEOREM 3. *Let* d *be a complete metric in* $\mathbb{R}$ *which is invariant (i.e.,* $d(x+z, y+z) = d(x,y)$ *for* $x,y,z \in \mathbb{R}$*) and* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *be a function such that*

$$M_0 := \{m \in \mathbb{N} : s((1+\sqrt{m})^2) + s(m) < 1\} \neq \emptyset \tag{10}$$

where

$$s(m) := \inf\{k \in \mathbb{R}_+ : h(mx) \leqslant kh(x) \text{ for all } x \in \mathbb{R}_+\}, \tag{11}$$

for $m \in \mathbb{N}$. *Suppose that* $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfies the inequality*

$$d\Big(f(x+y+2\sqrt{xy}),\, f(x)+f(y)\Big) \leqslant h(x)+h(y) \tag{12}$$

for all $x,y \in \mathbb{R}_+$. *Then there exists a unique solution* $R : \mathbb{R}_+ \to \mathbb{R}$ *to (3) such that*

$$d(f(x), R(x)) \leqslant s_0 h(x) \tag{13}$$

for all $x \in \mathbb{R}_+$, *where*

$$s_0 := \inf\Big\{\frac{1+s(m)}{1-s((1+\sqrt{m})^2)-s(m)} : m \in M_0\Big\}.$$

Proof. Let $m \in \mathbb{N}$. On letting $y = mx$ in the inequality (12), we will see that

$$d\Big(f((1+\sqrt{m})^2 x),\, f(x)+f(mx)\Big) \leqslant (1+s(m))h(x) \tag{14}$$

for all $x \in \mathbb{R}_+$. Since $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a function, for each $m \in \mathbb{N}$ $c_m : \mathbb{R}_+ \to \mathbb{R}_+$ can be defined by $c_m(x) = (1+s(m))h(x)$, for all $x \in \mathbb{R}_+$.

To apply Brzdęk fixed point method, for each $m \in \mathbb{N}$, we define two operators $\mathscr{T}_m : \mathbb{R}^{\mathbb{R}_+} \to \mathbb{R}^{\mathbb{R}_+}$ and $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_+} \to \mathbb{R}_+^{\mathbb{R}_+}$ by

$$\mathscr{T}_m\xi(x) := \xi\Big((1+\sqrt{m})^2 x\Big) - \xi(mx) \tag{15}$$

and

$$\Lambda_m\delta(x) := \delta\Big((1+\sqrt{m})^2x\Big) + \delta(mx) \tag{16}$$

for all $x \in \mathbb{R}_+$ and $\xi \in \mathbb{R}^{\mathbb{R}_+}, \delta \in \mathbb{R}_+^{\mathbb{R}_+}$. For each $m \in \mathbb{N}$, we also define mappings $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ $(j = 1,2)$ by

$$f_1(x) = (1+\sqrt{m})^2x,\ f_2(x) = mx$$

for all $x \in \mathbb{R}_+$. By the definition of Λ_m, it satisfies (8). Now we will check whether the condition (7) holds or not. Let $\xi, \mu \in \mathbb{R}^{\mathbb{R}_+}$. By using the property of d, we have

$$\begin{aligned}
&d(\mathscr{T}_m\xi(x),\, \mathscr{T}_m\mu(x))\\
&= d\Big(\xi(f_1(x)) - \xi(f_2(x)),\, \mu(f_1(x)) - \mu(f_2(x))\Big)\\
&\leqslant d\Big(\xi(f_1(x)) - \xi(f_2(x)),\, \mu(f_1(x)) - \xi(f_2(x))\Big)\\
&\quad + d\Big(\mu(f_1(x)) - \xi(f_2(x)),\, \mu(f_1(x)) - \mu(f_2(x))\Big)\\
&= d\Big(\xi(f_1(x)),\, \mu(f_1(x))\Big) + d\Big(\xi(f_2(x)),\, \mu(f_2(x))\Big)
\end{aligned}$$

for all $x \in \mathbb{R}_+$. Hence the inequality (7) holds. Now, we will show that the condition (9) also holds. By the assumption of d and the inequality (14), we have

$$d(\mathscr{T}_m f(x), f(x)) = d\Big(f((1+\sqrt{m})^2x), f(x) + f(mx)\Big) \leqslant c_m(x) \tag{17}$$

for all $x \in \mathbb{R}_+$, where $c_m(x) = (1+s(m))h(x)$. We note that

$$\begin{aligned}
\Lambda_m c_m(x) &= (1+s(m))\Big(h(f_1(x)) + h(f_2(x))\Big)\\
&\leqslant (1+s(m))\Big(s((1+\sqrt{m})^2) + s(m)\Big)h(x)\\
&= \Big(s((1+\sqrt{m})^2) + s(m)\Big)\, c_m(x)
\end{aligned}$$

for all $x \in \mathbb{R}_+$. Using the mathematical induction, we get that

$$\Lambda_m^n c_m(x) = \Big(s((1+\sqrt{m})^2) + s(m)\Big)^n c_m(x)$$

for all $x \in \mathbb{R}_+$ and each $n \in \mathbb{N}$. For each $m \in M_0$ and $x \in \mathbb{R}_+$, we will see that

$$c_m^*(x) := \sum_{j=0}^{\infty} (\Lambda_m^j c_m)(x) \leqslant \frac{1+s(m)}{1-s((1+\sqrt{m})^2) - s(m)}h(x) \tag{18}$$

where $\Lambda_m^0 c_m(x) = c_m(x)$. On letting $\varepsilon = c_m$ and $\phi = f$ in Theorem 2, Brzdęk fixed point method implies that

$$T_m(x) := \lim_{n\to\infty} \mathscr{T}_m^n f(x)$$

exists and $T_m(x)$ is a fixed point of $\mathscr{T}_m$ such that

$$d(f(x), T_m(x)) \leqslant c_m^*(x)$$

for all $m \in M_0$ and $x \in \mathbb{R}_+$.

Next, we will check that the T_m satisfies the equation (3) for each $m \in M_0$. Let $n \in \mathbb{N}_0$, $m \in M_0$ and $\mathscr{T}_m^0 f(x) = f(x)$. Now, we will show that

$$\begin{aligned} &d\Big(\mathscr{T}_m^n f(x+y+2\sqrt{xy}),\, \mathscr{T}_m^n f(x) + \mathscr{T}_m^n f(y)\Big) \\ &\leqslant \Big(s((1+\sqrt{m})^2) + s(m)\Big)^n (h(x)+h(y)) \end{aligned} \tag{19}$$

for all $x, y \in \mathbb{R}_+$. To prove the inequality (19), we will use the mathematical induction. Let $n = 0$. This case follows from the inequality (12). Assume that it holds when $n = k$. By using induction step and some properties of our assumption of d, we will see that

$$\begin{aligned} &d\Big(\mathscr{T}_m^{k+1} f(x+y+2\sqrt{xy}),\, \mathscr{T}_m^{k+1} f(x) + \mathscr{T}_m^{k+1} f(y)\Big) \\ &\leqslant d\Big(\mathscr{T}_m^k f((1+\sqrt{m})^2(x+y+2\sqrt{xy})) - \mathscr{T}_m^k f(m(x+y+2\sqrt{xy})), \\ &\qquad \mathscr{T}_m^k f((1+\sqrt{m})^2 x) - \mathscr{T}_m^k f(mx) + \mathscr{T}_m^k f((1+\sqrt{m})^2 y) - \mathscr{T}_m^k f(my)\Big) \\ &\leqslant d\Big(\mathscr{T}_m^k f((1+\sqrt{m})^2(x+y+2\sqrt{xy})) - \mathscr{T}_m^k f(m(x+y+2\sqrt{xy})), \\ &\qquad \mathscr{T}_m^k f((1+\sqrt{m})^2 x) + \mathscr{T}_m^k f((1+\sqrt{m})^2 y) - \mathscr{T}_m^k f(m(x+y+2\sqrt{xy}))\Big) \\ &\qquad + d\Big(\mathscr{T}_m^k f((1+\sqrt{m})^2 x) + \mathscr{T}_m^k f((1+\sqrt{m})^2 y) - \mathscr{T}_m^k f(m(x+y+2\sqrt{xy})), \\ &\qquad \mathscr{T}_m^k f((1+\sqrt{m})^2 x) - \mathscr{T}_m^k f(mx) + \mathscr{T}_m^k f((1+\sqrt{m})^2 y) - \mathscr{T}_m^k f(my)\Big) \\ &= d\Big(\mathscr{T}_m^k f((1+\sqrt{m})^2(x+y+2\sqrt{xy})), \mathscr{T}_m^k f((1+\sqrt{m})^2 x) + \mathscr{T}_m^k f((1+\sqrt{m})^2 y)\Big) \\ &\qquad + d\Big(\mathscr{T}_m^k f(m(x+y+2\sqrt{xy})), \mathscr{T}_m^k f(mx) + \mathscr{T}_m^k f(my)\Big) \\ &\leqslant \Big(s((1+\sqrt{m})^2) + s(m)\Big)^k \Big(h((1+\sqrt{m})^2 x) + h((1+\sqrt{m})^2 y)\Big) \\ &\qquad + \Big(s((1+\sqrt{m})^2) + s(m)\Big)^k \Big(h(mx) + h(my)\Big) \\ &\leqslant \Big(s((1+\sqrt{m})^2) + s(m)\Big)^{k+1} \Big(h(x) + h(y)\Big) \end{aligned}$$

for all $x, y \in \mathbb{R}_+$ and $m \in M_0$. On letting $n \to \infty$ in the inequality (19) we may obtain the following equality

$$T_m(x+y+2\sqrt{xy}) = T_m(x) + T_m(y)$$

for all $x, y \in \mathbb{R}_+$ and $m \in M_0$. For each $m \in M_0$, we may conclude that the mapping T_m is a solution of a radical functional equation (3), that is,

$$T_m(x) = T_m((1+\sqrt{m})^2 x) - T_m(mx)$$

for all $x \in \mathbb{R}_+$.

Now, we will show that the choice of $m \in M_0$ does not imply that the fixed point $T_m(x)$ of $\mathscr{T}_m$. In other words, we will show that a mapping $R : \mathbb{R}_+ \to \mathbb{R}$ satisfying the equation (3) and the following inequality

$$d(f(x), R(x)) \leqslant Lh(x) \tag{20}$$

is equal to T_m for each $m \in M_0$, where $L > 0$ is constant. Since the mapping R satisfies (3), we have

$$R(x+y+2\sqrt{xy}) = R(x)+R(y) \tag{21}$$

for all $x, y \in \mathbb{R}_+$.

Let $m_0 \in M_0$ be fixed. We note that

$$\begin{aligned}
d(R(x), T_{m_0}(x)) &\leqslant d(R(x), f(x)) + d(f(x), T_{m_0}(x))\\
&\leqslant \Big(L + \frac{1+s(m_0)}{1-s((1+\sqrt{m_0})^2)-s(m_0)}\Big)h(x)\\
&\leqslant h(x)L_0 \sum_{j=0}^{\infty}\Big(s((1+\sqrt{m_0})^2)+s(m_0)\Big)^j
\end{aligned}$$

where $L_0 = \Big(1-s((1+\sqrt{m_0})^2)-s(m_0)\Big)L + \Big(1+s(m_0)\Big)$. Next, we will show that for each $l \in \mathbb{N}_0$,

$$d\Big(R(x), T_{m_0}(x)\Big) \leqslant h(x)L_0 \sum_{j=l}^{\infty}\Big(s((1+\sqrt{m_0})^2)+s(m_0)\Big)^j \tag{22}$$

for all $x \in \mathbb{R}_+$. To show this, we will use the mathematical induction, again. The case $l = 0$ follows from the previous inequality. Assume that it holds when the case $l \in \mathbb{N}_0$. Now, for each $m_0 \in M_0$,

$$\begin{aligned}
&d(R(x), T_{m_0}(x))\\
&= d\Big(R((1+\sqrt{m_0})^2x) - R(m_0x), T_{m_0}((1+\sqrt{m_0})^2x) - T_{m_0}(m_0x)\Big)\\
&\leqslant h\Big((1+\sqrt{m_0})^2x\Big)L_0 \sum_{j=l}^{\infty}\Big(s((1+\sqrt{m_0})^2)+s(m_0)\Big)^j\\
&\quad +h\Big(m_0x\Big)L_0 \sum_{j=l}^{\infty}\Big(s((1+\sqrt{m_0})^2)+s(m_0)\Big)^j\\
&\leqslant h(x)L_0 \sum_{j=l+1}^{\infty}\Big(s((1+\sqrt{m_0})^2)+s(m_0)\Big)^j.
\end{aligned}$$

Hence the inequality (22) holds whenever $l \in \mathbb{N}_0$. On taking $l \to \infty$ in the inequality (22) we have

$$R = T_{m_0} \tag{23}$$

where $m_0 \in M_0$. This means that $T_m = T_{m_0}$ for each $m_0 \in M_0$. Hence we get that

$$d(f(x), T_m(x)) \leqslant \frac{1+s(m)}{1-s((1+\sqrt{m})^2)-s(m)}h(x)$$

for all $m \in M_0$ and $x \in \mathbb{R}_+$. Thus we may conclude that the inequality (13) holds with $R := T_m$ and also the uniqueness follows from the equality (23). $\square$

REMARK 1. As we discussed in Introduction Brzdęk, El-hady, and Schwaiger [10] presented very abstract stability result concerning the generalized radical functional equation (4) (see Theorem 5 in [10] for the precise stability result) with the concept of *Hyers-Ulam property* for the Cauchy additive equation $g(x+y) = g(x)+g(y)$ where g is a function from a nonempty groupoid P to another Q. As expected the result in Theorem 3 we just showed can be deduced from the corresponding one in Theorem 5 in [10]. Clearly, the complete and invariant metric d in Theorem 3 is subinvariant that is one of two conditions in Theorem 5 in [10] and hence for any $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$d\Big(f(x+y+2\sqrt{xy}), f(x)+f(y)\Big) \leqslant \psi(x,y), \quad x,y \in \mathbb{R}_+$$

there is a solution $R : \mathbb{R}_+ \to \mathbb{R}$ of the radical equation (3) such that

$$d(f(x),R(x)) \leqslant K(\Phi(\sqrt{x})+\chi(x)), \quad x \in \mathbb{R}_+$$

where $\chi(z) = \inf_{y_0\in\mathbb{R}_+} K[\psi_1(z,y_0)+\psi_2(z,y_0)]$ and $K \geqslant 1$. However, the main stability result on the radical functional equation (3) in Theorem 3 presents a specific expression for the control function ψ and an accurate estimation for the stability $K(\Phi(\sqrt{x})+\chi(x))$ in terms of the function $h(x)$ and the constant s_0 when $\Pi(x) = \sqrt{x}$ and $p(x) = x^2$ in the conditional equation (4). Also, Brzdęk [8] used a fixed point approach called Brzdęk fixed point method to investigate the stability of Cauchy additive functional equation $f(x+y) = f(x)+f(y)$, in the class of functions taking a commutative group into anther one and improved some earlier stability estimations. The stability result in the main theorem (see Theorem 5) in [8] involves two automorphisms u and u' on a group $(X,+)$ with a relationship $u'(x) = x-u(x)$ for $x \in X$. Those mappings on a group in the stability for Cauchy additive equation, as we proved above, correspond to the natural number m and $(1+\sqrt{m})^2$ on the set of nonnegative real numbers in the stability estimations of the radical functional equation (3).

COROLLARY 1. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping such that*

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}_+} \frac{h((1+\sqrt{n})^2)x)+h(nx)}{h(x)} = 0. \tag{24}$$

Suppose $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$d\Big(f(x+y+2\sqrt{xy}), f(x)+f(y)\Big) \leqslant h(x)+h(y), \tag{25}$$

for all $x, y \in \mathbb{R}_+$. *Then there exists a unique solution* $R : \mathbb{R}_+ \to \mathbb{R}$ *to* (3) *such that*

$$d(f(x), R(x)) \leqslant h(x) \tag{26}$$

for all $x \in \mathbb{R}_+$.

Proof. Let $n \in \mathbb{N}$ and let

$$a_n := \sup_{x \in \mathbb{R}_+} \frac{h((1+\sqrt{n})^2 x) + h(nx)}{h(x)}$$

for each $x \in \mathbb{R}_+$. By the definition $s(n)$ as in Theorem 3, we will see that

$$s((1+\sqrt{n})^2) \leqslant \sup_{x \in \mathbb{R}_+} \frac{h((1+\sqrt{n})^2 x)}{h(x)} \leqslant a_n$$

and

$$s(n) \leqslant \sup_{x \in \mathbb{R}_+} \frac{h(nx)}{h(x)} \leqslant a_n\,.$$

These inequalities imply that

$$s((1+\sqrt{n})^2) + s(n) \leqslant 2a_n \tag{27}$$

for all $x \in \mathbb{R}_+$. By our assumption of (24), the sequence $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = 0$, that is,

$$\lim_{k\to\infty} \sup_{x \in \mathbb{R}_+} \frac{h((1+\sqrt{n_k})^2 x) + h(n_k x)}{h(x)} = 0. \tag{28}$$

The inequalities (27) and (28) imply that

$$\lim_{k\to\infty} s((1+\sqrt{n_k})^2) = \lim_{k\to\infty} s(n_k) = 0\,,$$

that is,

$$\lim_{k\to\infty} \Big(s((1+\sqrt{n_k})^2 x) + s(n_k x) \Big) = 0\,.$$

Thus we have

$$\lim_{k\to\infty} \frac{1 + s(n_k)}{1 - s((1+\sqrt{n_k})^2) - s(n_k)} = 1\,.$$

On letting $s_0 = 1$ as in Theorem 3, the inequality (26) follows from the inequality (13). $\square$

3. Fixed point alternative method

Now, we will state the theorem, the alternative of fixed point in a generalized metric space. After then we will study the stability by using a fixed point alternative method. For a given mapping $f:\mathbb{R}_+\to\mathbb{R}$, let

$$Df(x,y)=f(x+y+2\sqrt{xy}))-f(x)-f(y),$$

$x,\, y\in\mathbb{R}_+$.

DEFINITION 2. Let X be a set. A function $d: X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,\, y)=0$ if and only if $x=y$;
(2) $d(x,\, y)=d(y,\, x)$ for all $x,\, y\in X$;
(3) $d(x,\, z)\leqslant d(x,\, y)+d(y,\, z)$ for all $x,\, y,\, z\in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity. Now, we will introduce one of fundamental results of fixed point theory. For the proof, refer to [16].

THEOREM 4. (The alternative of fixed point [16], [23]) *Suppose that we are given a complete generalized metric space* (Ω,d) *and a strictly contractive mapping* $T: X\to X$ *with Lipschitz constant* $0<L<1$. *Then for each given* $x\in X$, *either*

$$d(T^n x,\, T^{n+1}x)=\infty \text{ for all } n\geqslant 0,$$

or there exists a natural number n_0 *such that*

1. $d(T^n x,\, T^{n+1}x)<\infty$ *for all* $n\geqslant n_0$;
2. *The sequence* $\{T^n x\}$ *is convergent to a fixed point* y^* *of* J;
3. y^* *is the unique fixed point of* T *in the set*
$$Y=\{y\in X\,|\, d(T^{n_0}x,y)<\infty\};$$
4. $d(y,y^*)\leqslant\frac{1}{1-L}d(y,Ty)$ *for all* $y\in Y$.

THEOREM 5. *Let* $f:\mathbb{R}_+\to\mathbb{R}$ *be a function for which there exists a function* $\phi:\mathbb{R}_+^2\to[0,\infty)$ *such that there exists a constant* L, $0<L<1$, *satisfying the inequalities*

$$\| Df(x,y)\|\leqslant\phi(x,y) \tag{29}$$

$$\phi(4x,4y)\leqslant 2L\phi(x,y),$$

for all $x,y\in\mathbb{R}_+$. *Then there exists a unique radical function* $R:\mathbb{R}_+\to\mathbb{R}$ *defined by* $R(x)=\lim_{n\to\infty}\frac{1}{2^n}f(4^n x)$ *such that*

$$\| f(x)-R(x)\|\leqslant\frac{1}{2(1-L)}\phi(x,x), \tag{30}$$

for all $x\in\mathbb{R}_+$.

Proof. Consider the set

$$\Omega = \{g | g : \mathbb{R}_+ \to \mathbb{R}\}$$

and introduce the generalized metric on Ω,

$$d(g,h) = inf\{c \in (0,\infty) \mid \| g(x) - h(x) \| \leqslant c\,\phi(x,x), x \in \mathbb{R}_+\}.$$

It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \to \Omega$ by

$$Tg(x) = \frac{1}{2} g(4x), \quad g \in \Omega$$

for all $x \in \mathbb{R}_+$. Note that for all $g,h \in \Omega$, let $c \in (0,\infty)$ be an arbitrary constant with $d(g,h) \leqslant c$. Then

$$\| g(x) - h(x) \| \leqslant c\phi(x,x)$$

for all $x \in \mathbb{R}_+$. By replacing x by $4x$ and dividing $\frac{1}{2}$ in the previous inequality, we have

$$\left\| \frac{1}{2} g(4x) - \frac{1}{2} h(4x) \right\| \leqslant \frac{1}{2} c\phi(4x,4x) \leqslant Lc\phi(x,x)$$

for all $x \in \mathbb{R}_+$. Hence we have that

$$d(Tg, Th) \leqslant Ld(g,h),$$

for all $g,h \in \Omega$, that is, T is a strictly contractive mapping of Ω with the Lipschitz constant L. By setting $x = y$ in the inequality (29), then we have

$$\left\| f(x) - \frac{1}{2} f(4x) \right\| \leqslant \frac{1}{2} \phi(x,x),$$

for all $x \in \mathbb{R}_+$, that is, $d(Tf, f) \leqslant \frac{1}{2} < \infty$. We can apply the fixed point alternative method and since $\lim_{r\to\infty} d(T^r f, R) = 0$, there exists a fixed point R of T in Ω such that

$$R(x) = \lim_{n\to\infty} \frac{f(4^n x)}{2^n}, \tag{31}$$

for all $x \in \mathbb{R}_+$. Letting $x = 4^n x$ and $y = 4^n y$ in the equation (29) and dividing by $\frac{1}{2^n}$,

$$\begin{aligned} \| DR(x,y) \| &= \lim_{n\to\infty} \frac{\| Df(4^n x, 4^n y) \|}{2^n} \\ &\leqslant \lim_{n\to\infty} \frac{1}{2^n} \phi(4^n x, 4^n y) \\ &\leqslant \lim_{n\to\infty} L^n \phi(x,y) = 0, \end{aligned}$$

for all $x,y \in \mathbb{R}_+$; that is it satisfies the equation (3). Hence the R is a solution to (3). Also, the fixed point alternative guarantees that such a R is the unique function. Again using the fixed point alternative method, we have

$$d(f,R) \leqslant \frac{1}{1-L} d(Tf, f).$$

Hence we may conclude that

$$d(f,R) \leqslant \frac{1}{1-L} d(Tf,f) \leqslant \frac{1}{2(1-L)}.$$

That is, the previous inequality implies the equation (30), as desired. □

COROLLARY 2. *Let θ and L be positive real numbers with $0 < L < 1$ and let $f : \mathbb{R}_+ \to \mathbb{R}$ be a mapping such that*

$$||Df(x,y)|| \leqslant \theta(||x|| + ||y||) \tag{32}$$

for all $x, y \in \mathbb{R}_+$ Then there exists a unique solution $R : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$||f(x) - R(x)|| \leqslant \frac{\theta}{1-L} ||x||$$

for all $x \in \mathbb{R}_+$.

Proof. On taking $\phi(x,y) = \theta(||x|| + ||y||)$ for all $x, y \in \mathbb{R}_+$, it is easy to show that the inequality (32) holds. Similar to the proof of Theorem 5, we have

$$||f(x) - R(x)|| \leqslant \frac{1}{2(1-L)} \phi(x,x) = \frac{\theta}{1-L} ||x||$$

for all $x \in \mathbb{R}_+$. □

REMARK 2. From the main results of stability above in two fixed point methods, we consider the distinctive properties for each one here. First of all, Brzdęk fixed point approach requires the metric have the invariance property, i.e., $d(x+z, y+z) = d(x,y)$ for $x, y, z \in \mathbb{R}$ while the alternative fixed point one needs just the generalized metric although a strictly contractive mapping should be assumed. However, both methods in common apply scaling processes for h and T, respectively. Also the direct dependency of the equation in the alternative would be higher than Brzdęk fixed point approach given the scaling of $f(x)$ in the limiting process, i.e., $\dfrac{f(4^n x)}{2^n}$ where we might have different bases in scaling depending on the functional equations. Lastly, the use of $y = mx$ in the Brzdęk's method should be remarked. This linear relationship between variables x and y makes all the computation possibly simple enough to get the very nice stability of the radical equation. In the case of Cauchy additive equation on a commutative group $(X, +)$ in [8] this linear function can be generalized an automorphism $u(x)$ for $x \in X$ and so we would like to propose one open problem: What are the stability results by the Brzdęk fixed point theorem when we generalize the relation between x and y corresponding to the automorphism u for the stability of Cauchy equation?

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(Received May 8, 2021)

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