Mathematical
Inequalities
& Applications
Volume 25, Number 2 (2022), 447–465 doi:10.7153/mia-2022-25-28

FOURIER TRANSFORM OF VARIABLE ANISOTROPIC HARDY SPACES WITH APPLICATIONS TO HARDY–LITTLEWOOD INEQUALITIES

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(*Communicated by I. Perić*)

Abstract. Let $p(\cdot):\ \mathbb{R}^n \to (0,1]$ be a variable exponent function satisfying the globally log-Hölder continuous condition and A a general expansive matrix on $\mathbb{R}^n$. Let $H_A^{p(\cdot)}(\mathbb{R}^n)$ be the variable anisotropic Hardy space associated with A defined via the radial maximal function. In this article, via the known atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$ and establishing two useful estimates on anisotropic variable atoms, the author shows that the Fourier transform $\widehat{f}$ of $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ coincides with a continuous function F in the sense of tempered distributions, and F satisfies a pointwise inequality which contains a step function with respect to A as well as the Hardy space norm of f. As applications, the author also obtains a higher order convergence of the continuous function F at the origin. Finally, an analogue of the Hardy–Littlewood inequality in the variable anisotropic Hardy space setting is also presented. All these results are new even in the classical isotropic setting.

1. Introduction

The main purpose of this article is to investigate the Fourier transform on the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ from [22], where $p(\cdot):\ \mathbb{R}^n \to (0,1]$ is a variable exponent function satisfying the so-called globally log-Hölder continuous condition [see (5) and (6) below] and A is a general expansive matrix on $\mathbb{R}^n$ (see Definition 1 below). The problem of describing the Fourier transform of classical Hardy spaces $H^p(\mathbb{R}^n)$ originated from the fundamental work Fefferman and Stein [15], which has started an intensively studied area of real-variable Hardy spaces. First, using entire functions of exponential type, Coifman [8] characterized the Fourier transform $\widehat{f}$ of $f \in H^p(\mathbb{R})$ (namely, for the dimension $n=1$). For the study of the Fourier transform on Hardy spaces in the higher dimensions, we refer the reader to [5, 9, 16, 26] and their references.

In particular, the following well-known result was obtained by Taibleson and Weiss [26]: for each fixed $p \in (0,1]$, the Fourier transform $\widehat{f}$ of f which belongs to $H^p(\mathbb{R}^n)$ coincides with a continuous function F in the sense of tempered distributions and, for each $\xi \in \mathbb{R}^n$,

$$|F(\xi)| \leqslant C\|f\|_{H^p(\mathbb{R}^n)}|\xi|^{n(1/p-1)}, \tag{1}$$

Mathematics subject classification (2020): 42B35, 42B30, 42B10, 46E30.
Keywords and phrases: Expansive matrix, (variable) Hardy space, Fourier transform, Hardy-Littlewood inequality.

where C is a positive constant depending only on n and p. Moreover, the inequality (1) further implies the famous Hardy–Littlewood inequality for Hardy spaces, namely, for each given $p \in (0,1]$, there exists a positive constant R such that, for any $f \in H^p(\mathbb{R}^n)$,

$$\left[\int_{\mathbb{R}^n} |\xi|^{n(p-2)} |F(\xi)|^p \, d\xi\right]^{1/p} \leqslant R\|f\|_{H^p(\mathbb{R}^n)}, \tag{2}$$

where F is as in (1); see, for instance, [25, p. 128]. In addition, via the known atomic characterization of the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$, Bownik and Wang [5] proved that both inequalities (1) and (2) hold true for the Hardy space $H^p_A(\mathbb{R}^n)$. Very recently, these results were extended to the setting of Hardy spaces associated with ball quasi-Banach function spaces or the anisotropic mixed-norm Hardy space $H^{\vec{p}}_{\vec{a}}(\mathbb{R}^n)$, where

$$\vec{a} := (a_1, \ldots, a_n) \in [1,\infty)^n \quad \text{and} \quad \vec{p} := (p_1, \ldots, p_n) \in (0,1]^n$$

are two vectors; see [18, 17].

On another hand, as a generalization of the classical Hardy space $H^p(\mathbb{R}^n)$, the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, in which the constant exponent p is replaced by a variable exponent function $p(\cdot) : \mathbb{R}^n \to (0,\infty]$, was first studied by Nakai and Sawano [23] and, independently, by Cruz-Uribe and Wang [11] with some weaker assumptions on $p(\cdot)$ than those used in [23]. For more development about this Hardy space and other function spaces with variable exponents, we refer the reader to [1, 2, 10, 13, 14, 20, 21, 24, 27, 28, 29, 30, 31]. In addition, the anisotropic Hardy space $H^p_A(\mathbb{R}^n)$, with $p < \infty$, which was first investigated by Bownik [4], has proved important for the study of discrete groups of dilations in wavelet theory, and also includes both the classical Hardy space and the parabolic Hardy space of Calderón and Torchinsky [7] as special cases. Based on these work, recently, Liu et al. [22] introduced the variable anisotropic Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$ with respect to the expansive matrix A, and established its various real-variable characterizations. Nowadays, this anisotropic setting has proved useful not only in developing function spaces arising in harmonic analysis, but also in many other branches such as the wavelet theory (see, for instance, [3, 4, 12]) and partial differential equations (see, for instance, [6, 19]).

Motivated by the real-variable theory of the variable anisotropic Hardy space $H^{p(\cdot)}_A(\mathbb{R}^n)$ from [22] and the aforementioned results about the characterizations of the Fourier transform on classical Hardy spaces $H^p(\mathbb{R}^n)$ and anisotropic Hardy spaces $H^p_A(\mathbb{R}^n)$ as well as anisotropic mixed-norm Hardy spaces $H^{\vec{p}}_{\vec{a}}(\mathbb{R}^n)$, in this article, we first extend the inequality (1) to the setting of variable anisotropic Hardy spaces and then also give out some applications of our main result.

To be precise, in Section 2, we recall the notions of expansive matrices, variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and variable anisotropic Hardy spaces (see, Definitions 1 and 4 below).

The goal of Section 3 is to obtain the main result, namely, Theorem 1 below. For this purpose, we first establish two uniform pointwise estimates on anisotropic variable atoms (see Lemmas 1 and 2 below) as well as an auxiliary inequality (see Lemma 4 below). Using these and the known atomic characterization of $H^{p(\cdot)}_A(\mathbb{R}^n)$ from [22, Theorem 4.8], we then show that the Fourier transform $\widehat{f}$ of f which belongs to $H^{p(\cdot)}_A(\mathbb{R}^n)$

coincides with a continuous function F in the sense of tempered distributions. We also prove that this continuous function F, multiplied by a step function with respect to A, can be pointwisely controlled by a positive constant multiple of the Hardy space norm of f. This elucidates the necessity of vanishing moments of anisotropic variable atoms in some sense [see Remark 1(ii) below].

In Section 4, as applications, applying a technical inequality obtained in the proof of Theorem 1, we first show a higher order convergence of the continuous function F at the origin (see Theorem 2 below). Then we prove that the function F, multiplied by some power of a step function with respect to A, is p_+-integrable, and this integral can be controlled by a positive constant multiple of the Hardy space norm of f (see Theorem 3 below). This result is a generalization of the Hardy–Littlewood inequality for the present setting of variable anisotropic Hardy spaces.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1,2,\ldots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\mathbf{0}$ be the *origin* of $\mathbb{R}^n$. For each fixed multi-index $\alpha := (\alpha_1,\ldots,\alpha_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}_+^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. We use C to denote a positive constant which is independent of the main parameters, but may vary in different setting. The *symbol* $g \lesssim h$ means $g \leqslant Ch$ and, if $g \lesssim h \lesssim g$, then we write $g \sim h$. If $f \leqslant Ch$ and $h = g$ or $h \leqslant g$, we then write $f \lesssim h \sim g$ or $f \lesssim h \lesssim g$, *rather than* $f \lesssim h = g$ or $f \lesssim h \leqslant g$. In addition, for any set $E \subset \mathbb{R}^n$, we denote by $\mathbf{1}_E$ its *characteristic function*, by $E^{\complement}$ the set $\mathbb{R}^n \setminus E$ and by $|E|$ its *n-dimensional Lebesgue measure*. For any $d \in \mathbb{R}$, we denote by $\lfloor d \rfloor$ the *largest integer not greater than* d.

2. Preliminaries

In this section, we recall the notions of expansive matrices and variable anisotropic Hardy spaces (see, for instance, [4, 22]).

The following definition of expansive matrices is from [4].

DEFINITION 1. A real $n \times n$ matrix A is called an *expansive matrix* (shortly, a *dilation*) if

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

here and thereafter, $\sigma(A)$ denotes the *set of all eigenvalues of* A.

By [4, p. 5, Lemma 2.2], we know that there exist an open ellipsoid Δ, with $|\Delta| = 1$, and $r \in (1,\infty)$ such that $\Delta \subset r\Delta \subset A\Delta$. Thus, for any $i \in \mathbb{Z}$, $B_i := A^i\Delta$ is open, $B_i \subset rB_i \subset B_{i+1}$ and $|B_i| = b^i$ with $b := |\det A|$. For any $x \in \mathbb{R}^n$ and $i \in \mathbb{Z}$, an ellipsoid $x + B_i$ is called a *dilated ball*. Let $\mathfrak{B}$ be the set of all such dilated balls, namely,

$$\mathfrak{B} := \{x + B_i : \ x \in \mathbb{R}^n, \ i \in \mathbb{Z}\} \tag{3}$$

and let $\tau := \inf\{k \in \mathbb{Z} : \ r^k \geqslant 2\}$.

In [4, p. 6, Definition 2.3], the following homogeneous quasi-norm was introduced.

DEFINITION 2. Given a dilation A, a measurable mapping $\rho:\ \mathbb{R}^n \to [0,\infty)$ is called a *homogeneous quasi-norm*, with respect to A, if

(i) $x \neq \mathbf{0}$ implies $\rho(x) \in (0,\infty)$;

(ii) for any $x \in \mathbb{R}^n$, $\rho(Ax) = b\rho(x)$;

(iii) there exists a constant $C \in [1,\infty)$ such that, for any x, $y \in \mathbb{R}^n$, $\rho(x+y) \leqslant C[\rho(x)+\rho(y)]$.

For any given dilation A, in [4, p. 6, Lemma 2.4], it was proved that all homogeneous quasi-norms with respect to A are equivalent. Therefore, once A is fixed, we can use the *step homogeneous quasi-norm* ρ defined by setting, for any $x \in \mathbb{R}^n$,

$$\rho(x) := \begin{cases} b^i & \text{when} \quad x \in B_{i+1} \setminus B_i, \\ 0 & \text{when} \quad x = \mathbf{0} \end{cases}$$

for convenience.

Recall also that an infinitely differentiable function ϕ is called a *Schwartz function* if, for any $k \in \mathbb{Z}_+$ and multi-index $\gamma \in \mathbb{Z}_+^n$,

$$\|\phi\|_{\gamma,k} := \sup_{x\in\mathbb{R}^n} [\rho(x)]^k \, |\partial^\gamma \phi(x)| < \infty.$$

Let $\mathscr{S}(\mathbb{R}^n)$ be the set of all Schwartz functions as above, equipped with the topology determined by $\{\|\cdot\|_{\alpha,\ell}\}_{\alpha\in\mathbb{Z}_+^n,\ell\in\mathbb{Z}_+}$, and $\mathscr{S}'(\mathbb{R}^n)$ its *dual space*, equipped with the weak-$*$ topology. Throughout this article, for any $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $i \in \mathbb{Z}$, let $\phi_i(\cdot) := b^i \phi(A^i \cdot)$.

For any measurable function $p(\cdot):\ \mathbb{R}^n \to (0,\infty]$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad \underline{p} := \min\{p_-,1\}. \tag{4}$$

Denote by $\mathscr{P}(\mathbb{R}^n)$ the *set of all measurable functions* $p(\cdot)$ satisfying $0 < p_- \leqslant p_+ < \infty$.

Given a function $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, the *modular functional* $\rho_{p(\cdot)}$ and the *Luxemburg–Nakano quasi-norm* $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, with respect to $p(\cdot)$, are defined, respectively, by setting, for any measurable function f,

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx \quad \text{and} \quad \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty):\ \rho_{p(\cdot)}(f/\lambda) \leqslant 1\right\}.$$

Furthermore, the *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that $\rho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Let $C^{\log}(\mathbb{R}^n)$ be the set of all $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfying the *globally log-Hölder continuous condition*, which means there exist two positive constants $C_{\log}(p)$ and C_∞, and $p_\infty \in \mathbb{R}$ such that, for any x, $y \in \mathbb{R}^n$,

$$|p(x)-p(y)| \leqslant \frac{C_{\log}(p)}{\log(e+1/\rho(x-y))} \tag{5}$$

and

$$|p(x) - p_\infty| \leqslant \frac{C_\infty}{\log(e + \rho(x))}. \tag{6}$$

DEFINITION 3. Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ satisfy $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$. The *radial maximal function $M_\phi(f)$* of $f \in \mathscr{S}'(\mathbb{R}^n)$, with respect to ϕ, is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\phi(f)(x) := \sup_{i \in \mathbb{Z}} |f * \phi_i(x)|.$$

Applying [22, Definition 2.4 and Theorem 3.10], we now give an equivalent definition of variable anisotropic Hardy spaces as follows.

DEFINITION 4. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and ϕ be as in Definition 3. The *variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$* is defined by setting

$$H_A^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \ M_\phi(f) \in L^{p(\cdot)}(\mathbb{R}^n) \right\}$$

and, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$, let $\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} := \|M_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

3. Fourier transforms of $H_A^{p(\cdot)}(\mathbb{R}^n)$

In this section, we study the Fourier transform $\widehat{f}$, where the distribution f comes from the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$.

Recall that, for any $\phi \in \mathscr{S}(\mathbb{R}^n)$, its *Fourier transform*, denoted by $\mathfrak{F}\phi$ or $\widehat{\phi}$, is defined by setting, for any $v \in \mathbb{R}^n$,

$$\mathfrak{F}\phi(v) = \widehat{\phi}(v) := \int_{\mathbb{R}^n} \phi(x) e^{-2\pi \imath x \cdot v}\,dx.$$

here and thereafter, $\imath := \sqrt{-1}$ and $x \cdot v := \sum_{k=1}^n x_k v_k$ for any $x := (x_1, \ldots, x_n)$, $v := (v_1, \ldots, v_n) \in \mathbb{R}^n$. Moreover, for any $f \in \mathscr{S}'(\mathbb{R}^n)$, its *Fourier transform*, also denoted by $\mathfrak{F}f$ or $\widehat{f}$, is defined by setting, for any $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle \mathfrak{F}f, \phi \rangle = \langle \widehat{f}, \phi \rangle := \langle f, \widehat{\phi} \rangle.$$

We now present the main result of this article as follows: the Fourier transform $\widehat{f}$ of $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ coincides with a continuous function F in the sense of tempered distributions, and F satisfies a pointwise inequality.

THEOREM 1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $0 < p_- \leqslant p_+ \leqslant 1$, where p_-, p_+ are as in (4). Then, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that*

$$\widehat{f} = F \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n),$$

and there exists a positive constant C, depending only on A, p_- and p_+, such that, for any $x \in \mathbb{R}^n$,

$$|F(x)| \leqslant C\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1}\right\}, \tag{7}$$

here and thereafter, ρ_* is as in Definition 2 with A replaced by its adjoint matrix A^*.

To prove this theorem, we need some notions and technical lemmas. First, for any $r \in (0,\infty]$ and measurable set $E \subset \mathbb{R}^n$, the Lebesgue space $L^r(E)$ is defined to be the set of all measurable functions f such that, when $r \in (0,\infty)$,

$$\|f\|_{L^r(E)} := \left[\int_E |f(x)|^r\,dx\right]^{1/r} < \infty$$

and

$$\|f\|_{L^\infty(E)} := \operatorname*{ess\,sup}_{x\in E} |f(x)| < \infty.$$

In addition, the dilation operator D_A is defined by setting, for any measurable function f on $\mathbb{R}^n$,

$$D_A(f)(\cdot) := f(A\cdot).$$

Moreover, we have the following identity: for any $k \in \mathbb{Z}$, $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\widehat{f}(x) = b^k\left(D_{A^*}^k \mathfrak{F} D_A^k f\right)(x).$$

The succeeding notions of anisotropic $(p(\cdot),r,s)$-atoms and variable anisotropic atomic Hardy spaces $H_A^{p(\cdot),r,s}(\mathbb{R}^n)$ are from [22].

DEFINITION 5. (i) Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, $r \in (1,\infty]$,

$$s \in \left[\left\lfloor\left(\frac{1}{p_-}-1\right)\frac{\ln b}{\ln \lambda_-}\right\rfloor, \infty\right) \cap \mathbb{Z}_+, \tag{8}$$

where p_- is as in (4). A measurable function a on $\mathbb{R}^n$ is called an *anisotropic $(p(\cdot),r,s)$-atom* (shortly, a *$(p(\cdot),r,s)$-atom*) if

(i)$_1$ $\operatorname{supp} a \subset B$, where $B \in \mathfrak{B}$ with $\mathfrak{B}$ as in (3);

(i)$_2$ $\|a\|_{L^r(\mathbb{R}^n)} \leqslant \frac{|B|^{1/r}}{\|\mathbf{1}_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}$;

(i)$_3$ $\int_{\mathbb{R}^n} a(x)x^\gamma\,dx = 0$ for any $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \leqslant s$.

(ii) Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $r \in (1,\infty]$ and s be as in (8). The *variable anisotropic atomic Hardy space* $H_A^{p(\cdot),r,s}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathscr{S}'(\mathbb{R}^n)$ satisfying that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot),r,s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n).$$

Moreover, for any $f\in H_A^{p(\cdot),r,s}(\mathbb{R}^n)$, let

$$\|f\|_{H_A^{p(\cdot),r,s}(\mathbb{R}^n)}:=\inf\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where the infimum is taken over all the decompositions of f as above.

Following the proof of [5, Lemma 4], we easily obtain the following uniform estimate for atoms; the details are omitted.

LEMMA 1. *Let $p(\cdot)$, r and s be as in Definition 5(ii). Assume that a is a $(p(\cdot),r,s)$-atom supported in $x_0+B_{k_0}$ with some $x_0\in\mathbb{R}^n$ and $k_0\in\mathbb{Z}$. Then there exists a positive constant C, depending only on A and s, such that, for any $\alpha\in\mathbb{Z}_+^n$ with $|\alpha|\leqslant s$ and $x\in\mathbb{R}^n$,*

$$\left|\partial^\alpha\left(\mathfrak{F}D_A^{k_0}a\right)(x)\right|\leqslant Cb^{-k_0/r}\|a\|_{L^r(\mathbb{R}^n)}\min\left\{1,|x|^{s-|\alpha|+1}\right\}. \tag{9}$$

From Lemma 1, we deduce a uniform estimate on the Fourier transform of atoms, which is later used to prove Theorem 1.

LEMMA 2. *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $p_+\in(0,1]$, r and s be as in Definition 5(ii). Then there exists a positive constant C such that, for any $(p(\cdot),r,s)$-atom and $x\in\mathbb{R}^n$,*

$$|\widehat{a}(x)|\leqslant C\max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1},[\rho_*(x)]^{\frac{1}{p_+}-1}\right\}, \tag{10}$$

where ρ_* is the homogeneous quasi-norm with respect to A^* and p_-, p_+ are as in (4).

To show Lemma 2, we need the following inequalities, which are just [4, p. 11, Lemma 3.2].

LEMMA 3. *Let A be some fixed dilation. Then there exists a positive constant C, depending only on A, such that, for any $x\in\mathbb{R}^n$,*

$$\frac{1}{C}[\rho(x)]^{\ln\lambda_-/\ln b}\leqslant|x|\leqslant C[\rho(x)]^{\ln\lambda_+/\ln b}\qquad\text{when }\rho(x)\in(1,\infty),$$

and

$$\frac{1}{C}[\rho(x)]^{\ln\lambda_+/\ln b}\leqslant|x|\leqslant C[\rho(x)]^{\ln\lambda_-/\ln b}\qquad\text{when }\rho(x)\in[0,1].$$

We now give the proof of Lemma 2.

Proof of Lemma 2. Let a be a $(p(\cdot),r,s)$-atom supported in $x_0+B_{k_0}$ with some $x_0\in\mathbb{R}^n$ and $k_0\in\mathbb{Z}$. Then, from (9) with $\alpha=(\overbrace{0,\ldots,0}^{n\text{ times}})$, it follows that, for any $x\in\mathbb{R}^n$,

$$\begin{aligned}
|\widehat{a}(x)| &= \left|b^{k_0}\left(D_{A^*}^{k_0}\mathfrak{F}D_A^{k_0}a\right)(x)\right| \\
&= \left|b^{k_0}\left(\mathfrak{F}D_A^{k_0}a\right)\left((A^*)^{k_0}x\right)\right| \\
&\lesssim b^{k_0}b^{-k_0/r}\|a\|_{L^r(\mathbb{R}^n)}\min\left\{1,\left|(A^*)^{k_0}x\right|^{s+1}\right\},
\end{aligned}$$

which, together with the size condition of a, implies that

$$\begin{aligned}
|\widehat{a}(x)| &\lesssim b^{k_0}\left\|\mathbf{1}_{x_0+B_{k_0}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1}\min\left\{1,\left|(A^*)^{k_0}x\right|^{s+1}\right\} \\
&\lesssim b^{k_0}\max\left\{b^{-\frac{k_0}{p_-}},b^{-\frac{k_0}{p_+}}\right\}\min\left\{1,\left|(A^*)^{k_0}x\right|^{s+1}\right\}.
\end{aligned}\tag{11}$$

To obtain (10), we next consider two cases: $\rho_*(x)\leqslant b^{-k_0}$ and $\rho_*(x)>b^{-k_0}$.

Case 1). $\rho_*(x)\leqslant b^{-k_0}$. In this case, note that $\rho_*((A^*)^{k_0}x)\leqslant 1$. By (11), Lemma 3 and the fact that

$$1-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}\geqslant 1-\frac{1}{p_-}+(s+1)\frac{\ln\lambda_-}{\ln b}>0$$

[see (4) and (8)], we conclude that, for any $x\in\mathbb{R}^n$ satisfying $\rho_*(x)\leqslant b^{-k_0}$,

$$\begin{aligned}
|\widehat{a}(x)| &\lesssim b^{k_0}\max\left\{b^{-\frac{k_0}{p_-}},b^{-\frac{k_0}{p_+}}\right\}\left[\rho_*\left((A^*)^{k_0}x\right)\right]^{(s+1)\frac{\ln\lambda_-}{\ln b}} \\
&\sim \max\left\{b^{k_0[1-\frac{1}{p_-}+(s+1)\frac{\ln\lambda_-}{\ln b}]},b^{k_0[1-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}]}\right\}[\rho_*(x)]^{(s+1)\frac{\ln\lambda_-}{\ln b}} \\
&\lesssim \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1},[\rho_*(x)]^{\frac{1}{p_+}-1}\right\}.
\end{aligned}\tag{12}$$

This proves (10) for Case 1).

Case 2). $\rho_*(x)>b^{-k_0}$. In this case, note that $\rho_*((A^*)^{k_0}x)>1$. By (11), Lemma 3 again and the fact that

$$\frac{1}{p_-}-1\geqslant\frac{1}{p_+}-1\geqslant 0,$$

we know that, for any $x\in\mathbb{R}^n$ satisfying $\rho_*(x)>b^{-k_0}$,

$$\begin{aligned}
|\widehat{a}(x)| &\lesssim b^{k_0}\max\left\{b^{-\frac{k_0}{p_-}},b^{-\frac{k_0}{p_+}}\right\} \\
&\sim \max\left\{b^{(1-\frac{1}{p_-})k_0},b^{(1-\frac{1}{p_+})k_0}\right\} \\
&\lesssim \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1},[\rho_*(x)]^{\frac{1}{p_+}-1}\right\}.
\end{aligned}$$

This finishes the proof of (10) for Case 2) and hence of Lemma 2. $\square$

By borrowing some ideas from the proof of [30, Lemma 5.9], we obtain the following technical lemma.

LEMMA 4. *Let* $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ *with* $p_+\in(0,1]$. *Then, for any* $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ *and* $\{B^{(i)}\}_{i\in\mathbb{N}}\subset\mathfrak{B}$,

$$\sum_{i\in\mathbb{N}}|\lambda_i|\leqslant\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\underline{p}$ *is as in* (4).

Proof. Let $\lambda:=\sum_{i\in\mathbb{N}}|\lambda_i|$. Note that, for any $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ and $t\in(0,1]$,

$$\left(\sum_{i\in\mathbb{N}}|\lambda_i|\right)^t\leqslant\sum_{i\in\mathbb{N}}|\lambda_i|^t.$$

By the fact that $\underline{p}\in(0,1]$ [see (4)], we find that

$$\begin{aligned}\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\lambda\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}&\geqslant\left\|\sum_{i\in\mathbb{N}}\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\lambda\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\geqslant\sum_{i\in\mathbb{N}}\frac{|\lambda_i|}{\lambda}\left\|\frac{\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}=1,\end{aligned}$$

which implies the desired conclusion and hence completes the proof of Lemma 4. $\square$

To prove Theorem 1, we also need the following atomic characterizations of the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ established in [22, Theorem 4.8].

LEMMA 5. *Let* $p(\cdot)\in C^{\log}(\mathbb{R}^n)$, $r\in(\max\{p_+,1\},\infty]$ *with* p_+ *as in* (4), s *be as in* (8) *and* $N\in\mathbb{N}\cap[\lfloor(\frac{1}{\underline{p}}-1)\frac{\ln b}{\ln\lambda_-}\rfloor+2,\infty)$ *with* $\underline{p}$ *as in* (4). *Then* $H_A^{p(\cdot)}(\mathbb{R}^n)=H_A^{p(\cdot),r,s}(\mathbb{R}^n)$ *with equivalent quasi-norms.*

Now, we show Theorem 1.

Proof of Theorem 1. Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$, $r\in(\max\{p_+,1\},\infty]$, s be as in (8) and $f\in H_A^{p(\cdot)}(\mathbb{R}^n)$. Then, from Lemma 5 and Definition 5(ii), we deduce that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ and a sequence of $(p(\cdot),r,s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}}\subset\mathfrak{B}$ such that

$$f=\sum_{i\in\mathbb{N}}\lambda_i a_i\quad\text{in}\quad\mathscr{S}'(\mathbb{R}^n),$$

and

$$\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{13}$$

Therefore, by the continuity of Fourier transform on $\mathscr{S}'(\mathbb{R}^n)$, we know that

$$\widehat{f} = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i} \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n). \tag{14}$$

Moreover, for any $i \in \mathbb{N}$, $a_i \in L^1(\mathbb{R}^n)$ implies that $\widehat{a_i} \in L^\infty(\mathbb{R}^n)$. By this, Lemmas 2 and 4, and (13), we conclude that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} \sum_{i\in\mathbb{N}} |\lambda_i||\widehat{a_i}(x)| &\lesssim \sum_{i\in\mathbb{N}} |\lambda_i| \max\left\{ [\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1} \right\} \\ &\lesssim \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \max\left\{ [\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1} \right\} \\ &< \infty. \end{aligned} \tag{15}$$

Thus, for any $x \in \mathbb{R}^n$,

$$F(x) := \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}(x) \tag{16}$$

makes sense pointwisely and

$$|F(x)| \lesssim \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \max\left\{ [\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1} \right\}.$$

We next prove that the above function F is continuous on $\mathbb{R}^n$. To do this, we only need to show that F is continuous on any compact subset of $\mathbb{R}^n$. Note that, for any given compact subset X, there exists a positive constant $C_{(A,X)}$, depending only on the dilation A and X, such that $\rho_*(\cdot) \leqslant C_{(A,X)}$ holds true absolutely on X. From this and the estimate of (15), it follows that, for any $x \in X$,

$$\sum_{i\in\mathbb{N}} |\lambda_i||\widehat{a_i}(x)| \lesssim \max\left\{ \left[C_{(A,X)}\right]^{\frac{1}{p_-}-1}, \left[C_{(A,X)}\right]^{\frac{1}{p_+}-1} \right\} \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)},$$

Therefore, the summation $\sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i}(\cdot)$ converges uniformly on X. This, combined with the fact that, for any $i \in \mathbb{N}$, $\widehat{a_i}$ is continuous, implies that F is also continuous on any compact subset X and hence on $\mathbb{R}^n$.

Finally, to complete the proof of Theorem 1, by (14) and (16), it suffices to show that

$$F = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i} \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n). \tag{17}$$

To this end, by Lemma 2 and the definition of Schwartz functions, we find that, for any $\phi \in \mathscr{S}(\mathbb{R}^n)$ and $i \in \mathbb{N}$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} \widehat{a_i}(x)\phi(x)\,dx\right| \\
&\quad \leqslant \sum_{k=1}^{\infty} \int_{(A^*)^{k+1}B_0^*\setminus(A^*)^k B_0^*} \max\left\{[\rho_*(x)]^{\frac{1}{p_-}-1}, [\rho_*(x)]^{\frac{1}{p_+}-1}\right\} |\phi(x)|\,dx + \|\phi\|_{L^1(\mathbb{R}^n)} \\
&\quad \lesssim \sum_{k=1}^{\infty} b^k b^{k(\frac{1}{p_-}-1)} b^{-k(\lceil\frac{1}{p_-}-1\rceil+2)} + \|\phi\|_{L^1(\mathbb{R}^n)} \\
&\quad \sim 1,
\end{aligned}$$

where B_0^* is the unit dilated ball with respect to A^* and, for any $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the least integer not less than t. By this, Lemma 4 and (13), we further have

$$\lim_{K\to\infty} \sum_{i=K+1}^{\infty} |\lambda_i| \left|\int_{\mathbb{R}^n} \widehat{a_i}(x)\phi(x)\,dx\right| \lesssim \lim_{K\to\infty} \sum_{i=K+1}^{\infty} |\lambda_i| = 0,$$

which implies that, for any $\phi \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle F, \phi\rangle = \lim_{K\to\infty} \left\langle \sum_{i=1}^{K} \lambda_i \widehat{a_i}, \phi \right\rangle.$$

This finishes the proof of (17) and hence of Theorem 1. $\square$

REMARK 1. (i) When $p(\cdot) \equiv p \in (0,1]$, the Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ in Theorem 1 coincides with the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ from [4], and the inequality (7) becomes

$$|F(x)| \leqslant C\|f\|_{H_A^p(\mathbb{R}^n)} [\rho_*(x)]^{\frac{1}{p}-1}$$

with C as in (7). In this case, Theorem 1 is just [5, Theorem 1].

(ii) Let $f \in H_A^{p(\cdot)}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. In this case, we have $F = \widehat{f}$ and, using the inequality (7) with $x = \mathbf{0}$, we have $\widehat{f}(\mathbf{0}) = 0$. This implies that the function $f \in H_A^{p(\cdot)}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ has a vanishing moment, which elucidates the necessity of the vanishing moment of atoms in some sense.

(iii) Very recently, in [17, Theorem 2.4], Huang et al. obtained a result similar to Theorem 1 with the Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ replaced by the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$, where

$$\vec{a} := (a_1, \ldots, a_n) \in [1,\infty)^n \quad \text{and} \quad \vec{p} := (p_1, \ldots, p_n) \in (0,1]^n.$$

We should point out that the integrable exponent of the anisotropic mixed-norm Hardy space $H_{\vec{a}}^{\vec{p}}(\mathbb{R}^n)$ is a vector $\vec{p} \in (0,1]^n$, whose associated basic function space is the mixed-norm Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ which has different orders of integrability in different variables; however, the integrable exponent of the Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ investigated in the present article is a variable exponent function,

$$p(\cdot):\ \mathbb{R}^n \to (0,1],$$

whose associated basic function space is the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$; Obviously, $L^{\vec{p}}(\mathbb{R}^n)$ and $L^{p(\cdot)}(\mathbb{R}^n)$ cannot cover each other, so do [17, Theorem 2.4] and Theorem 1 of the present article.

4. Applications

As applications of Theorem 1, in this section, we first present a higher order convergence of the function F given in Theorem 1 at the point $\mathbf{0}$. Then we obtain an analogue of the Hardy–Littlewood inequality in the variable anisotropic Hardy space setting.

We begin with the following result.

THEOREM 2. *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ satisfy $0<p_-\leqslant p_+\leqslant 1$, where p_-, p_+ are as in (4). Then, for any $f\in H_A^{p(\cdot)}(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that $\widehat{f}=F$ in $\mathscr{S}'(\mathbb{R}^n)$ and*

$$\lim_{|x|\to 0^+}\frac{F(x)}{[\rho_*(x)]^{\frac{1}{p_+}-1}}=0, \tag{18}$$

where ρ_* is the homogeneous quasi-norm with respect to A^*.

Proof. Assume that $p(\cdot)\in C^{\log}(\mathbb{R}^n)$, $r\in(\max\{p_+,1\},\infty]$, s is as in (8) and $f\in H_A^{p(\cdot)}(\mathbb{R}^n)$. Then, by Lemma 5 and Definition 5(ii), we know that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ and a sequence of $(p(\cdot),r,s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}}\subset\mathfrak{B}$ such that

$$f=\sum_{i\in\mathbb{N}}\lambda_i a_i \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n),$$

and

$$\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}\sim\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{19}$$

Moreover, from Theorem 1 and its proof, we deduce that there exists a continuous function on $\mathbb{R}^n$, namely,

$$F=\sum_{i\in\mathbb{N}}\lambda_i\widehat{a_i} \tag{20}$$

such that $\widehat{f} = F$ in $\mathscr{S}'(\mathbb{R}^n)$.

Therefore, to complete the proof of Theorem 2, it suffices to show that (18) holds true for the function F as in (20). Indeed, for any $(p(\cdot), r, s)$-atom a supported in a dilated ball $x_0 + B_{k_0}$ with some $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$, by an argument similar to that used in Case 1) of the proof of Lemma 2, we conclude that, for any $x \in \mathbb{R}^n$ with $\rho_*(x) \leqslant b^{-k_0}$,

$$|\widehat{a}(x)| \lesssim \max\left\{ b^{k_0[1-\frac{1}{p_-}+(s+1)\frac{\ln\lambda_-}{\ln b}]}, b^{k_0[1-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}]} \right\} [\rho_*(x)]^{(s+1)\frac{\ln\lambda_-}{\ln b}}.$$

This, together with the fact that

$$1 - \frac{1}{p_-} + (s+1)\frac{\ln\lambda_-}{\ln b} > 0,$$

further implies that

$$\lim_{|x|\to 0^+} \frac{|\widehat{a}(x)|}{[\rho_*(x)]^{\frac{1}{p_+}-1}} = 0. \tag{21}$$

On another hand, from (20), it follows that, for any $x \in \mathbb{R}^n$,

$$\frac{|F(x)|}{[\rho_*(x)]^{\frac{1}{p_+}-1}} \leqslant \sum_{i\in\mathbb{N}} |\lambda_i| \frac{|\widehat{a_i}(x)|}{[\rho_*(x)]^{\frac{1}{p_+}-1}}. \tag{22}$$

In addition, by Lemma 4 and (19), we find that $\sum_{i\in\mathbb{N}} |\lambda_i| < \infty$. Thus, the equality (21) implies that, for any given $\varepsilon \in (0,1)$, there exists a positive constant ν such that, for any $i \in \mathbb{N}$ and $x \in \mathbb{R}^n$ with $|x| \leqslant \nu$,

$$\frac{|\widehat{a_i}(x)|}{[\rho_*(x)]^{\frac{1}{p_+}-1}} < \frac{\varepsilon}{\sum_{i\in\mathbb{N}} |\lambda_i| + 1}.$$

By this and (22), we know that, for any $x \in \mathbb{R}^n$ with $|x| \leqslant \nu$,

$$\frac{|F(x)|}{[\rho_*(x)]^{\frac{1}{p_+}-1}} < \varepsilon.$$

Thus,

$$\lim_{|x|\to 0^+} \frac{F(x)}{[\rho_*(x)]^{\frac{1}{p_+}-1}} = 0,$$

which completes the proof of (18) and hence of Theorem 2. $\square$

REMARK 2. (i) Similar to Remark 1, if $p(\cdot) \equiv p \in (0,1]$, then the Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ in Theorem 2 coincides with the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ from [4]. In this case, Theorem 2 is just [5, Corollary 6].

(ii) By Theorem 2 and Lemma 3, we find that

$$\lim_{|x|\to 0^+} \frac{F(x)}{|x|^{\frac{\ln b}{\ln \lambda_+}(\frac{1}{p_+}-1)}} = 0. \tag{23}$$

Note that, when $p(\cdot) \equiv p \in (0,1]$ and $A = d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, here and thereafter, $\mathrm{I}_{n\times n}$ denotes the $n\times n$ *unit matrix*, the Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ coincides with the classical Hardy space $H^p(\mathbb{R}^n)$ of Fefferman and Stein [15]. In this case, $\frac{\ln b}{\ln \lambda_+} = n$ and $p_+ = p$, and hence (23) goes back to the well-known result on $H^p(\mathbb{R}^n)$ (see [25, p. 128]).

As another application of Theorem 1, we also establish a variant of the Hardy–Littlewood inequality in the variable anisotropic Hardy space setting as follows.

THEOREM 3. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ satisfy $0 < p_- \leqslant p_+ \leqslant 1$, where p_-, p_+ are as in (4). Then, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$, there exists a continuous function F on $\mathbb{R}^n$ such that $\widehat{f} = F$ in $\mathscr{S}'(\mathbb{R}^n)$ and*

$$\left(\int_{\mathbb{R}^n} |F(x)|^{p_+} \min\left\{[\rho_*(x)]^{p_+-\frac{p_+}{p_-}-1},\,[\rho_*(x)]^{p_+-2}\right\} dx\right)^{\frac{1}{p_+}} \leqslant C\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}, \tag{24}$$

where ρ_* denotes the homogeneous quasi-norm with respect to A^* and C is a positive constant depending only on A, p_- and p_+.

Proof. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_+ \in (0,1]$, s be as in (8) and $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$. Then, by Lemma 5 and Definition 5(ii), it is easy to see that there exist a sequence $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot),2,s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, in $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in} \quad \mathscr{S}'(\mathbb{R}^n),$$

and

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant 2\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} < \infty. \tag{25}$$

Moreover, by Theorem 1 and its proof, we conclude that there exists a continuous function on $\mathbb{R}^n$, namely,

$$F = \sum_{i\in\mathbb{N}} \lambda_i \widehat{a_i} \tag{26}$$

such that $\widehat{f} = F$ in $\mathscr{S}'(\mathbb{R}^n)$.

Therefore, to prove Theorem 3, it suffices to show that (24) holds true for the function F as in (26). To this end, by the fact that $\underline{p} \leqslant p_+ \leqslant 1$ and the well-known inequality that, for any $\{\alpha_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $t \in (0,1]$,

$$\left[\sum_{i\in\mathbb{N}} |\alpha_i|\right]^t \leqslant \sum_{i\in\mathbb{N}} |\alpha_i|^t \tag{27}$$

as well as (25), we find that

$$\begin{aligned}
\left(\sum_{i\in\mathbb{N}} |\lambda_i|^{p_+}\right)^{1/p_+} &= \left(\sum_{i\in\mathbb{N}} \left\| \frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_+}\right)^{1/p_+} \\
&= \left(\sum_{i\in\mathbb{N}} \left\| \frac{|\lambda_i|^{p_+} \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_+}} \right\|_{L^{p(\cdot)/p_+}(\mathbb{R}^n)}\right)^{1/p_+} \\
&\leqslant \left\| \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_+} \right\|_{L^{p(\cdot)/p_+}(\mathbb{R}^n)}^{1/p_+} \\
&= \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{p_+} \right\}^{1/p_+} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leqslant \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \mathbf{1}_{B^{(i)}}}{\|\mathbf{1}_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\leqslant 2\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned} \tag{28}$$

On another hand, by (26), the fact that $p_+ \in (0,1]$, (27) and the Fatou lemma, it is easy to see that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |F(x)|^{p_+} \min\left\{ [\rho_*(x)]^{p_+ - \frac{p_+}{p_-} - 1}, [\rho_*(x)]^{p_+ - 2} \right\} dx \\
&\quad \leqslant \sum_{i\in\mathbb{N}} |\lambda_i|^{p_+} \int_{\mathbb{R}^n} \left[|\widehat{a}_i(x)| \min\left\{ [\rho_*(x)]^{1 - \frac{1}{p_-} - \frac{1}{p_+}}, [\rho_*(x)]^{1 - \frac{2}{p_+}} \right\} \right]^{p_+} dx.
\end{aligned} \tag{29}$$

If we can prove the following assertion: there exists a positive constant R such that, for any $(p(\cdot), 2, s)$-atom a,

$$\left(\int_{\mathbb{R}^n} \left[|\widehat{a}(x)| \min\left\{ [\rho_*(x)]^{1 - \frac{1}{p_-} - \frac{1}{p_+}}, [\rho_*(x)]^{1 - \frac{2}{p_+}} \right\} \right]^{p_+} dx \right)^{1/p_+} \leqslant R, \tag{30}$$

then, by this assertion, (29) and (28), we have

$$\left(\int_{\mathbb{R}^n}|F(x)|^{p_+}\min\left\{[\rho_*(x)]^{p_+-\frac{p_+}{p_-}-1},[\rho_*(x)]^{p_+-2}\right\}dx\right)^{1/p_+}$$
$$\leqslant R\left(\sum_{i\in\mathbb{N}}|\lambda_i|^{p_+}\right)^{1/p_+}\lesssim\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

This is the desired conclusion (24).

Thus, to complete the whole proof, it remains to show the assertion (30). Indeed, for any $(p(\cdot),2,s)$-atom a supported in a dilated ball $x_0+B_{k_0}$ with some $x_0\in\mathbb{R}^n$ and $k_0\in\mathbb{Z}$, we easily know that

$$\begin{aligned}&\left(\int_{\mathbb{R}^n}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}},[\rho_*(x)]^{1-\frac{2}{p_+}}\right\}\right]^{p_+}dx\right)^{1/p_+}\\&\lesssim\left(\int_{(A^*)^{-k_0+1}B_0^*}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}},[\rho_*(x)]^{1-\frac{2}{p_+}}\right\}\right]^{p_+}dx\right)^{1/p_+}\\&\quad+\left(\int_{((A^*)^{-k_0+1}B_0^*)^{\complement}}\left[|\widehat{a}(x)|\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}},[\rho_*(x)]^{1-\frac{2}{p_+}}\right\}\right]^{p_+}dx\right)^{1/p_+}\\&=:\mathrm{I}_1+\mathrm{I}_2,\end{aligned}\tag{31}$$

where B_0^* is the unit dilated ball with respect to A^*.

Let ε be a fixed positive constant such that

$$1-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}-\varepsilon\geqslant 1-\frac{1}{p_-}+(s+1)\frac{\ln\lambda_-}{\ln b}-\varepsilon>0.$$

Then, for I_1, from the estimate of (12), it follows that

$$\begin{aligned}\mathrm{I}_1&\lesssim b^{k_0[1+(s+1)\frac{\ln\lambda_-}{\ln b}]}\max\left\{b^{\frac{k_0}{p_-}},b^{\frac{k_0}{p_+}}\right\}\left(\int_{(A^*)^{-k_0+1}B_0^*}\right.\\&\quad\times\left.\left[\min\left\{[\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}},[\rho_*(x)]^{1-\frac{2}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}}\right\}\right]^{p_+}dx\right)^{1/p_+}\\&\lesssim b^{k_0[1+(s+1)\frac{\ln\lambda_-}{\ln b}]}\max\left\{b^{\frac{k_0}{p_-}},b^{\frac{k_0}{p_+}}\right\}\min\left\{b^{-k_0[1-\frac{1}{p_-}+(s+1)\frac{\ln\lambda_-}{\ln b}-\varepsilon]},\right.\\&\quad\times\left. b^{-k_0[1-\frac{1}{p_+}+(s+1)\frac{\ln\lambda_-}{\ln b}-\varepsilon]}\right\}\left(\int_{(A^*)^{-k_0+1}B_0^*}[\rho_*(x)]^{\varepsilon p_+-1}dx\right)^{1/p_+}\\&\sim b^{k_0\varepsilon}\left[\sum_{k=-\infty}^{0}b^{-k_0+k}b^{(-k_0+k)(\varepsilon p_+-1)}\right]^{1/p_+}\sim 1.\end{aligned}\tag{32}$$

To deal with I_2, by the Hölder inequality, the Plancherel theorem, the fact that $0<$

$p_- \leqslant p_+ \leqslant 1$ and the size condition of a, we conclude that

$$
\begin{aligned}
\mathrm{I}_2 &\lesssim \left\{ \int_{((A^*)^{-k_0+1}B_0^*)^\complement} |\widehat{a}(x)|^2\,dx \right\}^{\frac{1}{2}} \\
&\quad \times \left\{ \int_{((A^*)^{-k_0+1}B_0^*)^\complement} \left[\min\left\{ [\rho_*(x)]^{1-\frac{1}{p_-}-\frac{1}{p_+}}, [\rho_*(x)]^{1-\frac{2}{p_+}} \right\} \right]^{\frac{2p_+}{2-p_+}} dx \right\}^{\frac{2-p_+}{2p_+}} \\
&\lesssim \|a\|_{L^2(\mathbb{R}^n)} \left\{ \sum_{k=0}^{\infty} b^{-k_0+k} \left[\min\left\{ b^{(-k_0+k)(1-\frac{1}{p_-}-\frac{1}{p_+})}, b^{(-k_0+k)(1-\frac{2}{p_+})} \right\} \right]^{\frac{2p_+}{2-p_+}} \right\}^{\frac{2-p_+}{2p_+}} \\
&\lesssim \|a\|_{L^2(\mathbb{R}^n)} \left\{ b^{-k_0} \left[\min\left\{ b^{-k_0(1-\frac{1}{p_-}-\frac{1}{p_+})}, b^{-k_0(1-\frac{2}{p_+})} \right\} \right]^{\frac{2p_+}{2-p_+}} \right\}^{\frac{2-p_+}{2p_+}} \\
&\lesssim \max\left\{ b^{k_0(\frac{1}{2}-\frac{1}{p_-})}, b^{k_0(\frac{1}{2}-\frac{1}{p_+})} \right\} \min\left\{ b^{-k_0(\frac{1}{2}-\frac{1}{p_-})}, b^{-k_0(\frac{1}{2}-\frac{1}{p_+})} \right\} \\
&\sim 1.
\end{aligned}
$$

This, combined with (31) and (32), implies that (30) holds true and hence finishes the proof of Theorem 3. $\square$

REMARK 3. Recall that the well-known Hardy–Littlewood inequality for the classical Hardy space $H^p(\mathbb{R}^n)$ is as follows: Let $p \in (0,1]$. Then for each $f \in H^p(\mathbb{R}^n)$, we can find a continuous function F on $\mathbb{R}^n$ satisfying that $\widehat{f} = F$ in $\mathscr{S}'(\mathbb{R}^n)$ and

$$
\left[\int_{\mathbb{R}^n} |x|^{n(p-2)} |F(x)|^p\,dx \right]^{1/p} \leqslant C \|f\|_{H^p(\mathbb{R}^n)}, \tag{33}
$$

where C is a positive constant independent of f and F (see [25, p. 128]).

We point out that the inequality (24) in Theorem 3 is an analogue of the Hardy–Littlewood inequality in the present setting. Indeed, similar to Remark 1, when $p(\cdot) \equiv p \in (0,1]$, the Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ in Theorem 3 becomes the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ from [4]. In this case, $p_+ = p_- = p$ and hence Theorem 3 is just [5, Corollary 8]. Moreover, if $A = d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, then the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ (namely, the Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ with $p(\cdot) \equiv p \in (0,1]$) coincides with the classical Hardy space $H^p(\mathbb{R}^n)$ of Fefferman and Stein [15]. In this case, $\rho_*(x) \sim |x|^n$ for any $x \in \mathbb{R}^n$, and hence the Hardy–Littlewood inequality (24) is just (33).

Acknowledgements. The author would like to thank the referee for his/her carefully reading and several useful comments which indeed improve the quality of this article. This article is supported by the National Natural Science Foundation of China

(Grant No. 12001527), the Natural Science Foundation of Jiangsu Province (Grant No. BK20200647) and the Project Funded by China Postdoctoral Science Foundation (Grant No. 2021M693422).

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(Received May 16, 2021)

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