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ON HERMITE–HADAMARD INEQUALITIES FOR (k,h)–CONVEX SET–VALUED MAPS

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Abstract. We introduce the class of (k,h)-convex set-valued maps defined on k-convex domains by

$$h(t)G(x_1)+h(1-t)G(x_2)\subset G(k(t)x_1+k(1-t)x_2),\ x_1,x_2\in D,\ t\in[0,1],$$

and prove a Hermite-Hadamard-type theorem for such maps. Many other properties of (k,h)-convex set-valued maps are also presented.

1. Introduction

It is well known that if a function $f\colon I\to\mathbb{R}$ defined on an interval I is convex, that is

$$f(tx+(1-t)y)\leqslant tf(x)+(1-t)f(y),\ x,y\in I,\ t\in[0,1],$$

then for all $a,b\in I$ with $a<b$ it satisfies the following Hermite-Hadamard inequalities

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2}. \tag{1}$$

These inequalities, having natural geometrical interpretation and many applications, are one of the most known results on convex functions. There are many books and papers containing their historical background, various generalizations, refinements and extensions for other related classes of functions (see e.g. [3, 5, 7] and the references given there).

In the paper [4] by Micherda and Rajba, the so called k-convex sets and (k,h)-convex functions were introduced with the following definitions.

DEFINITION 1. Let $k\colon(0,1)\to\mathbb{R}$ be a given function. A subset D of a real linear space X is called k-convex if $k(t)x+k(1-t)y\in D$ for all $x,y\in D$ and $t\in(0,1)$.

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DEFINITION 2. Let $k,h\colon (0,1)\to\mathbb{R}$ be two given functions and suppose that $D\subset X$ is a k-convex set. A function $f\colon D\to\mathbb{R}$ is called (k,h)-convex if, for all $x,y\in D$ and $t\in(0,1)$

$$f\big(k(t)x+k(1-t)y\big)\leqslant h(t)f(x)+h(1-t)f(y). \tag{2}$$

If the inequality in (2) is reversed, f is called (k,h)-concave.

REMARK 1. The class of (k,h)-convex functions generalizes many other known classes of functions. It is evident that for $h(t)=k(t)=t$ we get the classical convexity. For $k(t)=t$, the notion of (k,h)-convexity agrees with the one of h-convexity introduced by Varošanec [10]. Recall that a nonnegative function $f\colon I\to\mathbb{R}$ is called h-convex if

$$f\big(tx+(1-t)y\big)\leqslant h(t)f(x)+h(1-t)f(y) \tag{3}$$

for all $x,y\in I$ and $t\in(0,1)$. This notion generalizes the concepts of classical convexity (for $h(t)=t$), s-Breckner-convexity (for $h(t)=t^s$ with some $s\in(0,1)$, P-functions (for $h(t)=1$ and Godunova-Levin functions (for $h(t)=t^{-1}$). If $s>0$, $k(t)=t^{1/s}$ and $h(t)=t$, then f is (k,h)-convex if and only if it is s-Orlicz-convex. For $k(t)=h(t)=1$, the class of (k,h)-convex functions consists of all mappings which are subadditive. If $k(t)=h(t)=1/2$ for all t, then (2) produces the family of Jensen-convex functions. If k is defined by the formula

$$k(t)=\begin{cases}2t & \text{for } t<1/2\\ 0 & \text{for } t\geqslant 1/2,\end{cases} \tag{4}$$

then f is a (k,k)-convex function if and only if it is star-shaped, i.e. $f(tx)\leqslant tf(x)$ for all $t\in[0,1]$ and $x\in D$.

In [4], the following results on the Hermite-Hadamard inequalities for (k,h)-convex functions are obtained (here D is a k-convex subset of $\mathbb{R}$ and the existence of integrals is assumed in the formulas).

THEOREM 2. [4, Corollary 3.6] *Let $f\colon D\to\mathbb{R}$ be a (k,h)-convex function, $h(1/2)>0$ and choose $a,b\in D$ such that $a<b$. Then*

$$\begin{aligned}\frac{1}{2h(1/2)}\int_0^1 f\big(k(1/2)[k(t)+k(1-t)]\cdot(a+b)\big)\,dt\\ \leqslant\int_0^1 f\big(k(t)a+k(1-t)b\big)\,dt\leqslant[f(a)+f(b)]\int_0^1 h(t)\,dt.\end{aligned} \tag{5}$$

THEOREM 3. [4, Corollary 3.2] *Let $f\colon D\to\mathbb{R}$ be a (k,h)-convex function with $h(1/2)>0$ and choose $a<b$ such that $[a,b]\subset D$. Then*

$$\frac{f\big(k(1/2)(a+b)\big)}{2h(1/2)}\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx. \tag{6}$$

REMARK 4. It is clear, that if f is a (k,h)-concave function, then the opposite signs hold in inequalities (5) and (6).

The main purpose of this paper is to prove set-valued counterparts of the above theorems. Our result generalizes some earlier Hermite-Hadamard type theorems for convex set-valued maps presented in [5, 9]. As corollaries, we obtain many similar results for various related classes of set-valued maps.

2. Preliminaries

Let $(Y,\|\cdot\|)$ be a Banach space and $n(Y)$ denote the family of all nonempty subsets of Y. Denote also by $cl(Y)$ the family of all closed members of $n(Y)$ and by $clcb(Y)$ the family of all closed convex and bounded members of $n(Y)$.

DEFINITION 3. Let D be a k-convex subset of a real vector space X and let $k,h\colon (0,1)\to\mathbb{R}$ be two given functions. We say that that a set-valued map $G\colon D\to n(Y)$ is (k,h)-*convex* if

$$h(t)G(x_1)+h(1-t)G(x_2)\subset G(k(t)x_1+k(1-t)x_2) \tag{7}$$

for all $x_1,x_2\in D$ and $t\in[0,1]$.

It is clear that for $k(t)=h(t)=t$, $t\in[0,1]$, condition (7) coincides with the well known definition of convex set-valued maps (see [1, 6]). Such maps arise naturally from, e.g., constraints of convex optimization problems and play an important role in various questions of convex analysis and mathematical economics.

Many of the known properties of convex functions and convex set-valued maps have their counterparts for (k,h)-convex set-valued maps. Some of them are listed below.

REMARK 5. 1. If G is (k,h)-convex and single-valued, then inclusion (7) reduces to equality and we get the (k,h)-affine functions.

2. If $g\colon D\to Y$ is a (k,h)-affine function and $A\subset Y$ is h-convex, then $G: D\to n(Y)$ defined by $G(x)=g(x)+A$ is (k,h)-convex.

3. If $g\colon D\to\mathbb{R}$ is (k,h)-convex, then $G: D\to n(\mathbb{R})$ defined by $G(x)=[g(x),\infty)$ is (k,h)-convex.

4. A set-valued map $G\colon D\to n(Y)$ is (k,k)-convex if and only if its graph $Gr\,G=\{(x,y)\colon x\in D, y\in G(x)\}$ is a k-convex subset of $X\times Y$.

5. If $G_1,G_2\colon D\to n(Y)$ are (k,h)-convex and $c\in\mathbb{R}$, then G_1+G_2 and $c\cdot G_1$ are (k,h)-convex.

6. If $G_1,G_2\colon D\to n(Y)$ are (k,h)-convex and $G_1(x)\cap G_2(x)\neq\emptyset$ for every $x\in D$, then $G_1\cap G_2 : D\to n(Y)$ defined by $(G_1\cap G_2)(x)=G_1(x)\cap G_2(x)$ is also (k,h)-convex. Note that the analogous property for $G_1\cup G_2$ does not hold. For instance, assume that $k(t)=h(t)=t$ and take $G_1,G_2\colon[-1,1]\to n(\mathbb{R})$ defined by $G_1(x)=[0,x+1]$, $G_2(x)=[0,1-x]$, $x\in[-1,1]$. Then G_1 and G_2 are (k,h)-convex, but $G_1\cup G_2$ is not.

7. If $G_1, G_2 \colon D \to n(Y)$ are (k,h)-convex, then $G_1 \times G_2 : D \to n(Y \times Y)$ defined by $(G_1 \times G_2)(x) = G_1(x) \times G_2(x)$ is also (k,h)-convex.

8. If $G\colon D \to n(Y)$ is (k,h)-convex, then $conv\, G : D \to n(Y)$ defined by $(conv\, G)(x) = conv\, G(x)$ is (k,h)-convex (because $conv\,(A+B) = conv\, A + conv\, B$ for all $A, B \subset Y$).

9. If $G\colon D \to n(Y)$ is (k,h)-convex and $int\, G(x) \neq \emptyset$ for every $x \in D$, then $int\, G\colon D \to n(Y)$ defined by $(int\, G)(x) = int\, G(x)$ is (k,h)-convex (because $int\, A + int\, B \subset int\,(A+B)$ for all $A, B \subset Y$).

10. If $G\colon D \to n(Y)$ is (k,h)-convex, then $cl\, G : D \to n(Y)$ defined by $(cl\, G)(x) = cl\, G(x)$ is (k,h)-convex (because $cl\, A + cl\, B \subset cl\,(A+B)$ for all $A, B \subset Y$).

LEMMA 6. *A set-valued map $G\colon D \to clcb(\mathbb{R})$ is (k,h)-convex if and only if it is of the form $G(x) = [g_1(x), g_2(x)]$, $x \in D$, where $g_1 : D \to \mathbb{R}$ is (k,h)-convex and $g_2 : D \to \mathbb{R}$ is (k,h)-concave.*

Proof. The "if" part is clear. To prove the "only if" part, consider the functions $g_1\colon D \to \mathbb{R}$ and $g_2\colon D \to \mathbb{R}$ defined by

$$g_1(x) = \inf G(x) \quad \text{and} \quad g_2(x) = \sup G(x),\ x \in D.$$

By the (k,h)-convexity of G and the fact that for any $A, B \subset \mathbb{R}$, $\inf(A+B) = \inf A + \inf B$ and $\sup(A+B) = \sup A + \sup B$, we obtain that g_1 is (k,h)-convex and g_2 is (k,h)-concave. Since the values of G are convex and closed, the result follows. □

From now we assume that Y is a separable Banach space and $I \subset \mathbb{R}$ is an interval. For a given set-valued map $G\colon I \to n(Y)$ the integral $\int_I G(t)dt$ is understood in the sense of Aumann, i.e. it is the set of integrals of all integrable (in the sense of Bochner) selections of the map G (cf. [1, 2]):

$$\int_I G(t)dt = \left\{ \int_I g(t)dt \ :\ g : I \to Y \text{ is integrable and } g(t) \in G(t),\ t \in I \right\}.$$

A set-valued map $G\colon I \to n(Y)$ is called *integrable bounded*, if there exists a non-negative integrable function $\psi : I \to \mathbb{R}$ such that $G(t) \subset \psi(t)B$, $t \in I$, where B is the unit ball in Y. In this case every measurable selection of G is integrable and, consequently, the Aumann integral of G is nonempty whenever G is measurable.

The following properties of the Aumann integral will be needed in our investigations:

LEMMA 7. [1, Theorems 8.6.3, 8.6.4] *Let $G\colon I \to cl(Y)$ be a measurable set-valued map.*

a) The closure of the integral of G is convex and

$$\overline{\int_I G(t)dt} = \overline{conv}\left(\int_I G(t)dt \right).$$

b) If Y is finite dimensional, then the integral of G is convex. In particular, if $Y = \mathbb{R}$ and $G(t) = [g_1(t), g_2(t)]$, $t \in I$, then

$$\int_I G(t)dt = \left[\int_I g_1(t)dt, \int_I g_2(t)dt \right].$$

c) *If G is integrable bounded, then*

$$\overline{\int_I G(t)dt} = \int_I \overline{conv} G(t)dt.$$

3. Hermite-Hadamard inequalities for (k,h)-convex set-valued maps

THEOREM 8. *Assume that D is a k-convex subset of real vector space. Let $G\colon D \to clcb(Y)$ be a (k,h)-convex integrable bounded set-valued map. Assume that $h(1/2) > 0$ and $a,b \in D$. Then*

$$\begin{aligned}[G(a)+G(b)]\cdot\int_0^1 h(t)\,dt &\subset \int_0^1 G\big(k(t)a{+}k(1-t)b\big)\,dt \\ &\subset \frac{1}{2h(1/2)}\int_0^1 G\big(k(1/2)[k(t)+k(1-t)](a+b)\big)\,dt. \end{aligned} \tag{8}$$

Proof. To prove the left inclusion, fix $a,b \in D$ and take arbitrary $u \in G(a)$ and $v \in G(b)$. Consider the function $g\colon [0,1] \to Y$ defined by

$$g(t) = h(t)u + h(1-t)v, \quad t \in [0,1].$$

By the (k,h)-convexity of G, we get

$$g(t) \in h(t)G(a) + h(1-t)G(b) \subset G\big(k(t)a + k(1-t)b\big),$$

which means that g is a selection of G. Simple calculations give

$$\int_0^1 g(t)dt = u\int_0^1 h(t)dt + v\int_0^1 h(1-t)dt = (u+v)\int_0^1 h(t)dt.$$

Hence,

$$(u+v)\int_0^1 h(t)dt \in \int_0^1 G\big(k(t)a + k(1-t)b)\big)dt,$$

which shows that

$$\big(G(a)+G(b)\big)\int_0^1 h(t)dt \subset \int_0^1 G\big(k(t)a + k(1-t)b)\big)dt.$$

The proof of the right inclusion is divided into two steps.

First, we assume that $Y = \mathbb{R}$. Then, by Lemma 6, G is of the form $G(x) = [g_1(x), g_2(x)]$, $x \in D$, where $g_1\colon D \to \mathbb{R}$ is (k,h)-convex and $g_2\colon D \to \mathbb{R}$ is (k,h)-concave. Let $g\colon [0,1] \to \mathbb{R}$ be any integrable selection of the set-valued map $t \mapsto G\big(k(t)a + k(1-t)b\big)$. Denote $\varphi(t) := k(1/2)[k(t)+k(1-t)](a+b)$. Then, by Theorem 2 (cf. also Remark 4), we get

$$\int_0^1 g(t)dt \geqslant \int_0^1 g_1\big(k(t)a + k(1-t)b)\big)dt \geqslant \frac{1}{2h(1/2)}\int_0^1 g_1\big(\varphi(t)\big)dt$$

and

$$\int_0^1 g(t)dt \leqslant \int_0^1 g_2\big(k(t)a+k(1-t)b\big)\big)dt \leqslant \frac{1}{2h(1/2)}\int_0^1 g_2\big(\varphi(t)\big)dt.$$

Hence, by Lemma 7(b),

$$\int_0^1 g(t)dt \in$$

$$\frac{1}{2h(1/2)}\cdot\Big[\int_0^1 g_1\big(\frac{1}{2h(1/2)}\int_0^1 G\big(\varphi(t)\big)dt\big)dt, \int_0^1 g_2\big(\frac{1}{2h(1/2)}\int_0^1 G\big(\varphi(t)\big)dt\big)dt\Big]$$
$$= \frac{1}{2h(1/2)}\int_0^1 G\big(\varphi(t)\big)dt.$$

Since this relation holds for any integrable selection g of the map $t\mapsto G\big(k(t)+k(1-t)\big)$, we get

$$\int_0^1 G\big(k(t)a+k(1-t)b\big)\big)dt \subset \frac{1}{2h(1/2)}\int_0^1 G\big(\varphi(t)\big)dt$$
$$= \frac{1}{2h(1/2)}\int_0^1 G\big(k(1/2)[k(t)+k(1-t)](a+b)\big)dt.$$

Now, assume that Y is an arbitrary separable Banach space. Take a nonzero continuous linear functional $y^*\in Y^*$ and consider the set-valued map $x\mapsto \overline{y^*(G(x))}$, $x\in D$. It is easy to check that this map is (k,h)-convex and has values in $clcb(\mathbb{R})$. So, by the previous step,

$$\int_0^1 \overline{y^*\Big(G\big(k(t)a+k(1-t)b\big)\Big)}dt \subset \frac{1}{2h(1/2)}\int_0^1 \overline{y^*\Big(G\big(\varphi(t)\big)\Big)}dt. \tag{9}$$

Take arbitrary $z\in\int_0^1 G\big(k(t)a+k(1-t)b\big)dt$. By the definition of the Aumann integral, there exists an integrable selection g of the set-valued map $t\mapsto G\big(k(t)+k(1-t)\big)$ such that $z=\int_0^1 g(t)dt$. From (9), we obtain

$$y^*(z) = y^*\Big(\int_0^1 g(t)dt\Big) = \int_0^1 y^*\big(g(t)\big)dt \in \frac{1}{2h(1/2)}\int_0^1 \overline{y^*\Big(G\big(\varphi(t)\big)\Big)}dt. \tag{10}$$

Since G is integrable bounded and the values $y^*\Big(G\big(\varphi(t)\big)\Big)$ are convex, by Lemma 7(c), we get

$$\int_0^1 \overline{y^*\Big(G\big(\varphi(t)\big)\Big)}dt = \overline{\int_0^1 y^*\Big(G\big(\varphi(t)\big)\Big)} = \overline{y^*\Big(\int_0^1 G\big(\varphi(t)\big)dt\Big)}$$
$$\subset \overline{y^*\Big(\overline{\int_0^1 G\big(\varphi(t)\big)dt}\Big)}. \tag{11}$$

From (10) and (11), we obtain

$$y^*(z) \in \frac{1}{2h(1/2)}\overline{y^*\Big(\overline{\int_0^1 G\big(\varphi(t)\big)dt}\Big)}.$$

Since this condition holds for arbitrary $y^* \in Y^*$ and, by Lemma 7(a), the set $\overline{\int_0^1 G(\varphi(t))\,dt}$ is convex and closed, by the separation theorem (see [8], Corollary 2.5.11), we obtain

$$z \in \overline{\frac{1}{2h(1/2)}\int_0^1 G(\varphi(t))\,dt}$$

and hence, using once more Lemma 7(c),

$$z \in \frac{1}{2h(1/2)}\int_0^1 G(\varphi(t))\,dt.$$

Consequently,

$$\begin{aligned}\int_0^1 G\big(k(t)a+k(1-t)b\big)dt &\subset \frac{1}{2h(1/2)}\int_0^1 G(\varphi(t))\,dt\\ &= \frac{1}{2h(1/2)}\int_0^1 G\big(k(1/2)[k(t)+k(1-t)](a+b)\big)dt,\end{aligned}$$

which finishes the proof. □

THEOREM 9. *Assume that D is a k-convex subset of $\mathbb{R}$ and choose $a<b$ such that $[a,b]\subset D$. Let $G\colon I\to clcb(Y)$ be a (k,h)-convex measurable set-valued map with $h(1/2)>0$. Then*

$$\frac{1}{b-a}\int_a^b G(x)\,dx \subset \frac{G\big(k(1/2)(a+b)\big)}{2h(1/2)}. \tag{12}$$

Proof. Let $z\in\frac{1}{b-a}\int_a^b G(x)\,dx$. Then, there exists an integrable selection g of G such that $z=\frac{1}{b-a}\int_a^b g(x)\,dx$. Suppose, contrary to our claim, that

$$z\notin \frac{G\big(k(1/2)(a+b)\big)}{2h(1/2)}.$$

By the separation theorem, there exists a linear continuous functional $y^*\in Y^*$ such that

$$y^*(z) < \inf y^*\Big(\frac{G\big(k(1/2)(a+b)\big)}{2h(1/2)}\Big). \tag{13}$$

Consider the function $f\colon D\to\mathbb{R}$ defined by

$$f(x)=\inf y^*\big(G(x)\big),\ x\in D.$$

It is easy to check that f is (k,h)-convex. Therefore, by Theorem 3,

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geqslant \frac{f\big(k(1/2)(a+b)\big)}{2h(1/2)}. \tag{14}$$

On the other hand, by (13), and taking into account that $y^*(g(x))\geqslant \inf y^*\big(G(x)\big)=f(x)$, we obtain

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{1}{b-a}\int_a^b y^*(g(x))\,dx = y^*(z) < \frac{f\big(k(1/2)(a+b)\big)}{2h(1/2)}.$$

This contradicts (14) and finishes the proof. □

4. Applications

As an immediate consequence of the Hermite-Hadamard-type inclusions proved above for (k,h)-convex set-valued maps, we can obtain many similar results for various related classes of set-valued maps. In this section we present a few examples of them. In the corollaries below D is assumed to be a k-convex subset of $\mathbb{R}$ (with the appropriate function k).

For $k(t) = h(t) = t$, $t \in (0,1)$, Theorem 8 as well as Theorem 9 reduces to the known Hermite-Hadamard inclusions for convex set-valued maps proved (under somewhat different assumptions on G) in [5, 9].

COROLLARY 10. *If $G\colon D \to clcb(Y)$ is convex and integrable bounded, then*

$$\frac{G(a)+G(b)}{2} \subset \frac{1}{b-a}\int_a^b G(x)\,dx \subset G\Big(\frac{a+b}{2}\Big),$$

for all $a,b \in D$, $a<b$.

If we assume that $k(t) = t$ for all $t \in (0,1)$, then from Theorem 8, we obtain a counterpart of the Hermite-Hadamard inequality for h-convex set-valued maps.

COROLLARY 11. *Let $G\colon D \to clcb(Y)$ be a h-convex integrable bounded set-valued map. Assume that $h(1/2) > 0$ and $a,b \in D$. Then*

$$[G(a)+G(b)]\int_0^1 h(t)\,dt \subset \frac{1}{b-a}\int_a^b G(x)\,dx \subset \frac{G\big(\frac{a+b}{2}\big)}{2h(1/2)}.$$

REMARK 12. Taking into account Remark 1, from Corollary 11, we can obtain further counterparts of the Hermite-Hadamard inequality for s-Breckner-convex set-valued maps (for $h(t) = t^s$ with some $s \in (0,1)$), P-convex set-valued maps (for $h(t) = 1$) and Godunova-Levin convex set-valued maps (for $h(t) = t^{-1}$).

If we assume that $k(t) = t^{1/s}$ and $h(t) = t$ for all $t \in (0,1)$, then from Theorems 8 and 9, we obtain counterparts of the Hermite-Hadamard inequalities for s-Orlicz-convex set-valued maps.

COROLLARY 13. *Let $G\colon D \to clcb(Y)$ be an s-Orlicz-convex integrable bounded set-valued map. Assume that $a,b \in D$. Then*

$$\begin{aligned}\frac{G(a)+G(b)}{2} &\subset \int_0^1 G\big(t^{1/s}a + (1-t)^{1/s}b\big)\,dt \\ &\subset \int_0^1 G\bigg(\frac{1}{2^{1/s}}\big[t^{1/s}+(1-t)^{1/s}\big](a+b)\bigg)\,dt\end{aligned}$$

and

$$\frac{1}{b-a}\int_a^b G(x)\,dx \subset G\left(\frac{a+b}{2^{1/s}}\right).$$

If we assume that $k(t)$ is defined by (4) and $h(t) = k(t)$ for all $t \in (0,1)$, then from the left-hand side of (8) in Theorem 8, we obtain a counterpart of the Hermite-Hadamard inequalities for star-shaped set-valued maps.

COROLLARY 14. *Let $G\colon D \to clcb(Y)$ be a star-shaped integrable bounded set-valued map. Assume that $a,b \in D \setminus \{0\}$. Then*

$$\frac{G(a)+G(b)}{2} \subset \frac{1}{a}\int_0^a G(t)\,dt + \frac{1}{b}\int_0^b G(t)\,dt.$$

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