ABSTRACT HARDY SPACES

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Abstract. In this paper, we study the abstract Hardy spaces on spaces of homogeneous X. Firstly, we give the definitions of the atomic Hardy spaces H_{ato}^p and the molecular Hardy spaces $H_{\varepsilon,mol}^p (0 < p < 1)$. Secondly, we give out the comparison between our Hardy spaces with some other Hardy spaces. Finally, we prove the continuity theorem of the sublinear operator on the Hardy spaces and give an example.

1. Introduction

The real variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ on the n-dimensional Euclidean space $\mathbb{R}^n$ has received more and more attention in recent decades, which initiated by E.M. Stein and G. Weiss [36], and then systematically developed by C. Fefferman and E.M. Stein [21]. C. Fefferman and E.M. Stein [21] brought real variable methods into this subject, eventually, the evolution of their ideas led to the atomic or molecular characterizations and the applications of Hardy spaces, see the articles [3, 6, 13, 15, 22, 25, 26, 31, 33, 38, 39, 41, 43, 45] and the references therein. Furthermore, the atomic and the molecular characterizations enabled the extension of the real theory of Hardy spaces on $\mathbb{R}^n$ to spaces of homogeneous type, which is a more general setting for function spaces than Euclidean space [11, 12].

There are lots of applications in various fields of analysis, for instance, harmonic analysis, functional analysis and partial differential equations, see the articles [1, 2, 4, 5, 8, 9, 18, 19, 20, 23, 29, 30, 32, 34, 40, 42] and the references therein. Moreover, it is well known that when $p \in (1, \infty)$, $L^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$ are essentially the same, however, when $p \in (0, 1]$, the space $H^p(\mathbb{R}^n)$ is more suitable for problems emerging in the theory of the boundedness of operators, because some singular integrals (such as Riesz transform) are bounded on $H^p(\mathbb{R}^n)$, but not on $L^p(\mathbb{R}^n)$.

In the Euclidean case (with the Lebesgue measure), the space $H^p(\mathbb{R}^n)$ has many different characterizations [14, 37]. One of the important characterizations is in terms of atoms:

Recall that a function a supported in a ball Q of $\mathbb{R}^n$ is called a (p,q)-atom if $\|a\|_{L^q(Q)} \leqslant |Q|^{1/q-1/p}$, and $\int_Q x^k a(x)dx = 0, 0 \leqslant k \leqslant [1/p] - 1$. It can be proved that any (p,q)-atom a is in $H^p(\mathbb{R}^n)$. Then the following atomic decomposition theorem [10, 28] holds: a tempered distribution $f \in \mathscr{S}'(\mathbb{R}^n)$ belongs to $H^p(\mathbb{R}^n)$ if and only if it has a decomposition

$$f = \sum_{k=0}^{\infty} \lambda_k a_k,$$

where the a_k's are (p,q)-atoms and $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$.

As for the spaces of homogeneous type, let us recall the definition of the atomic Hardy space in [12].

Let (Y,d,ν) be a space of homogeneous type and $\varepsilon > 0$ be a fixed parameter. A function $m \in L^2(Y)$ is called a $(p,2,\varepsilon)$-molecule associated to a ball Q if $\int_Y md\nu = 0$, for all $i \geqslant 0$,

$$\left(\int_{2^{i+1}Q \setminus 2^i Q} |m|^2 d\nu \right)^{1/2} \leqslant \nu(2^{i+1}Q)^{1/2-1/p} 2^{-\varepsilon i} \text{ and } \left(\int_Q |m|^2 d\nu \right)^{1/2} \leqslant \nu(Q)^{1/2-1/p}.$$

If in addition we assume $\text{supp}(m) \subset Q$, we call m a $(p,2)$-atom. Thus, a $(p,2)$-atom is exactly a $(p,2,\infty)$-molecule. Then a function h belongs to $H_{CW}^p(Y)$ (called the "Hardy space of Coifman-Weiss" on Y [12]) if there exists a decomposition

$$h = \sum_{i \in \mathbb{N}} \lambda_i m_i, \quad \nu - a.e.,$$

where m_i are $(p,2,\varepsilon)$-molecules and λ_i are coefficients which satisfy

$$\sum_{i \in \mathbb{N}} |\lambda_i|^p < \infty.$$

In 2008 and 2009, F. Bernicot and J. Zhao [6, 7] studied a kind of new abstract Hardy spaces $H_{\varepsilon,mol}^1$ which keep the main properties of the classical Hardy spaces H^1. In the present paper, we give a further research about the abstract Hardy spaces.

The paper is organized as follows.

In Section 2, first, we give some definitions and notations, such as the definitions of the atomic Hardy space H_{ato}^p and the molecular Hardy space $H_{\varepsilon,mol}^p(0 < p < 1)$, and then we introduce two useful properties about the spaces H_{ato}^p and $H_{\varepsilon,mol}^p$. In Section 3, we give out the comparison between our Hardy spaces with some other Hardy spaces. In Section 4, we prove the continuity theorem on the Hardy space, and then we give an example which satisfies the assumption of the continuity theorem. At the end of this section we give the embedding property.

Finally, we make some conventions on notations. If $f \leqslant Cg$, we write $f \lesssim g$; if $f \lesssim g \lesssim f$, we then write $f \sim g$. For any subset E of X, we use χ_E to denote its *characteristic function*.

2. Preliminaries

About the definition of spaces of homogeneous type, we consider a set X equipped with a quasi-metric d and a Borel measure μ.

Quasi-metric d is a function $d : X \times X \to [0, \infty)$ satisfying

(i) $d(x_1, x_2) = d(x_2, x_1) \geqslant 0$ for all $x_1, x_2 \in X$;

(ii) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$;

(iii) the quasi-triangle inequality: there is a constant $A_0 \in [1, \infty)$ such that for all $x_1, x_2, x_3 \in X$,

$$d(x_1, x_2) \leqslant A_0(d(x_1, x_3) + d(x_3, x_2)).$$

The nonzero measure μ satisfies the doubling property:

$$\exists A > 0, \exists \delta > 0, \forall x \in X, \forall r > 0, \forall t \geqslant 1, \quad \frac{\mu(B(x, tr))}{\mu(B(x, r))} \leqslant A t^\delta, \tag{1}$$

where $B(x, r)$ is the open ball with center $x \in X$ and radius $r > 0$. δ is the homogeneous dimension of X.

We say that (X, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss if d is a quasi-metric on X and μ is a nonzero Borel measure on X satisfying the doubling condition.

Here we are working with real valued functions and we will use "real" duality, and we have the same results with complex duality and complex valued functions.

For Q a ball, and $i \geqslant 0$, we write $S_i(Q)$ the scaled corona around the ball Q:

$$S_i(Q) := \left\{ x, 2^i \leqslant 1 + \frac{d(x, c(Q))}{r_Q} < 2^{i+1} \right\},$$

where r_Q is the radius of the ball Q and $c(Q)$ its center. It is easy to see that $S_0(Q)$ corresponds to the ball Q and $S_i(Q) \subset 2^{i+1}Q$ for $i \geqslant 1$, where λQ is as usual the ball with center $c(Q)$ and radius λr_Q.

Denote $\mathscr{Q}$ by the collection of all balls: $\mathscr{Q} := \{B(x, r), x \in X, r > 0\}$, and $\mathbb{B} := (B_Q)_{Q \in \mathscr{Q}}$ a collection of L^2-bounded linear operator, indexed by the collection $\mathscr{Q}$. We assume that these operators B_Q are uniformly bounded on L^2: there exists a constant $0 < A' < \infty$ so that:

$$\forall f \in L^2, \forall Q \text{ ball}, \ \big\| B_Q(f) \big\|_{L^2} \leqslant A' \|f\|_{L^2}. \tag{2}$$

In the rest of the paper, we allow the constants to depend on A, A' and δ.

Now, we define atoms and molecules by using the collection $\mathbb{B}$.

DEFINITION 1. Let $0 < p < 1$ and $\varepsilon > 0$ be a fixed parameter. A function $m \in L^2$ is called a $(p, 2, \varepsilon)$-molecule associated to a ball Q if there exists a real function f_Q such that

$$m = B_Q(f_Q),$$

with

$$\forall i \geqslant 0, \|f_Q\|_{2, S_i(Q)} \leqslant \mu(2^i Q)^{\frac{1}{2} - \frac{1}{p}} 2^{-\varepsilon i}.$$

We call $m = B_Q(f_Q)$ a $(p,2)$-atom if in addition we have $\text{supp}(f_Q) \subset Q$. So an atom is exactly a $(p,2,\infty)$-molecule.

The functions f_Q in this definition are normalized in L^p. It is easy to see that

$$\|f_Q\|_{L^p} \lesssim 1 \quad \text{and} \quad \|f_Q\|_{L^2} \lesssim \mu(Q)^{\frac{1}{2} - \frac{1}{p}}.$$

So by the L^2-boundedness of the operators B_Q, we know that each molecule belongs to the space L^2.

Next, we define the atomic Hardy space H_{ato}^p and the molecular Hardy space $H_{\varepsilon,mol}^p$. First, we recall some notions related to quasi-Banach spaces [24].

DEFINITION 2. A quasi-Banach space $\mathscr{B}$ is a vector space endowed with a quasi-norm $\|\cdot\|_{\mathscr{B}}$ which is non-negative, non-degenerate(i.e. $\|f\|_{\mathscr{B}} = 0$ if and only if $f = 0$), homogeneous, and obeys the quasi-triangle inequality, that is, there exists a constant $K \in [1, \infty)$ such that, for all $f, g \in \mathscr{B}$,

$$\|f + g\|_{\mathscr{B}} \leqslant K(\|f\|_{\mathscr{B}} + \|g\|_{\mathscr{B}}).$$

DEFINITION 3. Let $r \in (0,1]$. A quasi-Banach space $\mathscr{B}_r$ with the quasi-norm $\|\cdot\|_{\mathscr{B}_r}$ is called an r-quasi-Banach space if $\|f + g\|_{\mathscr{B}_r}^r \leqslant \|f\|_{\mathscr{B}_r}^r + \|g\|_{\mathscr{B}_r}^r$ for all $f, g \in \mathscr{B}_r$. Hereafter, $\|\cdot\|_{\mathscr{B}_r}^r$ is called the r-quasi-norm of the r-quasi-Banach space $\mathscr{B}_r$.

DEFINITION 4. Let $0 < p < 1$. A measurable function f is said to belong to the space $\mathbb{H}_{ato}^p$ if there exists a sequence of $(p,2)$-atoms $\{m_i\}_{i=1}^\infty$, such that $f = \sum_{i \in \mathbb{N}} \lambda_i m_i$ in L^2, where for all i, λ_i are real numbers satisfying

$$\sum_{i \in \mathbb{N}} |\lambda_i|^p < \infty.$$

Furthermore, define

$$\|f\|_{H_{ato}^p} := \inf_{f = \sum_{i \in \mathbb{N}} \lambda_i m_i} \left(\sum_i |\lambda_i|^p\right)^{1/p}.$$

The atomic Hardy space H_{ato}^p is then defined as the completion of $\mathbb{H}_{ato}^p$ with respect to the p-quasi-norm $\|\cdot\|_{H_{ato}^p}^p$.

Similarly, we have

DEFINITION 5. Let $0 < p < 1$ and $\varepsilon > 0$. A measurable function f is said to belong to the space $\mathbb{H}_{\varepsilon,mol}^p$ if there exists a sequence of $(p,2,\varepsilon)$-molecules $\{m_i\}_{i=1}^\infty$, such that $f = \sum_{i \in \mathbb{N}} \lambda_i m_i$ in L^2, where for all i, λ_i are real numbers satisfying

$$\sum_{i \in \mathbb{N}} |\lambda_i|^p < \infty.$$

Furthermore, define

$$\|f\|_{H_{\varepsilon,mol}^p} := \inf_{f = \sum_{i \in \mathbb{N}} \lambda_i m_i} \left(\sum_i |\lambda_i|^p\right)^{1/p}.$$

The molecular Hardy space $H_{\varepsilon,mol}^p$ is then defined as the completion of $\mathbb{H}_{\varepsilon,mol}^p$ with respect to the p-quasi-norm $\|\cdot\|_{H_{\varepsilon,mol}^p}^p$.

REMARK 1. (i) Using the theorem of completion of K. Yosida [44], we know that $\mathbb{H}_{ato}^p$ has a completion space H_{ato}^p, that is, for any $h \in H_{ato}^p$, there exists a Cauchy sequence $\{h_k\}_{k=1}^\infty$ in $\mathbb{H}_{ato}^p$ such that

$$\lim_{k\to\infty} \|h_k - h\|_{H_{ato}^p}^p = 0. \tag{3}$$

Moreover, if $\{h_k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{H}_{ato}^p$, then there exists a unique $h \in H_{ato}^p$ such that (3) holds true. Similarly, we have the conclusion for $\mathbb{H}_{\varepsilon,mol}^p$ and $H_{\varepsilon,mol}^p$.

(ii) Let $0 < p < 1$. We have the following inclusions:

$$\forall\, 0 < \varepsilon < \varepsilon', \;\; \mathbb{H}_{ato}^p \hookrightarrow \mathbb{H}_{\varepsilon',mol}^p \hookrightarrow \mathbb{H}_{\varepsilon,mol}^p.$$

In fact, the space $\mathbb{H}_{ato}^p$ corresponds to the space $\mathbb{H}_{\infty,mol}^p$.

DEFINITION 6. According to the atomic Hardy space H_{ato}^p, we define that a function $f \in H_{F,ato}^p$ if f admits a finite atomic decomposition. We equip this space with the norm

$$\|f\|_{H_{F,ato}^p} := \inf_{f=\sum_{i=1}^N \lambda_i m_i} \left(\sum_{i=1}^N |\lambda_i|^p \right)^{1/p},$$

where the infimum is taken over all the finite atomic decompositions. Similarly we define the space $H_{F,\varepsilon,mol}^p$ with the finite molecular decompositions.

REMARK 2. Since each molecule and each atom belongs to L^2, it is easy to see that these previous spaces are included into L^2 (not continuously). The space $H_{F,ato}^p$ is dense into H_{ato}^p and similarly for the molecular spaces.

At the end of this section, like [22], we give two useful properties about the atomic Hardy space H_{ato}^p and the molecular Hardy space $H_{\varepsilon,mol}^p$.

THEOREM 1. *Let* (X,d,μ) *be a space of homogeneous type and* $0 < p < 1$. *Then* $f \in H_{ato}^p$ *if and only if there exist* $(p,2)$-*atoms* $\{m_i\}_{i=1}^\infty$ *such that*

$$f = \sum_{i=1}^\infty \lambda_i m_i \;\; \text{in}\;\; H_{ato}^p, \tag{4}$$

and $\sum_{i=1}^\infty |\lambda_i|^p < \infty$. *Moreover,*

$$\|f\|_{H_{ato}^p}^p = \inf \left\{ \sum_{i=1}^\infty |\lambda_i|^p \right\},$$

where the infimum is taken over all possible decompositions of f *as in* (4).

Proof. First, we assume that $f \in H_{ato}^p$. Note that if (4) holds true, it is obvious that

$$\|f\|_{H_{ato}^p}^p \leqslant \inf\left\{ \sum_{i=1}^{\infty} |\lambda_i|^p \right\}, \tag{5}$$

where the infimum is taken over all possible decompositions of f as in (4). It remains to verify (4) and the reverse inequality of (5). For any $f \in H_{ato}^p$, we consider the following two cases.

Case (i) When $f \in \mathbb{H}_{ato}^p$. By Definition 4, there exists a sequence of $(p,2)$-atoms, $\{m_i\}_{i=1}^{\infty}$, such that $f = \sum_{i=1}^{\infty} \lambda_i m_i$ in L^2 and $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$.

Now we claim that (4) holds true.

Indeed, for every $M \in \mathbb{N}$, $f - \sum_{i=1}^{M} \lambda_i m_i = \sum_{i=M+1}^{\infty} \lambda_i m_i$ in L^2. Then we have that

$$\left\| f - \sum_{i=1}^{M} \lambda_i m_i \right\|_{H_{ato}^p}^p \leqslant \sum_{i=M+1}^{\infty} |\lambda_i|^p \to 0 \ \text{ as } \ M \to \infty.$$

Thus, the claim holds true. Again, by Definition 4 and (5), we get the desired result for *Case* (i).

Case (ii) When $f \in H_{ato}^p \setminus \mathbb{H}_{ato}^p$. Using (i) of Remark 1, there exists a Cauchy sequence $\{f_k\}_{k=1}^{\infty}$ in $\mathbb{H}_{ato}^p$ such that

$$\|f - f_k\|_{H_{ato}^p}^p \leqslant 2^{-k-2} \|f\|_{H_{ato}^p}^p.$$

It is easy to see that $f = \sum_{k=1}^{\infty} (f_k - f_{k-1})$ in H_{ato}^p, where we let $f_0 := 0$. Since $f_k - f_{k-1} \in \mathbb{H}_{ato}^p$ for all $k \in \mathbb{N}$, by Definition 4 and *Case* (i), we know that for any $\varepsilon \in (0, \infty)$ and any $k \in \mathbb{N}$, there exists a sequence of $(p,2)$-atoms $\{m_{k,i}\}_{i=1}^{\infty}$, such that

$$f_k - f_{k-1} = \sum_{i=1}^{\infty} \lambda_{k,i} m_{k,i} \ \text{ in } \ L^2 \ \text{ and } \ H_{ato}^p$$

and

$$\sum_{i=1}^{\infty} |\lambda_{k,i}|^p < \|f_k - f_{k-1}\|_{H_{ato}^p}^p + \frac{\varepsilon}{2^k}.$$

Using this and $f = \sum_{k=1}^{\infty} (f_k - f_{k-1})$ in H_{ato}^p, we further prove that

$$f = \sum_{k=1}^{\infty} (f_k - f_{k-1}) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \lambda_{k,i} m_{k,i} \ \text{ in } \ H_{ato}^p,$$

and

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |\lambda_{k,i}|^p \leqslant \sum_{k=1}^{\infty} \|f_k - f_{k-1}\|_{H_{ato}^p}^p + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k}$$

$$\leqslant \sum_{k=1}^{\infty} \left[\|f_k - f\|_{H_{ato}^p}^p + \|f_{k-1} - f\|_{H_{ato}^p}^p \right] + \varepsilon$$

$$\leqslant \sum_{k=1}^{\infty} 2^{-k} \|f\|_{H_{ato}^p}^p + \varepsilon = \|f\|_{H_{ato}^p}^p + \varepsilon,$$

which together with the arbitrariness of ε, completes the proof of *Case* (ii) and hence the necessity of Theorem 1.

Conversely, given $f = \sum_{i=1}^{\infty} \lambda_i m_i$ in H_{ato}^p and $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$. Then for any $k \in \mathbb{N}$, $f_k = \sum_{i=1}^{k} \lambda_i m_i \in \mathbb{H}_{ato}^p$ and $\lim_{k \to \infty} f_k = f$ in H_{ato}^p. Therefore, $f \in H_{ato}^p$, which completes the proof of the sufficiency and hence Theorem 1. $\square$

Similarly, we have

THEOREM 2. *Let (X, d, μ) be a space of homogeneous type, $0 < p < 1$ and $\varepsilon > 0$. Then $f \in H_{\varepsilon, mol}^p$ if and only if there exist $(p, 2, \varepsilon)$-molecules $\{m_i\}_{i=1}^{\infty}$ such that*

$$f = \sum_{i=1}^{\infty} \lambda_i m_i \ \text{in} \ H_{\varepsilon, mol}^p, \tag{6}$$

and $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$. Moreover,

$$\|f\|_{H_{\varepsilon, mol}^p}^p = \inf\left\{ \sum_{i=1}^{\infty} |\lambda_i|^p \right\},$$

where the infimum is taken over all possible decompositions of f as in (6).

3. Comparison with other Hardy spaces

In [6], F. Bernicot and J. Zhao gave the comparison between their Hardy spaces H^1 and other Hardy spaces. In this section, we will also study the comparison between our Hardy spaces $H^p (0 < p < 1)$ and some other Hardy spaces.

3.1. The space of Coifman-Weiss

If we choose B_Q as follows:

$$B_Q(f)(x) = f(x)\chi_Q(x) - |Q|^{-1}(\int_Q f)\chi_Q(x),$$

then the Hardy space $H_{CW}^p(\mathbb{R}^n)$ $(0 < p < 1)$ of Coifman-Weiss is obtained and our atoms are the same as the ones defined in [35]. However, since the B_Q satisfies that supp $B_Q(f) \subset Q$ for any f, so our molecule is different from the one in [35] and our atomic and molecular spaces are the same if B_Q satisfies this special property.

3.2. Hardy spaces for Schrödinger operators with non-negative polynomial potentials

Let $X = \mathbb{R}^n$ and V be a non-negative function on X. We consider the following Schrödinger operator

$$L(f)(x) := -\Delta f(x) + V(x)f(x),$$

where V is a non-negative polynomial [16, 17].

We all know that $-L$ generates a semigroup $(T_t)_{t>0}$, which is L^2-bounded and satisfies some gaussian estimates. J. Dziubański define a Hardy space H_L^p by a maximal operator as follows: a function f belongs to H_L^p if

$$\|f\|_{H_L^p} := \left\| \sup_{t>0} |T_t f(x)| \right\|_{L^p} < \infty, \ \ 0 < p < 1.$$

Denote

$$m(x,V) := \sum_{\beta} |D^{\beta} V(x)|^{1/(|\beta|+2)},$$

and $m(x,V) \geqslant C$, where C is a constant. In [16], J. Dziubański gives an atomic decomposition of this space with the following definition: a function a is an H_L^p-atom if there exists a ball $Q = B(y_0, r)$ with

$$\text{supp}(a) \subset Q, \ \ \|a\|_{L^2} \leqslant |Q|^{1/2-1/p}, \ \ r \leqslant m(y_0, V)^{-1},$$

and if $r \leqslant \frac{1}{4} m(y_0, V)^{-1}$, then

$$\int_Q a(x) dx = 0.$$

This definition of atoms is a special case of ours if we define B_Q for $Q = B(y_0, r)$ a ball by

$$B_Q(f)(x) := \begin{cases} f(x)\chi_Q(x), & \frac{1}{4} m(y_0, V)^{-1} < r \leqslant m(y_0, V)^{-1}, \\ f(x)\chi_Q(x) - |Q|^{-1}(\int_Q f)\chi_Q(x), & r \leqslant \frac{1}{4} m(y_0, V)^{-1}, \\ 0, & r > m(y_0, V)^{-1}, \end{cases}$$

With this choice we have

$$H_L^p = H_{ato}^p.$$

3.3. Hardy spaces associated to divergence form elliptic operators

Let $X = \mathbb{R}^n$ and A be an $n \times n$ matrix with entries

$$a_{jk} : \mathbb{R}^n \to \mathbb{C}, \ \ j = 1, \cdots, n, \ k = 1, \cdots, n,$$

satisfying the ellipticity condition

$$\lambda |\xi|^2 \leqslant \mathfrak{Re} A\xi \cdot \overline{\xi} \ \text{ and } \ |A\xi \cdot \overline{\zeta}| \leqslant \Lambda |\xi||\zeta|, \ \ \forall \xi, \zeta \in \mathbb{C}^n,$$

for some constants $0 < \lambda \leqslant \Lambda < \infty$.

Denote the second order divergence form operator by

$$Lf := -\text{div}(A\nabla f).$$

In [27], S. Hofmann, S. Mayboroda and A. McIntosh define the space H_L^p $(0 < p < 1)$ associated to this operator. For $f \in L^2(\mathbb{R}^n)$,

$$\|f\|_{H_L^p} := \left\| \left(\int \int_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \right\|_{L^p},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |x - y| < t\}$.

They also give a molecular decomposition with the following definition: let $0 < p < 1$, $\varepsilon > 0$ and $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$, a function $m \in L^2$ is an H_L^p-molecule if there exists a ball $Q \subset \mathbb{R}^n$ such that

$$\|(r_Q^{-2}L^{-1})^k m\|_{2,S_i(Q)} \leqslant 2^{-i\varepsilon}|2^{i+1}Q|^{\frac{1}{2}-\frac{1}{p}}, i = 0, 1, 2, \cdots, k = 0, 1, \cdots, M.$$

If we choose B_Q as follows:

$$B_Q(f) := (r_Q^2 L)^M e^{-r_Q^2 L}(f) \text{ or } B_Q(f) := (Id - (Id + r_Q^2 L)^{-1})^M(f),$$

then our $(p, 2, \varepsilon)$-molecules are H_L^p-molecules. Therefore, we have

$$H_{\varepsilon,mol}^p \hookrightarrow H_L^p.$$

Here we have no idea about whether the inverse inclusion relation is right between the spaces.

3.4. Hardy spaces associated to a general semigroup

In [13], X. T. Duong and J. Li defined a space H_L^p ($0 < p < 1$) with a linear operator L of type ω on $L^2(X)$ with $\omega \in (0, \pi/2)$. They assume that L generates a holomorphic semigroup e^{-zl} with $0 \leqslant |\text{Arg}(z)| < \pi/2 - \omega$, which has a bounded H_∞-calculus on $L^2(X)$ and Davies-Gaffney estimate for its kernel. They define a Hardy space H_L^p: for $f \in L^2(X)$,

$$\|f\|_{H_L^p(X)} := \left\|\left(\int\int_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t}\right)^{1/2}\right\|_{L^p(X)},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}^n \times (0,\infty) : |x - y| < t\}$.

They also give a molecular decomposition by the definition: let $0 < p < 1$, $\varepsilon > 0$ and $M > [\frac{n(2-p)}{4p}]$, a function $m \in L^2$ is an H_L^p-molecule if there exist a function $b \in \mathscr{D}(L^M)$ and a ball $Q \subset X$ such that

$$m = L^M(b),$$

and for every $j = 0, 1, 2, \cdots$ and $k = 0, 1, \cdots, M$,

$$\|(r_Q^2 L)^k b\|_{2,S_j(Q)} \leqslant r_Q^{2M} 2^{-j\varepsilon} V(2^j Q)^{\frac{1}{2}-\frac{1}{p}},$$

where $\mathscr{D}(T)$ represents the domain of an unbounded operator T.

If we choose B_Q as follows:

$$B_Q(f) := (r_Q^2 L)^M e^{-r_Q^2 L}(f) \text{ or } B_Q(f) := (Id - (Id + r_Q^2 L)^{-1})^M(f),$$

then our $(p, 2, \varepsilon)$-molecules are H_L^p-molecules. Therefore, we have

$$H_{\varepsilon,mol}^p \hookrightarrow H_L^p.$$

Here we have no idea about whether the inverse inclusion relation is right between the spaces.

4. Continuity theorem on the Hardy space

When $0 < p \leqslant 1$, it is well known that a Calderón-Zygmund operator is continuous from the Coifman-Weiss space H_{CW}^p to L^p. In [6], F. Bernicot and J. Zhao proposed some general conditions which guarantee the continuity from their Hardy spaces into L^1. In this section, we make some changes about the conditions, which also guarantee the continuity from our Hardy spaces into $L^p (0 < p < 1)$.

We have the two following results:

THEOREM 3. *Let $0 < p < 1$ and T be an L^2-bounded sublinear operator satisfying the following "off-diagonal" estimates: for all ball Q and all $j \geqslant 2$, there exist some coefficients $\alpha_j(Q)$ such that for all L^2-functions f supported in Q*

$$\left(\frac{1}{\mu(2^{j+1}Q)} \int_{S_j(Q)} |T(B_Q(f))|^2 d\mu \right)^{1/2} \leqslant \alpha_j(Q) \left(\frac{1}{\mu(Q)} \int_Q |f|^2 d\mu \right)^{1/2}.$$

If the coefficients $\alpha_j(Q)$ satisfy

$$\Lambda := \sup_{Q \text{ ball}} \sum_{j \geqslant 2} \frac{\mu(2^{j+1}Q)}{\mu(Q)} (\alpha_j(Q))^p < \infty,$$

then there exists a constant C such that

$$\forall f \in H_{ato}^p, \quad \|T(f)\|_{L^p} \leqslant C \|f\|_{H_{ato}^p}.$$

THEOREM 4. *Let $0 < p < 1$ and T be an L^2-bounded sublinear operator satisfying the following "off-diagonal" estimates: for all ball Q and all $k \geqslant 0, j \geqslant 2$, there exist some coefficients $\alpha_{j,k}(Q)$ such that for every L^2-function f supported in $S_k(Q)$*

$$\left(\frac{1}{\mu(2^{j+k+1}Q)} \int_{S_j(2^kQ)} |T(B_Q(f))|^2 d\mu \right)^{1/2} \leqslant \alpha_{j,k}(Q) \left(\frac{1}{\mu(2^{k+1}Q)} \int_{S_k(Q)} |f|^2 d\mu \right)^{1/2}.$$

$$(7)$$

If the coefficients $\alpha_{j,k}(Q)$ satisfy

$$\Lambda := \sup_{k \geqslant 0} \sup_{Q \text{ ball}} \left\{ \sum_{j \geqslant 2} \frac{\mu(2^{j+k+1}Q)}{\mu(2^{k+1}Q)} (\alpha_{j,k}(Q))^p \right\} < \infty,$$

then for all $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that

$$\forall f \in H_{\varepsilon,mol}^p, \quad \|T(f)\|_{L^p} \leqslant C \|f\|_{H_{\varepsilon,mol}^p}.$$

REMARK 3. Note that when $\varepsilon = \infty$, Theorem 4 becomes Theorem 3. So it suffices to prove Theorem 4.

Proof of Theorem 4. First, we prove that there exists a constant $C = C(\varepsilon)$ such that for all $(p, 2, \varepsilon)$-molecules m:

$$\left\| T(m) \right\|_{L^p}^p \leqslant C(\Lambda + \|T\|_{L^2 \to L^2}^p). \tag{8}$$

By Definition 1 we know that there exists a ball Q and a function f_Q such that

$$m = B_Q(f_Q).$$

By decomposing the space X with the scaled coronas around Q and using the linearity of B_Q, we have

$$m = B_Q(f_Q) = \sum_{k \geqslant 0} B_Q(\chi_{S_k(Q)} f_Q).$$

Using the sublinearity of T, we get that

$$\left| T(m) \right| \leqslant \sum_{k \geqslant 0} \left| T B_Q(\chi_{S_k(Q)} f_Q) \right|.$$

By decomposing the integral with the coronas $(S_j(2^k Q))_{j \geqslant 0}$ which is a partition of X, we obtain that

$$\left\| T(m) \right\|_{L^p}^p \leqslant \sum_{k \geqslant 0} \left\| T B_Q(\chi_{S_k(Q)} f_Q) \right\|_{L^p}^p \leqslant \sum_{\substack{k \geqslant 0 \\ j \geqslant 0}} \int_{S_j(2^k Q)} \left| T\left(B_Q(\chi_{S_k(Q)} f_Q) \right) \right|^p d\mu$$

$$\leqslant \sum_{\substack{k \geqslant 0 \\ j \geqslant 0}} \mu(2^{k+j+1} Q) \frac{1}{\mu(2^{k+j+1} Q)} \int_{S_j(2^k Q)} \left| T\left(B_Q(\chi_{S_k(Q)} f_Q) \right) \right|^p d\mu$$

$$\leqslant \sum_{\substack{k \geqslant 0 \\ j \geqslant 0}} \mu(2^{k+j+1} Q) \frac{1}{\mu(2^{k+j+1} Q)} \left(\int_{S_j(2^k Q)} \left| T\left(B_Q(\chi_{S_k(Q)} f_Q) \right) \right|^{p \cdot (2/p)} d\mu \right)^{p/2}$$

$$\times \mu(2^{k+j+1} Q)^{1-p/2}$$

$$\leqslant \sum_{\substack{k \geqslant 0 \\ j \geqslant 0}} \mu(2^{k+j+1} Q) \left(\frac{1}{\mu(2^{k+j+1} Q)} \int_{S_j(2^k Q)} \left| T\left(B_Q(\chi_{S_k(Q)} f_Q) \right) \right|^2 d\mu \right)^{p/2}.$$

Using the "off-diagonal" estimates (7) on T and the doubling condition (1) about the measure μ (for the parts $j \leqslant 1$), we have

$$\left\| T(m) \right\|_{L^p}^p$$

$$\leqslant \sum_{\substack{k \geqslant 0 \\ j \geqslant 2}} \mu(2^{k+1} Q) \frac{\mu(2^{k+j+1} Q)}{\mu(2^{k+1} Q)} \left(\alpha_{j,k}(Q) \right)^p \left(\frac{1}{\mu(2^{k+1} Q)} \int_{2^{k+1} Q} \left| \chi_{S_k(Q)} f_Q \right|^2 d\mu \right)^{p/2}$$

$$+ \sum_{\substack{k \geqslant 0 \\ j \leqslant 1}} A 2^{j\delta} \mu(2^{k+1} Q) \left(\frac{1}{\mu(2^{k+1} Q)} \int_X \left| T\left(B_Q(\chi_{S_k(Q)} f_Q) \right) \right|^2 d\mu \right)^{p/2}$$

$$\leqslant \sum_{\substack{k\geqslant 0 \\ j\geqslant 2}} \mu(2^{k+1}Q)\frac{\mu(2^{k+j+1}Q)}{\mu(2^{k+1}Q)}(\alpha_{j,k}(Q))^p \mu(2^{k+1}Q)^{-p/2}\|f_Q\|^p_{2,S_k(Q)}$$

$$+ \sum_{\substack{k\geqslant 0 \\ j\leqslant 1}} A 2^{j\delta}\mu(2^{k+1}Q)^{1-(p/2)}\|TB_Q\|^p_{L^2\to L^2}\|f_Q\|^p_{2,S_k(Q)}$$

Using (2) and the L^2-decay on f_Q, we obtain that

$$\|T(m)\|^p_{L^p}$$

$$\leqslant \sum_{\substack{k\geqslant 0 \\ j\geqslant 2}} \mu(2^{k+1}Q)\frac{\mu(2^{k+j+1}Q)}{\mu(2^{k+1}Q)}(\alpha_{j,k}(Q))^p \mu(2^{k+1}Q)^{-p/2}\mu(2^{k+1}Q)^{p/2-1}2^{-p\varepsilon k}$$

$$+ A'\|T\|^p_{L^2\to L^2}\sum_{\substack{k\geqslant 0 \\ j\leqslant 1}} 2^{-p\varepsilon k+j\delta}$$

$$\lesssim \sum_{k\geqslant 0} 2^{-p\varepsilon k}\left(\sum_{j\geqslant 2}\frac{\mu(2^{k+j+1}Q)}{\mu(2^{k+1}Q)}(\alpha_{j,k}(Q))^p + 2^{\delta+1}\|T\|^p_{L^2\to L^2}\right)$$

$$\lesssim \Lambda + \|T\|^p_{L^2\to L^2}.$$

We have that (8) holds true.

For any $f\in\mathbb{H}^p_{\varepsilon,mol}$, there exists a sequence of $(p,2,\varepsilon)$-molecules $\{m_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty \lambda_i m_i$ in L^2 and

$$\sum_{i=1}^\infty |\lambda_i|^p \sim \|f\|^p_{H^p_{\varepsilon,mol}}.$$

Since T is L^2-bounded, we have that, for any $N\in\mathbb{N}$,

$$\left\|\sum_{i=1}^N T(\lambda_i m_i) - Tf\right\|_{L^2} = \left\|T\left(\sum_{i=1}^N \lambda_i m_i - f\right)\right\|_{L^2} \lesssim \left\|\sum_{i=1}^N \lambda_i m_i - f\right\|_{L^2} \to 0, \ N\to\infty.$$

Moreover, for all $\eta\in(0,\infty)$,

$$\mu\left(\left\{x\in X : \left|\sum_{i=1}^N T(\lambda_i m_i)(x) - Tf(x)\right| > \eta\right\}\right) \to 0, \ N\to\infty.$$

According to the Riesz theorem, we get that there exists a subsequence $\left\{\sum_{i=1}^{N_k} T(\lambda_i m_i)\right\}_k$, such that

$$Tf = \lim_{k\to\infty}\sum_{i=1}^{N_k} T(\lambda_i m_i) \ \mu - a.e. \text{ on } X,$$

which together with the Fatou lemma and (8), implies that

$$\|Tf\|_{L^p}^p \leqslant \liminf_{k\to\infty} \int_X \sum_{i=1}^{N_k} |T(\lambda_i m_i)|^p d\mu$$

$$\leqslant \sum_{i=1}^{\infty} |\lambda_i|^p \|T(m_i)\|_{L^p}^p \lesssim \sum_{i=1}^{\infty} |\lambda_i|^p (\Lambda + \|T\|_{L^2\to L^2}^p)$$

$$\lesssim \Lambda + \|T\|_{L^2\to L^2}^p.$$

Moreover, using the density argument and the method similar to the *Case* (ii) of Theorem 1, we extend T to be a bounded linear operator from $H_{\varepsilon,mol}^p$ to L^p. $\square$

EXAMPLE 1. If we choose B_Q as follows:

$$B_Q(f)(x) = f(x)\chi_Q(x) - |Q|^{-1}(\int_Q f)\chi_Q(x),$$

then our atoms are the same as the ones defined in [35].

Thus, we set

$$B_Q(f)(x) = f(x)\chi_Q(x) - \left(\mu(Q)^{-1}\int_Q f d\mu\right)\chi_Q(x),$$

and let T be an L^2-bounded operator. If we suppose that the kernel $K(x,z)$ of T satisfies the Calderón-Zygmund condition: there exists $\theta > 0$ such that for all $x \neq z$ and $z' \in B(z, d(x,z)/2)$

$$|K(x,z)| \lesssim \frac{1}{\mu(B(x,d(x,z)))},$$

$$|K(x,z) - K(x,z')| \lesssim \frac{d(z,z')^\theta}{\mu(B(x,d(x,z)))d(x,z)^\theta},$$

then we have that T satisfies the assumptions in Theorem 3 for $2/3 < p < 1$. Therefore, T is continuous from $H_{CW}^p = H_{ato}^p$ into L^p ($2/3 < p < 1$). The following is the process of proof.

Proof. First, we have that

$$\left(\frac{1}{\mu(2^{j+1}Q)}\int_{S_j(Q)} |T(B_Q(f)(x))|^2 d\mu(x)\right)^{1/2}$$

$$= \left(\frac{1}{\mu(2^{j+1}Q)}\int_{S_j(Q)} |\int_Q [k(x,z) - k(x,z')](B_Q(f)(z))dz|^2 d\mu(x)\right)^{1/2}$$

$$\leqslant \left(\frac{1}{\mu(2^{j+1}Q)}\int_{S_j(Q)} [\int_Q |k(x,z) - k(x,z')||B_Q(f)(z)|dz]^2 d\mu(x)\right)^{1/2}$$

$$\lesssim \left(\frac{1}{\mu(2^{j+1}Q)}\int_{S_j(Q)} [\int_Q \frac{d(z,z')^\theta}{\mu(B(x,d(x,z)))d(x,z)^\theta}|B_Q(f)(z)|dz]^2 d\mu(x)\right)^{1/2}$$

$$\lesssim \left(\frac{1}{\mu(2^{j+1}Q)}\int_{S_j(Q)}\Big[\int_Q \frac{1}{\mu(B(x,d(x,z)))}|B_Q(f)(z)|dz\Big]^2 d\mu(x)\right)^{1/2}$$

$$\leqslant \frac{1}{\mu(2^{j+1}Q)^{1/2}}\int_Q \left(\int_{S_j(Q)}\frac{1}{\mu\big(B(x,d(x,z))\big)^2}|B_Q(f)(z)|^2 d\mu(x)\right)^{1/2} dz$$

$$\leqslant \frac{1}{\mu(2^{j+1}Q)^{1/2}}\frac{1}{\mu(2^{j+1}Q)}\int_Q \left(\int_{S_j(Q)}|B_Q(f)(z)|^2 d\mu(x)\right)^{1/2} dz$$

$$\leqslant \frac{1}{\mu(2^{j+1}Q)^{3/2}}\|f_Q\|_{2,Q}\mu(Q) = \frac{\mu(Q)}{\mu(2^{j+1}Q)^{3/2}}\left(\int_Q |f|^2 d\mu(x)\right)^{1/2}$$

$$= \frac{\mu(Q)^{3/2}}{\mu(2^{j+1}Q)^{3/2}}\left(\frac{1}{\mu(Q)}\int_Q |f|^2 d\mu(x)\right)^{1/2}.$$

Let $\alpha_j(Q) = \frac{\mu(Q)^{3/2}}{\mu(2^{j+1}Q)^{3/2}}$, then we get that

$$\sum_{j\geqslant 2}\frac{\mu(2^{j+1}Q)}{\mu(Q)}(\alpha_j(Q))^p = \sum_{j\geqslant 2}\frac{\mu(2^{j+1}Q)}{\mu(Q)}\left(\frac{\mu(Q)^{3/2}}{\mu(2^{j+1}Q)^{3/2}}\right)^p$$

$$= \sum_{j\geqslant 2}\left(\frac{\mu(2^{j+1}Q)}{\mu(Q)}\right)^{1-\frac{3}{2}p} \lesssim \sum_{j\geqslant 2}\left(2^{j\delta}\right)^{1-\frac{3}{2}p}.$$

Therefore, when $2/3 < p < 1$, we obtain that T is continuous from H_{ato}^p into L^p. $\quad\square$

Now we establish the $(H_{ato,\mathbb{B}}^p, H_{\varepsilon',mol,T\mathbb{B}}^p)$-boundedness of L^2-bounded operator T, where $\varepsilon' > 0, \mathbb{B} := (B_Q)_{Q\in\mathscr{Q}}$, and $T\mathbb{B} := (TB_Q)_{Q\in\mathscr{Q}}$.

THEOREM 5. *Let $0 < p < 1$ and T be an L^2-bounded sublinear operator. Assume that $\varepsilon' > 0$, then T is bounded from $H_{ato,\mathbb{B}}^p$ into $H_{\varepsilon',mol,T\mathbb{B}}^p$.*

Proof. We claim that if $m = B_Q f_Q$ is a $(p,2)$ atom of $H_{ato,\mathbb{B}}^p$, then Tm is a $(p,2,\varepsilon')$-molecule of $H_{\varepsilon',mol,T\mathbb{B}}^p$ and

$$\big\|T(m)\big\|_{H_{\varepsilon',mol,T\mathbb{B}}^p} = 1. \tag{9}$$

Indeed, suppose that (9) holds true. For each $f \in \mathbb{H}_{ato,\mathbb{B}}^p$, there exists a sequence of $(p,2)$-atoms $\{m_i\}_{i=1}^{\infty}$ and real numbers $\{\lambda_i\}_{i=1}^{\infty}$ such that $f = \sum_{i=1}^{\infty}\lambda_i m_i$ in L^2 with

$$\|f\|_{H_{ato,\mathbb{B}}^p}^p \sim \sum_{i=1}^{\infty}|\lambda_i|^p.$$

Since T is bounded on L^2, we have that

$$\left\|\sum_{i=1}^N T(\lambda_i m_i) - Tf\right\|_{L^2} = \left\|T\Big(\sum_{i=1}^N \lambda_i m_i - f\Big)\right\|_{L^2} \lesssim \left\|\sum_{i=1}^N \lambda_i m_i - f\right\|_{L^2} \to 0, \ N\to\infty.$$

Therefore, $Tf = \sum_{i=1}^{N} T(\lambda_i m_i)$ in L^2. And also we know that

$$\|Tf\|_{H^p_{\varepsilon',mol,T\mathbb{B}}}^p \leqslant \sum_{i=1}^{\infty} \|T(\lambda_i m_i)\|_{H^p_{\varepsilon',mol,T\mathbb{B}}}^p \lesssim \sum_{i=1}^{\infty} |\lambda_i|^p \|Tm_i\|_{H^p_{\varepsilon',mol,T\mathbb{B}}}^p$$

$$\lesssim \sum_{i=1}^{\infty} |\lambda_i|^p \sim \|f\|_{H^p_{ato,\mathbb{B}}}^p.$$

Moreover, using a density argument and the method similar to the *Case* (ii) of Theorem 1, we extend T to be a bounded linear operator from $H^p_{ato,\mathbb{B}}$ to $H^p_{\varepsilon',mol,T\mathbb{B}}$.

Now we prove the claim. First, Since T, B_Q and f_Q are all L^2 bounded, so we have that Tm is also L^2 bounded.

Furthermore, we know that $\mathrm{supp} f_Q \subset Q$, thus

$$\|f_Q\|_{2,S_i(Q)} \leqslant \mu(2^i Q)^{\frac{1}{2}-\frac{1}{p}} 2^{-\varepsilon' i}.$$

As a consequence, we get that Tm is a $(p,2,\varepsilon')$-molecule of $H^p_{\varepsilon',mol,T\mathbb{B}}$. By the definition of $\|\cdot\|_{H^p_{\varepsilon,mol}}$, we have that

$$\|T(m)\|_{H^p_{\varepsilon',mol,T\mathbb{B}}} = 1. \quad \square$$

Next, in order to discuss the embedding of our Hardy spaces into L^p, additionally, we assume that $\mathbb{B} = (B_Q)_Q$ satisfies some decay estimates: for M'' a large enough exponent, there exists a constant C such that

$$\forall i \geqslant 0, \forall k \geqslant 0, \forall f \in L^2, \mathrm{supp}(f) \subset 2^i Q, \|B_Q(f)\|_{2,S_k(2^i Q)} \leqslant C 2^{-M'' k} \|f\|_{2,2^i Q}. \quad (10)$$

We have the following embedding property.

THEOREM 6. *Let $0 < p < 1$, then we have the following inclusions:*

$$\forall \varepsilon > 0, H^p_{ato} \hookrightarrow H^p_{\varepsilon,mol} \hookrightarrow L^p.$$

Proof. First, we show that all $(p,2,\varepsilon)$-molecules (and atoms) are bounded in L^p. Using (10), we have

$$\|B_Q(f_Q)\|_{L^p}^p \leqslant \sum_{j \geqslant 0} \|B_Q(f_Q \chi_{S_j(Q)})\|_{L^p}^p \leqslant \sum_{j \geqslant 0} \sum_{k \geqslant 0} \|B_Q(f_Q \chi_{S_j(Q)})\|_{L^p,S_k(2^j Q)}^p$$

$$\lesssim \sum_{j \geqslant 0} \sum_{k \geqslant 0} \left(\int_{S_k(2^j Q)} |B_Q(f_Q \chi_{S_j(Q)})|^{p\cdot(2/p)} d\mu \right)^{p/2} \mu(2^{j+k}Q)^{1-p/2}$$

$$\lesssim \sum_{j \geqslant 0} \sum_{k \geqslant 0} \mu(2^{j+k}Q)^{1-p/2} 2^{-M'' kp} \|f_Q\|_{2,S_j(Q)}^p$$

$$\lesssim \sum_{j \geqslant 0} \sum_{k \geqslant 0} 2^{k\delta(1-p/2)} 2^{-M'' kp} 2^{-\varepsilon jp} \lesssim 1.$$

In the above we use the doubling property of μ and the fact that M'' is large enough ($M'' > \delta/p$ works). Thus, we get that all $(p,2,\varepsilon)$-molecules are bounded in L^p, and we can conclude the embedding property by using the definition of the Hardy spaces. $\square$

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