Mathematical
Inequalities
& Applications
Volume 25, Number 2 (2022), 535–549 doi:10.7153/mia-2022-25-33

COMPACTNESS OF THE TWO–DIMENSIONAL RECTANGULAR HARDY OPERATOR

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(*Communicated by L. E. Persson*)

Abstract. Criteria in terms of weights v and w are given for the compactness of the two–dimensional Hardy operator I_2 from Lebesgue space $L_v^p(\mathbb{R}_+^2)$ to $L_w^q(\mathbb{R}_+^2)$ for $1<p\leqslant q<\infty$. A two–sided estimate is found for the measure of non–compactness of $I_2 : L_v^p(\mathbb{R}_+^2)\to L_w^q(\mathbb{R}_+^2)$ for the same case of summation parameters p and q. The situation when $q<p$ is also discussed.

1. Introduction

Let $1<p,q<\infty$ and two non–negative on $\mathbb{R}_+^2 := (0,\infty)^2$ functions v and w (weights) be fixed. Consider an integral Hardy operator of the form

$$(I_2 f)(x,y):\ = \int_0^x \int_0^y f(s,t)\,ds dt, \qquad (x,y)\in\mathbb{R}_+^2, \tag{1}$$

acting from weighted Lebesgue space $L_v^p(\mathbb{R}_+^2)$ to analogous function space $L_w^q(\mathbb{R}_+^2)$. The space $L_v^p(\mathbb{R}_+^2)$ consists of all Lebesgue measurable functions f on $\mathbb{R}_+^2$ satisfying $\|f\|_{p,v}^p := \int_{\mathbb{R}_+^2} |f|^p v<\infty$. The conjugate (or dual) to I_2 operator I_2^* has the form

$$(I_2^* f)(x,y):\ = \int_x^\infty \int_y^\infty f(s,t)\,ds dt, \qquad (x,y)\in\mathbb{R}_+^2.$$

In this work we study the compactness property of $I_2 \colon L_v^p(\mathbb{R}_+^2)\to L_w^q(\mathbb{R}_+^2)$.

The boundedness of the operator I_2 in Lebesgue spaces was investigated in [8, 14, 16, 17, 20]. In particular, E. Sawyer in [17] obtained a criterion for the inequality

$$\|(I_2 f)\|_{q,w} \leqslant C\|f\|_{p,v} \tag{2}$$

to hold in the case $p\leqslant q$ for all $f\geqslant 0$ with a constant $C>0$ independent of f.

Mathematics subject classification (2020): 47G10, 47B38.
Keywords and phrases: Two–dimensional Hardy operator, compactness, Lebesgue space.
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THEOREM 1. (E. Sawyer [17, Theorem 1A]) *Let $1 < p \leqslant q < \infty$ and assume v, w are given weights. Put $p' := p/(p-1)$ and $\sigma := v^{1-p'}$. The inequality (2) holds if and only if*

$$A_1 := \sup_{(s,t)\in\mathbb{R}^2_+} A_1\big[(s,t);\sigma,w\big] := \sup_{(s,t)\in\mathbb{R}^2_+} \big[I_2^* w(s,t)\big]^{\frac{1}{q}} \big[I_2\sigma(s,t)\big]^{\frac{1}{p'}} < \infty, \tag{3}$$

$$A_2 := \sup_{(s,t)\in\mathbb{R}^2_+} A_2\big[(s,t);\sigma,w\big] := \sup_{(s,t)\in\mathbb{R}^2_+} \left(\int_0^s \int_0^t (I_2\sigma)^q w\right)^{\frac{1}{q}} \big[I_2\sigma(s,t)\big]^{-\frac{1}{p}} < \infty, \tag{4}$$

$$A_3 := \sup_{(s,t)\in\mathbb{R}^2_+} A_3\big[(s,t);\sigma,w\big] := \sup_{(s,t)\in\mathbb{R}^2_+} \left(\int_s^\infty \int_t^\infty (I_2^* w)^{p'} \sigma\right)^{\frac{1}{p'}} \big[I_2^* w(s,t)\big]^{-\frac{1}{q'}} < \infty. \tag{5}$$

Moreover, the best constant $C > 0$ in the inequality (2), which coincides with the norm $\|I_2\|_{L^p_v(\mathbb{R}^2_+)\to L^q_w(\mathbb{R}^2_+)} := \sup_{f\in L^p_v(\mathbb{R}^2_+)} \frac{\|I_2 f\|_{q,w}}{\|f\|_{p,v}}$ of the operator I_2, is equivalent to $A_1 + A_2 + A_3$ with equivalence constants depending of summation parameters p and q only.

The main feature of the results by E. Sawyer is the absence of any constraint condition on weights v and w, except the ones necessarily following from the finiteness of the functionals A_1, A_2 and A_3 given by (3)–(5). On the contrary, in papers [16, 20] one of the two weight functions must be factorizable, that is, representable as a product of one–dimensional functions. But, in contrast to E. Sawyer's theorem, the results from [16, 20] can be generalized to $n-$dimensional Hardy operators for all natural $n > 2$.

The one–dimensional analog of the condition (3) is the finiteness of the Muckenhoupt constant (see [15], [6] and monographs [10, 11]). Characteristics (4) and (5) are two–dimensional generalizations of the Tomaselli functional [19, definition (11)] in its direct and dual forms. Observe that boundedness conditions for the operator $(I_1 f)(x) = \int_0^x f(t)\,dt$, which correspond to the characteristics (3)–(5), are equivalent to each other in one–dimensional weighted Lebesgue spaces, that is $C \approx A_1 \approx A_2 \approx A_3$ (in one–dimensional interpretation) (see [5]). In the two–dimensional case this, generally speaking, is not true. Moreover, for $p = q$ there is no pair of constants amongst A_i, $i = 1,2,3$, to be sufficient for the validity of the inequality (2) (see examples in [17, § 4]). However, an important refinement of Sawyer's theorem was obtained in the recent paper [18] for the case $1 < p < q < \infty$. More precisely, Theorem 2 below stands that E. Sawyer's result is actual for $p = q$ only, while for p strictly less than q the finiteness of the only functional $A := A_1$ is necessary and sufficient for the boundedness of the operator I_2 from $L^p_v(\mathbb{R}^2_+)$ to $L^q_w(\mathbb{R}^2_+)$.

THEOREM 2. ([18, Theorem 2]) *Let $1 < p < q < \infty$. Denote $\alpha := \alpha(p,q) := \frac{p^2(q-1)}{q-p}$, $\alpha' = \alpha(q',p')$ and*

$$\mathbb{C}_{\alpha,\alpha'} := 3^{3q}\left[\left(\frac{2^4}{3}\right)^q \max\left\{\alpha, 2q(q')^{\frac{q}{p'}}\right\}\left(\frac{2^{p-1}}{2^{p-1}-1}\right)^{\frac{q}{p}} + 3^{\frac{1}{p}+\frac{1}{q'}}(\alpha')^{\frac{1}{p'}}\right].$$

The inequality (2) holds if and only if $A < \infty$. Besides, the following estimate is true *for the best constant C in* (2):

$$A \leqslant C \leqslant \mathbb{C}_{\alpha,\alpha'} A.$$

In [18], a connection was also established between the characterization of the inequality (2) in the case $p<q$ by three constants, as in Theorem 1, with an equivalent characteristic with the only constant A, as in Theorem 2:

$$A_1 \leqslant C_2 \leqslant \mathbb{C}_{1,1}\left[A_1+A_2+A_3\right] \leqslant \mathbb{C}_{1,1}\left[1+\alpha(p,q)^{\frac{1}{q}}+\alpha(q',p')^{\frac{1}{p'}}\right]A_1,$$

besides,

$$\lim_{p\uparrow q}\left[\alpha(p,q)+\alpha(q',p')\right]=\infty.$$

According to the information available to the authors, the study of the compactness of the two–dimensional rectangular integration operator I_2 of the form (1) has not been carried out before. In § 2 of this paper, the compactness criteria for $I_2 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ in the case $p<q$ are obtained. § 3 complements the work with the case $p=q$. Based on the results of [18] concerning the fulfillment of the inequality (2) for $q<p$, we give in § 4 a necessary condition and a sufficient condition for the compactness of the operator I_2 for $1<q<p$. A two–sided estimate for the measure of non-compactness for $I_2 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ is contained in § 5.

Throughout the work, relations of the form $\Phi \lesssim \Psi$ mean the fulfillment of inequalities $\Phi \leqslant c\Psi$ with $c>0$ independent of Φ and Ψ. We write $\Phi \approx \Psi$ in the case of $\Phi \lesssim \Psi \lesssim \Phi$. The symbols $:=$ and $=:$ are used to define new values. The notation $\langle F,G\rangle$ means an integral of the form $\int_{\mathbb{R}^2_+} F(x,y)G(x,y)\,dxdy$, symbol $\bigsqcup$ is used for disjoint union of sets.

2. A general scheme and the main result

Let $p<q$ and the operator I_2 is bounded from $L^p_v(\mathbb{R}^2_+)$ to $L^q_w(\mathbb{R}^2_+)$. Then $A<\infty$ by Theorem 2, from which it follows that $\sigma \in L^1\big((0,x)\times(0,y)\big)$ and $w \in L^1\big((x,\infty)\times(y,\infty)\big)$ for all $(x,y)\in\mathbb{R}^2_+$. Without loss of generality we assume that pre–images of the operators I_2 and I_2^* are non–negative functions.

Let $a,b,c,d \in (0,\infty)$, where $a<c$ and $b<d$. Put

$$\mathbf{\Omega}_0 := \big\{(0,a)\times(0,\infty)\big\} \sqcup \big\{[a,\infty)\times(0,b)\big\},$$

$$\mathbf{\Omega}_\infty := \big\{[a,\infty)\times(d,\infty)\big\} \sqcup \big\{(c,\infty)\times(b,d]\big\} =: W_1 \sqcup W_2$$

and

$$\mathbf{\Omega} := [a,c]\times[b,d].$$

Then $\mathbb{R}^2_+ = \mathbf{\Omega}_0 \sqcup \mathbf{\Omega} \sqcup \mathbf{\Omega}_\infty$.

Further, introduce the projectors $P_{\mathbf{\Omega}_0} f := \chi_{\mathbf{\Omega}_0} f$, $P_{\mathbf{\Omega}} f := \chi_{\mathbf{\Omega}} f$, $P_{\mathbf{\Omega}_\infty} f := \chi_{\mathbf{\Omega}_\infty} f$. Notice that the operator of the form $P_{\mathbf{\Omega}} I_2 P_{\mathbf{\Omega}} : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ coincides with $I_2 : L^p_v(\mathbf{\Omega}) \hookrightarrow L^q_w(\mathbf{\Omega})$ and is compact (see Remark after Theorem 2 in [7, Chapter XI,

§ 3.2]). This follows from the fact that a kernel $k_{\mathbf{\Omega}}$ of the mapping $P_{\mathbf{\Omega}} I_2 P_{\mathbf{\Omega}}$ of the form

$$k_{\mathbf{\Omega}}(x,y;s,t) = \chi_{\mathbf{\Omega}}(x,y) w^{\frac{1}{q}}(x,y) v^{-\frac{1}{p}}(s,t) \chi_{(a,x)\times(b,y)}(s,t)$$

satisfies the condition

$$\mathbb{K} := \left(\int_{\mathbb{R}^2_+} \left[\int_{\mathbb{R}^2_+} |k_{\mathbf{\Omega}}(x,y;s,t)|^{p'} dsdt \right]^{\frac{q}{p'}} dxdy \right)^{\frac{1}{q}} < \infty,$$

since

$$\mathbb{K} = \left(\int_a^c \int_b^d w(x,y) \left[\int_a^x \int_b^y \sigma(s,t)\, dsdt \right]^{\frac{q}{p'}} dxdy \right)^{\frac{1}{q}} \leqslant \left[I_2^* w(a,b) \right]^{\frac{1}{q}} \left[I_2 \sigma(c,d) \right]^{\frac{1}{p'}} < \infty.$$

In what follows we shall use a scheme for approximation of the I_2, in its operator norm, by compact operators (see e.g. [12]). To this end, we write

$$I_2 - P_{\mathbf{\Omega}} I_2 P_{\mathbf{\Omega}} = P_{\mathbf{\Omega_0}} I_2 P_{\mathbf{\Omega_0}} + P_{\mathbf{\Omega}} I_2 P_{\mathbf{\Omega_0}} + P_{\mathbf{\Omega_\infty}} I_2 P_{\mathbf{\Omega_0}} + P_{\mathbf{\Omega_\infty}} I_2 P_{\mathbf{\Omega}} + P_{\mathbf{\Omega_\infty}} I_2 P_{\mathbf{\Omega_\infty}}. \qquad (6)$$

Consider rectangles

$$W = [\boldsymbol{a}, \boldsymbol{b}] \times [\boldsymbol{c}, \boldsymbol{d}] \quad \text{and} \quad V = (\alpha, \beta) \times (\gamma, \delta)$$

in $\mathbb{R}^2_+$, and let

$$k_{W,V}(x,y;s,t) = \chi_W(x,y) w^{\frac{1}{q}}(x,y) v^{-\frac{1}{p}}(s,t) \chi_V(s,t).$$

Since

$$\mathbb{K}_{W,V} := \left(\int_{\mathbb{R}^2_+} \left[\int_{\mathbb{R}^2_+} |k_{W,V}(x,y;s,t)|^{p'} dsdt \right]^{\frac{q}{p'}} dxdy \right)^{\frac{1}{q}} < \infty,$$

then the operator $P_W I_2 P_V : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ is compact. Indeed, the kernel $k_{W,V}$ of the operator $P_W I_2 P_V$ in its precise form is given by

$$k_{W,V}(x,y;s,t) = \chi_W(x,y) w^{\frac{1}{q}}(x,y) v^{-\frac{1}{p}}(s,t) \chi_{V \cap [(0,x)\times(0,y)]}(s,t).$$

In such a case

$$\mathbb{K}^q_{W,V} = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \int_{\boldsymbol{c}}^{\boldsymbol{d}} w(x,y) \left(\int_\alpha^x \int_\gamma^y \sigma \right)^{\frac{q}{p'}} dxdy < \infty$$

if

$$\boldsymbol{a} \cdot \boldsymbol{c} > 0, \min(\boldsymbol{b}, \beta) \cdot \min(\boldsymbol{d}, \delta) < \infty, \qquad (7)$$

and the compactness of $P_W I_2 P_V : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ follows, for example, from [13, Theorem 7.3]. We have for the second term in (6) that

$$P_{\mathbf{\Omega}} I_2 P_{\mathbf{\Omega_0}} = P_{[a,c]\times[b,d]} I_2 P_{(0,a)\times(0,\infty)} + P_{[a,c]\times[b,d]} I_2 P_{[a,\infty)\times(0,b)}, \qquad (8)$$

and we see that the conditions (7) are satisfied for all the operators on the right hand side of (8). Therefore, $P_{\mathbf{\Omega}} I_2 P_{\mathbf{\Omega_0}} : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ is compact. Analogously one can confirm the compactness of $P_{\mathbf{\Omega}_\infty} I_2 P_{\mathbf{\Omega}}$. For the study of $P_{\mathbf{\Omega}_\infty} I_2 P_{\mathbf{\Omega}_0}$ we use the representation

$$\mathbf{\Omega}_0 = \bigsqcup_{j=1}^{4} V_j$$

with

$$V_1 := (0,a)\times(0,d),\, V_2 := (a,c)\times(0,b),\, V_3 := (0,a)\times(d,\infty),\, V_4 := (c,\infty)\times(0,b).$$

Then

$$P_{\mathbf{\Omega}_\infty} I_2 P_{\mathbf{\Omega_0}} = P_{W_1} I_2 (P_{V_1} + P_{V_2}) + P_{W_2} I_2 (P_{V_1} + P_{V_2} + P_{V_3}) + P_{W_1} I_2 (P_{V_3} + P_{V_4}) + P_{W_2} I_2 P_{V_4}. \quad (9)$$

It can be easily checked by applying the criterion (7) that the first two terms on the right hand side of (9) are compact. Thus,

$$I_2 - P_{\mathbf{\Omega}} I_2 P_{\mathbf{\Omega}} - P_{\mathbf{\Omega}} I_2 P_{\mathbf{\Omega_0}} - P_{\mathbf{\Omega_\infty}} I_2 P_{\mathbf{\Omega}} - P_{W_1} I_2 (P_{V_1} + P_{V_2}) - P_{W_2} I_2 (P_{V_1} + P_{V_2} + P_{V_3}) = P_{\mathbf{\Omega_0}} I_2 P_{\mathbf{\Omega_0}} + P_{W_1} I_2 (P_{V_3} + P_{V_4}) + P_{W_2} I_2 P_{V_4} + P_{\mathbf{\Omega_\infty}} I_2 P_{\mathbf{\Omega_\infty}}. \quad (10)$$

Denote

$$\|T\| := \|T\|_{L^p_v(\mathbb{R}^2_+)\to L^q_w(\mathbb{R}^2_+)}.$$

By Theorem 2,

$$\|P_{\mathbf{\Omega_0}} I_2 P_{\mathbf{\Omega_0}}\| \leqslant \mathbb{C}_{\alpha,\alpha'} \sup_{(u,z)\in\mathbb{R}^2_+} A_1\big[(u,z);\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}\big] \leqslant \mathbb{C}_{\alpha,\alpha'} \sup_{(u,z)\in\mathbf{\Omega}_0} A_1\big[(u,z);\sigma,w\big]. \quad (11)$$

An additional condition of the form

$$\lim_{\mathbf{\Omega}_0\downarrow\varnothing} \sup_{(u,z)\in\mathbf{\Omega}_0} A_1\big[(u,z);\sigma,w\big] = 0 \quad (12)$$

reduces the norm $\|P_{\mathbf{\Omega}_0} I_2 P_{\mathbf{\Omega}_0}\|$ to infinitesimal value. Analogously, since $I_2(\sigma\chi_{\mathbf{\Omega_\infty}})(u,z) = 0$ for all $(u,z) \notin \mathbf{\Omega_\infty}$, then

$$\|P_{\mathbf{\Omega_\infty}} I_2 P_{\mathbf{\Omega_\infty}}\| \leqslant \mathbb{C}_{\alpha,\alpha'} \sup_{(u,z)\in\mathbb{R}^2_+} A_1\big[(u,z);\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}\big] \leqslant \mathbb{C}_{\alpha,\alpha'} \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_1\big[(u,z);\sigma,w\big],$$

and the requirement

$$\lim_{\mathbf{\Omega}_\infty\uparrow\varnothing} \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_1\big[(u,z);\sigma,w\big] = 0 \quad (13)$$

makes the norm $\|P_{\mathbf{\Omega}_\infty} I_2 P_{\mathbf{\Omega}_\infty}\|$ infinitely small as well.

For the rest terms on the right hand side of (10) we find

$$\|P_{W_1} I_2 P_{V_3}\| \leqslant \mathbb{C}_{\alpha,\alpha'} \sup_{z\geqslant d} A_1\big[(a,z);\sigma,w\big] \to 0, \quad d\to\infty, \quad (14)$$

$$\|P_{W_1} I_2 P_{V_4}\| \leqslant \mathbb{C}_{\alpha,\alpha'} \sup_{u\geqslant c} A_1\big[(u,d);\sigma,w\big] \to 0, \quad c\to\infty, \quad (15)$$

$$\|P_{W_2} I_2 P_{V_4}\| \leqslant \mathbb{C}_{\alpha,\alpha'} \sup_{u\geqslant c} A_1\big[(u,b);\sigma,w\big] \to 0, \quad c\to\infty. \quad (16)$$

Let us demonstrate now the necessity of the condition (12) for the compact $I_2 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$. Let

$$f_{(u,z)}(s,t) := \chi_{(0,u)\times(0,z)}(s,t)\sigma(s,t)\left(\int_0^u\int_0^z \sigma\right)^{-1/p}, \qquad (u,z) \in \mathbf{\Omega}_0.$$

Then

$$\|f_{(u,z)}\|_{p,v} = 1.$$

For any linear functional $G \in [L^p_v(\mathbb{R}^2_+)]^*$ there exists an element $g \in L^{p'}_{v^{1-p'}}(\mathbb{R}^2_+)$ such that

$$\begin{aligned}\langle G, f_{(u,z)}\rangle &= \int_{\mathbb{R}^2_+} f_{(u,z)} g \leqslant \int_{\mathbf{\Omega}_0} f_{(u,z)} g \\ &\leqslant \left(\int_0^u\int_0^z |g|^{p'}\sigma\right)^{1/p'} \to 0, \qquad \mathbf{\Omega}_0 \downarrow \varnothing,\end{aligned}$$

which means that any sequence from the family $f_{(u,z)}$ with $(u,z) \in \mathbf{\Omega}_0$ converges weakly to 0 in $L^p_v(\mathbb{R}^2_+)$. By the compactness of I_2, this entails strong convergence of $I_2 f_{(u,z)}$, that is

$$\|I_2 f_{(u,z)}\|_{q,w} \to 0, \qquad \mathbf{\Omega}_0 \downarrow \varnothing. \tag{17}$$

On the other side,

$$\int_0^u\int_0^z f_{(u,z)} = \left(\int_0^u\int_0^z \sigma\right)^{\frac{1}{p'}}$$

and

$$\begin{aligned}\|I_2 f_{(u,z)}\|^q_{q,w} &\geqslant \int_u^\infty\int_z^\infty w\left(\int_0^u\int_0^z f_{(u,z)}\right)^q \\ &= \int_u^\infty\int_z^\infty w\left(\int_0^u\int_0^z \sigma\right)^{\frac{q}{p'}} \\ &= A_1^q\big[(u,z);\sigma,w\big],\end{aligned}$$

and (12) now follows from (17). The necessity of the condition (13) can be proven analogously, while the same for (14)–(16) is a consequence of (13).

Thus, the following statement has been proven in detail above.

THEOREM 3. *Let $1 < p < q < \infty$ and v, w be weight functions. Suppose the two–dimensional rectangular operator I_2 of the form* (1) *is bounded from $L^p_v(\mathbb{R}^2_+)$ to $L^q_w(\mathbb{R}^2_+)$. Then $I_2 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ is compact if and only if $A := A_1 < \infty$ and the conditions* (12) *and* (13) *are satisfied.*

3. The case $p = q$

We shall use the representation $\mathbb{R}^2_+ = \mathbf{\Omega}_0 \cup \mathbf{\Omega} \cup \mathbf{\Omega}_\infty$ from § 2.

THEOREM 4. *Let $p > 1$. Suppose that the Hardy operator I_2 of the form (1) is bounded from $L^p_v(\mathbb{R}^2_+)$ to $L^p_w(\mathbb{R}^2_+)$, where v,w are weights. Then $I_2 : L^p_v(\mathbb{R}^2_+) \to L^p_w(\mathbb{R}^2_+)$ is compact if and only if $\sum_{i=1,2,3} A_i < \infty$ and the conditions (12) and (13) in combination with*

$$\lim_{\mathbf{\Omega}_0 \downarrow \varnothing} \sup_{(u,z)\in\mathbf{\Omega}_0} A_2\big[(u,z);\sigma,w\big] = 0, \tag{18}$$

$$\lim_{\mathbf{\Omega}_\infty \uparrow \varnothing} \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_2\big[(u,z);\sigma\chi_{\mathbf{\Omega}_\infty},w\chi_{\mathbf{\Omega}_\infty}\big] = 0 \tag{19}$$

and

$$\lim_{\mathbf{\Omega}_0 \downarrow \varnothing} \sup_{(u,z)\in\mathbf{\Omega}_0} A_3\big[(u,z);\sigma\chi_{\mathbf{\Omega}_0},w\chi_{\mathbf{\Omega}_0}\big] = 0, \tag{20}$$

$$\lim_{\mathbf{\Omega}_\infty \uparrow \varnothing} \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_3\big[(u,z);\sigma,w\big] = 0 \tag{21}$$

are satisfied.

Proof. Sufficiency. We repeat the proof of the previous theorem up to the equality (10). After this it remains only to show that the norms of the operators on the right hand side of (10) tends to zero under the influence of the conditions (12), (13), (18)–(21). To this end, according to Theorem 1, one has to use estimates for operator norms of the type $\|I_2\|_{L^p_v(\mathbb{R}^2_+)\to L^p_w(\mathbb{R}^2_+)}$ by sums of the form $A_1 + A_2 + A_3$.

We write for the first operator on the right hand side of (10):

$$\|P_{\mathbf{\Omega}_0} I_2 P_{\mathbf{\Omega}_0}\| \lesssim \sum_{i=1}^{3} A_i(\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}),$$

where

$$\begin{aligned} A_1(\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}) &= \sup_{(s,t)\in\mathbb{R}^2_+} A_1\big[(s,t);\sigma\chi_{\mathbf{\Omega}_0},w\chi_{\mathbf{\Omega}_0}\big] \\ &= \sup_{(s,t)\in\mathbb{R}^2_+} \big[I_2^* w(s,t)\chi_{\mathbf{\Omega}_0}(s,t)\big]^{\frac{1}{p}} \big[I_2\sigma(s,t)\chi_{\mathbf{\Omega}_0}(s,t)\big]^{\frac{1}{p'}}, \end{aligned}$$

$$\begin{aligned} A_2(\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}) &= \sup_{(s,t)\in\mathbb{R}^2_+} A_2\big[(s,t);\sigma\chi_{\mathbf{\Omega}_0},w\chi_{\mathbf{\Omega}_0}\big] \\ &= \sup_{(s,t)\in\mathbb{R}^2_+} \left(\int_0^s\int_0^t (I_2(\sigma\chi_{\mathbf{\Omega}_0}))^p w\chi_{\mathbf{\Omega}_0}\right)^{\frac{1}{p}} \big[I_2(\sigma\chi_{\mathbf{\Omega}_0})(s,t)\big]^{-\frac{1}{p}}, \end{aligned}$$

$$A_3(\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}) = \sup_{(s,t)\in\mathbb{R}^2_+} A_3\big[(s,t);\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}\big]$$
$$= \sup_{(s,t)\in\mathbb{R}^2_+}\left(\int_s^\infty\int_t^\infty (I_2^*(w\chi_{\mathbf{\Omega}_0}))^{p'}\sigma\chi_{\mathbf{\Omega}_0}\right)^{\frac{1}{p'}}\big[I_2^*(w\chi_{\mathbf{\Omega}_0})(s,t)\big]^{-\frac{1}{q'}}.$$

The estimate

$$A_1(\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}) \lesssim \sup_{(u,z)\in\mathbf{\Omega}_0} A_1\big[(u,z);\sigma,w\big]$$

can be obtained analogously to (11) in the proof of Theorem 3. For the constant $A_2(\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0})$ we write

$$A_2(\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}) \approx \Big[\sup_{(s,t)\in\mathbf{\Omega}_0} + \sup_{(s,t)\notin\mathbf{\Omega}_0}\Big]\left(\int_0^s\int_0^t (I_2(\sigma\chi_{\mathbf{\Omega}_0}))^p w\chi_{\mathbf{\Omega}_0}\right)^{\frac{1}{p}}\times$$
$$\times\big[I_2(\sigma\chi_{\mathbf{\Omega}_0})(s,t)\big]^{-\frac{1}{p}} =: J_1 + J_2.$$

The term J_1 is regulated by the condition (18). To evaluate J_2 we notice that for $(s,t)\notin\mathbf{\Omega}_0$

$$\mathbf{\Omega}_0\cap\big\{(0,s)\times(0,t)\big\}\subset\big\{(0,a)\times(0,t)\big\}\cup\big\{(0,s)\times(0,b)\big\},$$

$$\mathbf{\Omega}_0\cap\big\{(0,s)\times(0,t)\big\}\supset\big\{(0,a)\times(0,t)\big\},\quad \mathbf{\Omega}_0\cap\big\{(0,s)\times(0,t)\big\}\supset\big\{(0,s)\times(0,b)\big\}.$$

It holds

$$J_2 \lesssim \sup_{(s,t)\notin\mathbf{\Omega}_0}\left[\left(\int_0^a\int_0^t (I_2(\sigma\chi_{\mathbf{\Omega}_0}))^p w\chi_{\mathbf{\Omega}_0}\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_0^s\int_0^b (I_2(\sigma\chi_{\mathbf{\Omega}_0}))^p w\chi_{\mathbf{\Omega}_0}\right)^{\frac{1}{p}}\right]\big[I_2(\sigma\chi_{\mathbf{\Omega}_0})(s,t)\big]^{-\frac{1}{p}} =: J_{21} + J_{22}.$$

Further,

$$J_{21} \lesssim \sup_{(s,t)\notin\mathbf{\Omega}_0}\left(\int_0^a\int_0^t (I_2(\sigma\chi_{\mathbf{\Omega}_0}))^p w\chi_{\mathbf{\Omega}_0}\right)^{\frac{1}{p}}\big[I_2(\sigma\chi_{\mathbf{\Omega}_0})(a,t)\big]^{-\frac{1}{p}},$$

$$J_{22} \lesssim \sup_{(s,t)\notin\mathbf{\Omega}_0}\left(\int_0^s\int_0^b (I_2(\sigma\chi_{\mathbf{\Omega}_0}))^p w\chi_{\mathbf{\Omega}_0}\right)^{\frac{1}{p}}\big[I_2(\sigma\chi_{\mathbf{\Omega}_0})(s,b)\big]^{-\frac{1}{p}}.$$

Since for $(s,t)\notin\mathbf{\Omega}_0$ the points (a,t) and (s,b) belong to the boundary of $\mathbf{\Omega}_0$, then

$$J_{21} + J_{22} \lesssim J_1.$$

Since

$$\mathbf{\Omega}_0\cap\big\{(s,\infty)\times(t,\infty)\big\} = \varnothing$$

for $(s,t) \notin \mathbf{\Omega}_0$, then

$$A_3(\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}) = \sup_{(s,t)\in\mathbf{\Omega}_0} A_3\left[(s,t);\sigma\chi_{\mathbf{\Omega}_0}, w\chi_{\mathbf{\Omega}_0}\right],$$

which is governed by the condition (20).

Analogous estimates are true for the operator $P_{\mathbf{\Omega}_\infty} I_2 P_{\mathbf{\Omega}_\infty}$. Indeed, we have

$$\|P_{\mathbf{\Omega}_\infty} I_2 P_{\mathbf{\Omega}_\infty}\| \lesssim \sum_{i=1}^{3} A_i(\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}),$$

where

$$\begin{aligned} A_1(\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}) &= \sup_{(s,t)\in\mathbb{R}^2_+} A_1\left[(s,t);\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}\right] \\ &= \sup_{(s,t)\in\mathbb{R}^2_+} \left[I_2^* w(s,t)\chi_{\mathbf{\Omega}_\infty}(s,t)\right]^{\frac{1}{p}} \left[I_2\sigma(s,t)\chi_{\mathbf{\Omega}_\infty}(s,t)\right]^{\frac{1}{p'}}, \end{aligned}$$

$$\begin{aligned} A_2(\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}) &= \sup_{(s,t)\in\mathbb{R}^2_+} A_2\left[(s,t);\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}\right] \\ &= \sup_{(s,t)\in\mathbb{R}^2_+} \left(\int_0^s \int_0^t \left(I_2(\sigma\chi_{\mathbf{\Omega}_\infty})\right)^p w\chi_{\mathbf{\Omega}_\infty}\right)^{\frac{1}{p}} \left[I_2(\sigma\chi_{\mathbf{\Omega}_\infty})(s,t)\right]^{-\frac{1}{p}}, \end{aligned}$$

$$\begin{aligned} A_3(\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}) &= \sup_{(s,t)\in\mathbb{R}^2_+} A_3\left[(s,t);\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}\right] \\ &= \sup_{(s,t)\in\mathbb{R}^2_+} \left(\int_s^\infty \int_t^\infty \left(I_2^*(w\chi_{\mathbf{\Omega}_\infty})\right)^{p'} \sigma\chi_{\mathbf{\Omega}_\infty}\right)^{\frac{1}{p'}} \left[I_2^*(w\chi_{\mathbf{\Omega}_\infty})(s,t)\right]^{-\frac{1}{q'}}. \end{aligned}$$

Since

$$\left\{(0,s)\times(0,t)\right\} \cap \mathbf{\Omega}_\infty = \varnothing$$

for $(s,t) \notin \mathbf{\Omega}_\infty$, then

$$A_1(\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}) \lesssim \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_1\left[(u,z);\sigma, w\right]$$

and

$$A_2(\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}) \lesssim \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_2\left[(u,z);\sigma, w\right],$$

where in the both inequalities their right hand sides are governed by (13) and (19), respectively. For $A_3(\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty})$ we write

$$\begin{aligned} A_3(\sigma\chi_{\mathbf{\Omega}_\infty}, w\chi_{\mathbf{\Omega}_\infty}) \approx &\left[\sup_{(s,t)\in\mathbf{\Omega}_\infty} + \sup_{(s,t)\notin\mathbf{\Omega}_\infty}\right] \left(\int_s^\infty \int_t^\infty (I_2^*(w\chi_{\mathbf{\Omega}_\infty}))^{p'} \sigma\chi_{\mathbf{\Omega}_\infty}\right)^{\frac{1}{p'}} \times \\ &\times \left[I_2^*(w\chi_{\mathbf{\Omega}_\infty})(s,t)\right]^{-\frac{1}{q'}} =: H_1 + H_2. \end{aligned}$$

The H_1 is controlled by (21). If $(s,t)\notin \mathbf{\Omega}_\infty$ then $(s,t)\in \mathbf{\Omega}\cup\mathbf{\Omega}_0$. Let $(s,t)\in\mathbf{\Omega}$. Therefore,

$$\mathbf{\Omega}_\infty\cap\big\{(s,\infty)\times(t,\infty)\big\}\subset\big\{(s,\infty)\times(d,\infty)\big\}\cup\big\{(c,\infty)\times(t,\infty)\big\},$$
$$\mathbf{\Omega}_\infty\cap\big\{(s,\infty)\times(t,\infty)\big\}\supset\big\{(s,\infty)\times(d,\infty)\big\},$$
$$\mathbf{\Omega}_\infty\cap\big\{(s,\infty)\times(t,\infty)\big\}\supset\big\{(c,\infty)\times(t,\infty)\big\}.$$

We have

$$H_2\lesssim\sup_{(s,t)\in\mathbf{\Omega}}\left[\left(\int_s^\infty\int_d^\infty\big(I_2^*(w\chi_{\mathbf{\Omega}_\infty})\big)^{p'}\sigma\chi_{\mathbf{\Omega}_\infty}\right)^{\frac{1}{p'}}\right.$$
$$\left.+\left(\int_c^\infty\int_t^\infty\big(I_2^*(w\chi_{\mathbf{\Omega}_\infty})\big)^{p'}\sigma\chi_{\mathbf{\Omega}_\infty}\right)^{\frac{1}{p'}}\right]\big[I_2^*(w\chi_{\mathbf{\Omega}_\infty})(s,t)\big]^{-\frac{1}{q'}}=:H_{21}+H_{22}.$$

Further,

$$H_{21}\lesssim\sup_{(s,t)\in\mathbf{\Omega}}\left(\int_s^\infty\int_d^\infty\big(I_2^*(w\chi_{\mathbf{\Omega}_\infty})\big)^{p'}\sigma\chi_{\mathbf{\Omega}_\infty}\right)^{\frac{1}{p'}}\big[I_2^*(w\chi_{\mathbf{\Omega}_\infty})(s,d)\big]^{-\frac{1}{q'}},$$

$$H_{22}\lesssim\sup_{(s,t)\in\mathbf{\Omega}}\left(\int_c^\infty\int_t^\infty\big(I_2^*(w\chi_{\mathbf{\Omega}_\infty})\big)^{p'}\sigma\chi_{\mathbf{\Omega}_\infty}\right)^{\frac{1}{p'}}\big[I_2^*(w\chi_{\mathbf{\Omega}_\infty})(c,t)\big]^{-\frac{1}{q'}}.$$

Since for $(s,t)\notin\mathbf{\Omega}_\infty$ the points (c,t) and (s,d) belong to the boundary of $\mathbf{\Omega}_\infty$, then

$$H_{21}+H_{22}\lesssim H_1.$$

For $(s,t)\in\mathbf{\Omega}_0$ argumentation is similar.

For the remaining operators on the right–hand side of (10) we use the one–dimensional Hardy inequality. For the operator $P_{W_1}I_2P_{V_3}$ we write

$$\|P_{W_1}I_2P_{V_3}\|=\sup_{f\geqslant 0}\frac{\left(\int_a^\infty\int_d^\infty\left(\int_0^a\int_d^y f(s,t)\,dsdt\right)^p w(x,y)\,dxdy\right)^{\frac{1}{p}}}{\left(\int_0^a\int_d^\infty f^p(s,t)v(s,t)\,dsdt\right)^{\frac{1}{p}}}$$
$$=:\sup_{f\geqslant 0}\frac{\left(\int_d^\infty\left(\int_d^y F(t)dt\right)^p W(y)dy\right)^{\frac{1}{p}}}{\left(\int_0^a\int_d^\infty f^p(s,t)v(s,t)\,dsdt\right)^{\frac{1}{p}}}$$
$$\lesssim\sup_{u\geqslant d}\left(\int_u^\infty W(y)dy\right)^{\frac{1}{p}}\left(\int_d^u V^{1-p'}(t)dt\right)^{\frac{1}{p'}}\times$$
$$\times\sup_{f\geqslant 0}\frac{\left(\int_d^\infty F^p(t)V(t)dt\right)^{\frac{1}{p}}}{\left(\int_0^a\int_d^\infty f^p(s,t)v(s,t)\,dsdt\right)^{\frac{1}{p}}},$$

where

$$W(y)=\int_a^\infty w(x,y)dx,\quad F(t)=\int_0^a f(s,t)ds,\quad V(t)=\left(\int_0^a\sigma(s,t)ds\right)^{-\frac{p}{p'}}.$$

By applyng Hölder's inequality, we find

$$\int_d^\infty F^p(t)V(t)dt = \int_d^\infty \left(\int_0^a f(s,t)ds\right)^p V(t)dt \leqslant \int_d^\infty \int_0^a f^p(s,t)v(s,t)\,dsdt.$$

Thus,

$$\begin{aligned}\|P_{W_1}I_2P_{V_3}\| &\lesssim \sup_{u\geqslant d}\left(\int_u^\infty \int_a^\infty w(x,y)\,dxdy\right)^{\frac{1}{p}} \left(\int_d^u \int_0^a \sigma(s,t)\,dsdt\right)^{\frac{1}{p'}}\\ &\leqslant \sup_{u\geqslant d} A_1[(u,a);\sigma,w],\end{aligned}$$

which means that the norm of $P_{W_1}I_2P_{V_3}$ is regulated by (13).

Similarly, we can write for $P_{W_1}I_2P_{V_4}$:

$$\begin{aligned}\|P_{W_1}I_2P_{V_4}\| &= \sup_{f\geqslant 0}\frac{\left(\int_a^\infty \int_d^\infty \left(\int_c^x \int_0^b f(s,t)\chi_{(c,\infty)}(s,t)\,dsdt\right)^p w(x,y)\,dxdy\right)^{\frac{1}{p}}}{\left(\int_c^\infty \int_0^b f^p(s,t)v(s,t)\,dsdt\right)^{\frac{1}{p}}}\\ &=: \sup_{f\geqslant 0}\frac{\left(\int_c^\infty \left(\int_c^x F(s)ds\right)^p W(x)dx\right)^{\frac{1}{p}}}{\left(\int_c^\infty \int_0^b f^p(s,t)v(s,t)\,dsdt\right)^{\frac{1}{p}}}\\ &\lesssim \sup_{u\geqslant c}\left(\int_z^\infty W(y)dy\right)^{\frac{1}{p}} \left(\int_d^u V^{1-p'}(t)dt\right)^{\frac{1}{p'}} \times\\ &\quad\times \sup_{f\geqslant 0}\frac{\left(\int_d^\infty F^p(s)V(s)ds\right)^{\frac{1}{p}}}{\left(\int_c^\infty \int_0^b f^p(s,t)v(s,t)\,dsdt\right)^{\frac{1}{p}}},\end{aligned}$$

where

$$W(x) = \int_d^\infty w(x,y)dy, \quad F(s) = \int_0^b f(s,t)dt, \quad V(s) = \left(\int_0^b \sigma(s,t)dt\right)^{-\frac{p}{p'}}.$$

From this one can obtain by Hölder's inequality

$$\int_c^\infty F^p(s)V(s)ds = \int_c^\infty \left(\int_0^b f(s,t)ds\right)^p V(s)ds \leqslant \int_c^\infty \int_0^b f^p(s,t)v(s,t)dsdt.$$

Therefore,

$$\begin{aligned}\|P_{W_1}I_2P_{V_4}\| &\lesssim \sup_{u\geqslant c}\left(\int_u^\infty \int_d^\infty w(x,y)\,dxdy\right)^{\frac{1}{p}} \left(\int_c^u \int_0^b \sigma(s,t)\,dsdt\right)^{\frac{1}{p'}}\\ &\leqslant \sup_{u\geqslant c} A_1[(u,d);\sigma,w],\end{aligned}$$

that is the norm of $P_{W_1}I_2P_{V_4}$ is controlled by (13).

By similar argumentation the estimate

$$\|P_{W_2}I_2P_{V_4}\| \lesssim \sup_{u\geqslant c} A_1[(u,b);\sigma,w]$$

can be done, and the sufficiency of the conditions (12), (13), (18)–(21) for the compactness of the bounded operator $I_2 : L^p_v(\mathbb{R}^2_+) \to L^p_w(\mathbb{R}^2_+)$ is now performed.

(*Necessity*) Sequences of test functions to be applied for establishing the necessity of the conditions (12), (18) and (13), (21) can be defined analogously to those in the proof of Theorem 3. To confirm that (19) and (20) are necessary as well, it is enough to choose

$$f_{(u,z)}(s,t) = \chi_{[(0,u)\times(0,z)]\cap\mathbf{\Omega}_\infty}(s,t)\sigma(s,t)\left(\int_0^u\int_0^z \chi_{\mathbf{\Omega}_\infty}\sigma\right)^{-1/p} \text{ for } (u,z)\in\mathbf{\Omega}_\infty,$$

and, respectively,

$$g_{(u,z)}(x,y) = \chi_{[(u,\infty)\times(z,\infty)]\cap\mathbf{\Omega}_0}(s,t)w(s,t)\left(\int_u^\infty\int_z^\infty \chi_{\mathbf{\Omega}_0}w\right)^{-1/p'} \text{ if } (u,z)\in\mathbf{\Omega}_0. \quad \square$$

4. The case $q<p$

THEOREM 5. *Let $1<q<p<\infty, 1/r := 1/q-1/p$, and v, w be weights. Suppose that the two–dimensional rectangular Hardy operator I_2 is given by (1). Then $I_2 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ is compact if*

$$B_v := \left(\int_{\mathbb{R}^2_+}\sigma(u,z)\left(\int_u^\infty\int_z^\infty (I_2\sigma)^{q-1}w\right)^{\frac{r}{q}} du\,dz\right)^{\frac{1}{r}} < \infty. \tag{22}$$

Conversaly, if $I_2 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ is compact then $B<\infty$, where

$$\begin{aligned} B :=& \left(\int_{\mathbb{R}^2_+} d_y\left[I_2\sigma(x,y)\right]^{\frac{r}{p'}} d_x\left(-\left[I_2^*w(x,y)\right]^{\frac{r}{q}}\right)\right)^{\frac{1}{r}} \\ =& \left(\int_{\mathbb{R}^2_+}\left[I_2\sigma(x,y)\right]^{\frac{r}{p'}} d_x d_y\left[I_2^*w(x,y)\right]^{\frac{r}{q}}\right)^{\frac{1}{r}} \\ =& \left(\int_{\mathbb{R}^2_+}\left[I_2^*w(x,y)\right]^{\frac{r}{q}} d_x d_y\left[I_2\sigma(x,y)\right]^{\frac{r}{p'}}\right)^{\frac{1}{r}}, \end{aligned}$$

and the two last equalities follow by integration by parts.

Proof. It is well known (see [1], [9, § 5.3]) that for $1<q<p$ a regular integral operator from $L^p_v(\mathbb{R}^2_+)$ to $L^q_w(\mathbb{R}^2_+)$ is compact if it is bounded. Thus, the assertions of Theorem follow from [18, Theorem 3]. $\square$

5. Measure of non–compactness

Define

$$\boldsymbol{\alpha}(T) := \inf \|T - P\|,$$

where the infimum is taken over all bounded linear mappings $P : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ of finite rank. The quantity $\boldsymbol{\alpha}(T)$ coincides with the so called set of measure of non–compactness of an operator T bounded from $L^p_v(\mathbb{R}^2_+)$ to $L^q_w(\mathbb{R}^2_+)$ (see [3, § 2] and [2, Proposition 3.1]). In order to obtain a two–sided estimate on $\boldsymbol{\alpha}(I_2)$ we put

$$A_1(\mathbf{\Omega}_0) := \sup_{(u,z)\in\mathbf{\Omega}_0} A_1\big[(u,z);\sigma,w\big], \quad A_1(\mathbf{\Omega}_\infty) := \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_1\big[(u,z);\sigma,w\big];$$
$$A_2(\mathbf{\Omega}_0) := \sup_{(u,z)\in\mathbf{\Omega}_0} A_2\big[(u,z);\sigma,w\big], \quad A_2(\mathbf{\Omega}_\infty) := \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_2\big[(u,z);\sigma\chi_{\mathbf{\Omega}_\infty},w\chi_{\mathbf{\Omega}_\infty}\big];$$
$$A_3(\mathbf{\Omega}_0) := \sup_{(u,z)\in\mathbf{\Omega}_0} A_3\big[(u,z);\sigma\chi_{\mathbf{\Omega}_0},w\chi_{\mathbf{\Omega}_0}\big], \quad A_3(\mathbf{\Omega}_\infty) := \sup_{(u,z)\in\mathbf{\Omega}_\infty} A_3\big[(u,z);\sigma,w\big].$$

Let

$$J(\mathbf{\Omega}_0) := A_1(\mathbf{\Omega}_0) \quad \text{and} \quad J(\mathbf{\Omega}_\infty) := A_1(\mathbf{\Omega}_\infty) \tag{23}$$

in the case $1 < p < q < \infty$; for $p = q$ we set

$$J(\mathbf{\Omega}_0) := \sum_{i=1}^{3} A_i(\mathbf{\Omega}_0) \quad \text{and} \quad J(\mathbf{\Omega}_\infty) := \sum_{i=1}^{3} A_i(\mathbf{\Omega}_\infty). \tag{24}$$

Denote

$$J_0 := \lim_{\mathbf{\Omega}_0\downarrow\varnothing} J(\mathbf{\Omega}_0) \quad \text{and} \quad J_\infty := \lim_{\mathbf{\Omega}_\infty\downarrow\varnothing} J(\mathbf{\Omega}_\infty),$$

where the above limits exist on the strength of monotonicity (23)–(24) with respect to vanishing of the sets $\mathbf{\Omega}_0$ and $\mathbf{\Omega}_\infty$. Recall that $\mathbb{R}^2_+ = \mathbf{\Omega}_0 \cup \mathbf{\Omega} \cup \mathbf{\Omega}_\infty$ (see § 2).

By taking into account notations on the right hand side of (10), we can formulate the following statement, which is the result of reasoning in §§ 1–2 and Theorems 1, 2.

COROLLARY 1. *Let* $1 < p \leqslant q < \infty$. *Then* (i) $P_{\mathbf{\Omega}_0} I_2 P_{\mathbf{\Omega}_0} : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ *is bounded if and only if* $J(\mathbf{\Omega}_0) < \infty$, *moreover,*

$$\|P_{\mathbf{\Omega_0}} I_2 P_{\mathbf{\Omega_0}}\| \approx J(\mathbf{\Omega}_0);$$

(ii) $P_{\mathbf{\Omega}_\infty} I_2 P_{\mathbf{\Omega}_\infty} : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ *is bounded if and only if* $J(\mathbf{\Omega}_\infty) < \infty$, *and*

$$\|P_{\mathbf{\Omega}_\infty} I_2 P_{\mathbf{\Omega}_\infty}\| \approx J(\mathbf{\Omega}_\infty).$$

The following estimate is also true:

$$P_{W_1} I_2 (P_{V_3} + P_{V_4}) + P_{W_2} I_2 P_{V_4} \lesssim A_1(\mathbf{\Omega}_\infty).$$

The required estimate on $\boldsymbol{\alpha}(I_2)$ is established in the next Theorem 6.

THEOREM 6. *Let* $1 < p \leqslant q < \infty$. *Then*

$$\boldsymbol{\alpha}(I_2) \approx J_0 + J_\infty.$$

Proof. The upper estimate on $\boldsymbol{\alpha}(I_2)$ follows from (10) and Corollary 1.

To prove the lower estimate we choose $\lambda > \boldsymbol{\alpha}(I_2)$. Then there exists an operator $F : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ of finite rank such that the inequality

$$\|I_2F - Ff\|_{q,w} \leqslant \lambda \|f\|_{p,v}$$

is valid for all $f \in L^p_v(\mathbb{R}^2_+)$. On the strength of [4, Lemma 2.2], for a given $\varepsilon > 0$ there are $F_0 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ of finite rank and a compact subset $\boldsymbol{K}$ of $\mathbb{R}^2_+$ such that $\|F - F_0\| < \varepsilon$ and $\operatorname{supp} F_0 f \subset \boldsymbol{K}$ for all $f \in L^p_v(\mathbb{R}^2_+)$. Therefore,

$$\|I_2 f - F_0 f\|_{q,w} \leqslant (\lambda + \varepsilon)\|f\|_{p,v} \qquad \text{for all } f \in L^p_v(\mathbb{R}^2_+).$$

Choose $\boldsymbol{\Omega} \supset \boldsymbol{K}$ and let $f \geqslant 0$ be such that $\operatorname{supp} f \subset \{\boldsymbol{\Omega}_0 \cup \boldsymbol{\Omega}_\infty\}$. Then

$$(\lambda+\varepsilon)\|f\|_{p,v} \geqslant \|I_2 f\|_{q,w} = \|I_2 P_{\boldsymbol{\Omega}_0 f + } I_2 P_{\boldsymbol{\Omega}_\infty}\| \geqslant \|P_{\boldsymbol{\Omega}_0} I_2 P_{\boldsymbol{\Omega}_0} f + P_{\boldsymbol{\Omega}_\infty} I_2 P_{\boldsymbol{\Omega}_\infty} f\|_{q,w}.$$

From this

$$(\lambda+\varepsilon)\|f\|_{p,v} \geqslant \|P_{\boldsymbol{\Omega}_0} I_2 P_{\boldsymbol{\Omega}_0} f\|_{q,w}$$

for all $0 \leqslant f \in L^p_v(\mathbb{R}^2_+)$ having supports in $\boldsymbol{\Omega}_0$, and

$$(\lambda+\varepsilon)\|f\|_{p,v} \geqslant \|P_{\boldsymbol{\Omega}_\infty} I_2 P_{\boldsymbol{\Omega}_\infty} f\|_{q,w}$$

for all $0 \leqslant f \in L^p_v(\mathbb{R}^2_+)$ with supports in $\boldsymbol{\Omega}_\infty$. The two last evaluations and Corollary 1 imply the required lower estimate, since ε can be chosen arbitrarily small. □

Acknowledgement. The work of the authors was supported by the Russian Science Foundation (project 22-21-00579).

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(Received September 2, 2021)

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