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REGULARITY OF COMMUTATOR OF BILINEAR MAXIMAL OPERATOR WITH LIPSCHITZ SYMBOLS

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Abstract. This paper is devoted to exploring the regularity properties of the commutator of maximal operator in the bilinear setting. More precisely, we introduce the commutator of bilinear maximal operator and bilinear maximal commutator. We establish some new boundedness and continuity for the above operators on the Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces under the condition that the symbol function belongs to the Lipschitz space.

1. Introduction

In a very recent article [15], Liu and Wang established the Sobolev regularity properties of the commutator of Hardy–Littlewood maximal operator and the maximal commutator as well as their fractional variants with Lipschitz symbols. The first motivation of this paper is to explore the above results to the bilinear setting. In addition, we prove the bounds and continuity for the commutator of bilinear maximal operator and bilinear maximal commutator on the fractional Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces. Let us recall some definitions and backgrounds.

DEFINITION 1. Let $0 \leqslant \alpha < n$. The bilinear fractional maximal operator $\mathfrak{M}_\alpha$ is defined as

$$\mathfrak{M}_\alpha(f,g)(x) = \sup_{r>0} \frac{1}{|B(O,r)|^{1-\alpha/n}} \int_{B(O,r)} |f(x+y)g(x-y)|dy,$$

where $x \in \mathbb{R}^n$ and $O = (0,0,\dots,0) \in \mathbb{R}^n$. Let b be a locally integral function defined on $\mathbb{R}^n$. The commutators of the bilinear fractional maximal operator with b in the i-th entry $(i = 1,2)$ are defined by

$$[\mathfrak{M}_\alpha, b]_1(f,g) = \mathfrak{M}_\alpha(bf,g) - b\mathfrak{M}_\alpha(f,g), \quad [\mathfrak{M}_\alpha, b]_2(f,g) = \mathfrak{M}_\alpha(f,bg) - b\mathfrak{M}_\alpha(f,g).$$

The bilinear fractional maximal commutator $\mathfrak{M}_{b,\alpha}$ is given by

$$\mathfrak{M}_{b,\alpha}(f,g)(x) = \sup_{r>0} \frac{1}{|B(O,r)|^{1-\alpha/n}} \int_{B(O,r)} |(b(x) - b(x+y))f(x+y)g(x-y)|dy.$$

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When $\alpha = 0$, the operator $\mathfrak{M}_\alpha$ reduces to the usual bilinear maximal operator $\mathfrak{M}$. In this case the operator $[\mathfrak{M}_\alpha, b]_i$ reduces to the commutator of bilinear maximal operator in the i-th entry $(i = 1,2)$, which is denoted by $[\mathfrak{M}, b]_i$. Meanwhile, the operator $\mathfrak{M}_{b,\alpha}$ becomes the bilinear maximal commutator $\mathfrak{M}_b$. It should be pointed out that the commutators of bilinear operators were originally introduced by Pérez and Torres [21] who studied the boundedness for the commutators of the bilinear Calderón–Zygmund operator $[T,b]_i$ $(i = 1,2)$, where T is the bilinear Calderón–Zygmund operator.

The regularity theory of maximal operators is an active topic of current research. The first work was due to Kinnunen [8] who showed that the Hardy–Littlewood maximal operator

$$\mathscr{M} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

is bounded on the first order Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leqslant \infty$, where $n \geqslant 1$, $B(x,r)$ denotes the open ball in $\mathbb{R}^n$ centered at x with radius r and $|B(x,r)|$ is the volume of $B(x,r)$. Here $W^{1,p}(\mathbb{R}^n)$ is defined as

$$W^{1,p}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} : \ \|f\|_{W^{1,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\nabla f = (D_1 f, \ldots, D_n f)$ is the weak gradient of f. Kinnunen's work has initiated a new research direction in harmonic analysis. We can consult [1, 2, 3, 9, 10, 14, 22, 23] for related works. Since the derivatives of the maximal operator are not sublinearity, the continuity of $\mathscr{M} : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ is certainly a nontrivial issue. This question was addressed by Luiro [19] and later extensions were given in [20, 1]. The regularity properties of maximal operators on other smooth function spaces have also been studied by many authors. For example, see [12] for the fractional Sobolev spaces, [11, 20, 14, 16, 26] for the Triebel–Lizorkin spaces and Besov spaces. Particularly, Carneiro and Moreira [1] showed that if $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$, then the map $\mathfrak{M} : W^{1,p_1}(\mathbb{R}^n) \times W^{1,p_2}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is bounded and continuous. Moreover, if $f \in W^{1,p_1}(\mathbb{R}^n)$ and $g \in W^{1,p_2}(\mathbb{R}^n)$, then

$$|\nabla \mathfrak{M}(f,g)(x)| \leqslant \mathfrak{M}(|\nabla f|, g)(x) + \mathfrak{M}(f, |\nabla g|)(x),$$

for almost everywhere $x \in \mathbb{R}^n$. Very recently, Liu, Liu and Zhang [14] extended the above result to the fractional version.

THEOREM A. ([14]) *Let $1 < p_1, p_2, p_1 p_2/(p_1 + p_2) < \infty$, $0 \leqslant \alpha < n(1/p_1 + 1/p_2)$ and $1/q = 1/p_1 + 1/p_2 - \alpha/n$. Then the map $\mathfrak{M}_\alpha : W^{1,p_1}(\mathbb{R}^n) \times W^{1,p_2}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ is bounded and continuous. Particularly, if $f \in W^{1,p_1}(\mathbb{R}^n)$ and $g \in W^{1,p_2}(\mathbb{R}^n)$, then*

$$\|\mathfrak{M}_\alpha(f,g)\|_{W^{1,q}(\mathbb{R}^n)} \lesssim_{\alpha,n,p_1,p_2} \|f\|_{W^{1,p_1}(\mathbb{R}^n)} \|g\|_{W^{1,p_2}(\mathbb{R}^n)}.$$

Moreover, the following pointwise estimate holds:

$$|\nabla \mathfrak{M}_\alpha(f,g)(x)| \leqslant \mathfrak{M}_\alpha(|\nabla f|, g)(x) + \mathfrak{M}_\alpha(f, |\nabla g|)(x),$$

for almost everywhere $x \in \mathbb{R}^n$.

In [14], the authors also established the boundedness and continuity of bilinear maximal operator on the inhomogeneous Triebel–Lizorkin spaces $F_s^{p,q}(\mathbb{R}^n)$ and the inhomogeneous Besov spaces $B_s^{p,q}(\mathbb{R}^n)$.

THEOREM B. ([14]) (i) *Let* $0 < s < 1,\ 1 < p_1, p_2, p, q < \infty$ *and* $1/p = 1/p_1 + 1/p_2$. *Then the map* $\mathfrak{M} : F_s^{p_1,q}(\mathbb{R}^n) \times F_s^{p_2,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$ *is bounded and continuous. Moreover, there exists a constant* $C > 0$ *such that*

$$\|\mathfrak{M}(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}$$

for all $f \in F_s^{p_1,q}(\mathbb{R}^n)$ *and* $g \in F_s^{p_2,q}(\mathbb{R}^n)$.

(ii) *Let* $0 < s < 1,\ 1 < p_1, p_2, p, q < \infty$ *and* $1/p = 1/p_1 + 1/p_2$. *Then the map* $\mathfrak{M} : B_s^{p_1,q}(\mathbb{R}^n) \times B_s^{p_2,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$ *is bounded and continuous. Moreover, there exists a constant* $C > 0$ *such that*

$$\|\mathfrak{M}(f,g)\|_{B_s^{p,q}(\mathbb{R}^n)} \leqslant C\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}$$

for all $f \in B_s^{p_1,q}(\mathbb{R}^n)$ *and* $g \in B_s^{p_2,q}(\mathbb{R}^n)$.

We would like to remark that the one dimensional case of $\mathfrak{M}$ was originally introduced by Calderón in 1964 when he raised the striking conjecture whether the map $\mathfrak{M} : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{R})$ is bounded. In the remarkable paper [13] Lacey addressed the above conjecture by showing that $\mathfrak{M}$ is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ for $1/p_1 + 1/p_2 = 1/p,\ 1 < p_1, p_2 < \infty$ and $2/3 < p \leqslant 1$. For $n \geqslant 2$, we get by Hölder's inequality that

$$\|\mathfrak{M}(f,g)\|_{L^p(\mathbb{R}^n)} \lesssim_{n,p_1,p_2} \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}, \tag{1}$$

provided that $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$. It was shown in [14] that

$$\|\mathfrak{M}_\alpha(f,g)\|_{L^p(\mathbb{R}^n)} \lesssim_{\alpha,n,p_1,p_2} \|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}, \tag{2}$$

and

$$\mathfrak{M}_\alpha : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \text{ is continuous,} \tag{3}$$

provided that $1 < p_1, p_2, p_1p_2/(p_1 + p_2) < \infty$, $0 \leqslant \alpha < n(1/p_1 + /1/p_2)$ and $1/q = 1/p_1 + 1/p_2 - \alpha/n$.

On the other hand, an interesting extension of regularity theory is the investigation on the regularity properties for the commutators of maximal operators. Let b be a locally integral function defined on $\mathbb{R}^n$. The commutator of the Hardy–Littlewood maximal operator is defined by

$$[b, \mathscr{M}](f)(x) = b(x)\mathscr{M}f(x) - \mathscr{M}(bf)(x), \quad x \in \mathbb{R}^n.$$

The maximal commutator of $\mathscr{M}$ with b is defined as

$$\mathrm{M}_b f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy, \quad x \in \mathbb{R}^n.$$

Liu, Xue and Zhang [18] studied the regularity for the commutators of maximal operators. They proved that if $1 < p_1, p_2, p < \infty$, $1/p = 1/p_1 + 1/p_2$ and $b \in W^{1,p_2}(\mathbb{R}^n)$, both $[b, \mathscr{M}]$ and M_b are bounded from $W^{1,p_1}(\mathbb{R}^n)$ to $W^{1,p}(\mathbb{R}^n)$. The above result was later extended to the fractional version by Liu and Xi in [17]. More precisely, let $0 \leqslant \alpha < n$. For a locally integral function b defined on $\mathbb{R}^n$, the commutator of the fractional maximal operator is defined by

$$[b, \mathscr{M}_\alpha](f)(x) = b(x)\mathscr{M}_\alpha f(x) - \mathscr{M}_\alpha(bf)(x), \quad x \in \mathbb{R}^n.$$

where the fractional maximal operator $\mathscr{M}_\alpha$ is defined as

$$\mathscr{M}_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^n.$$

The maximal commutator of $\mathscr{M}_\alpha$ with b is defined by

$$\mathrm{M}_{b,\alpha} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |b(x)-b(y)||f(y)|dy, \quad x \in \mathbb{R}^n.$$

Clearly, $[b, \mathscr{M}_\alpha] = [b, \mathscr{M}]$ and $\mathrm{M}_{b,\alpha} = \mathrm{M}_b$ when $\alpha = 0$. Liu and Xi [17] proved that if $1 < p_1, p_2, p_1p_2/(p_1+p_2) < \infty$, $0 \leqslant \alpha < n/p_1$, $1/q = 1/p_1 + 1/p_2 - \alpha/n$ and $b \in W^{1,p_2}(\mathbb{R}^n)$, then both $[b, \mathscr{M}_\alpha]$ and $\mathrm{M}_{b,\alpha}$ are bounded from $W^{1,p_1}(\mathbb{R}^n)$ to $W^{1,q}(\mathbb{R}^n)$. Very recently, Liu and Wang [15] investigated the Sobolev regularity of $[b, \mathscr{M}_\alpha]$ and $\mathrm{M}_{b,\alpha}$ with $b \in \mathrm{Lip}(\mathbb{R}^n)$. Here $\mathrm{Lip}(\mathbb{R}^n)$ denotes the *inhomogeneous* Lipschitz space, which is defined as

$$\mathrm{Lip}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \text{ continuous} : \ \ \|f\|_{\mathrm{Lip}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{Lip(\mathbb{R}^n)} < \infty,$$

and

$$\|f\|_{Lip(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \sup_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f(x+h)-f(x)|}{|h|}.$$

We now list partial result of [15] as follows:

THEOREM C. ([15]) *Let $1 < p < \infty$, $0 \leqslant \alpha < n/p$ and $1/q = 1/p - \alpha/n$. If $b \in \mathrm{Lip}(\mathbb{R}^n)$, then the map $[b, M_\alpha] : W^{1,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ is bounded and continuous. In particular, if $f \in W^{1,p}(\mathbb{R}^n)$, then for each $i \in \{1, \dots, n\}$ and almost every $x \in \mathbb{R}^n$,*

$$|\nabla([b, \mathscr{M}_\alpha](f))(x)| \leqslant 2\sqrt{n}\|b\|_{Lip(\mathbb{R}^n)} \mathscr{M}_\alpha f(x) + 2\|b\|_{L^\infty(\mathbb{R}^n)} \mathscr{M}_\alpha |\nabla f|(x).$$

Consequently,

$$\|[b, \mathscr{M}_\alpha](f)\|_{W^{1,q}(\mathbb{R}^n)} \lesssim_{\alpha,n,p} \|b\|_{\mathrm{Lip}(\mathbb{R}^n)} \|f\|_{W^{1,p}(\mathbb{R}^n)}.$$

The above boundedness also holds for $\mathrm{M}_{b,\alpha}$.

The first one of main motivations is to extend Theorem C to the bilinear setting. Before formulating our main results, let us point out the following comments, which are very useful in our proofs of main theorems.

REMARK 1. Let $1 < p_1, p_2, p_1p_2/(p_1+p_2) < \infty$, $0 \leqslant \alpha < n(1/p_1+1/p_2)$ and $1/q = 1/p_1+1/p_2-\alpha/n$ and $b \in L^\infty(\mathbb{R}^n)$. If $f \in L^{p_1}(\mathbb{R}^n)$ and $g \in L^{p_2}(\mathbb{R}^n)$, then the following facts are valid:

(i) For any fixed $i = 1,2$, in view of (2) and Hölder's inequality,

$$\|[\mathfrak{M}_\alpha, b]_i(f,g)\|_{L^q(\mathbb{R}^n)} \lesssim_{\alpha,n,p_1,p_2} \|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{4}$$

Combining (4) with the sublinearity of $\mathfrak{M}_\alpha$ implies that

$$[\mathfrak{M}_\alpha, b]_i : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \text{ is continuous.}$$

(ii) Observe that

$$\mathfrak{M}_{b,\alpha}(f,g)(x) \leqslant |b(x)|\mathfrak{M}_\alpha(f,g)(x) + \mathfrak{M}_\alpha(bf,g)(x),$$

which together with (2) and the sublinearity of $\mathfrak{M}_\alpha$ implies that

$$\|\mathfrak{M}_{b,\alpha}(f,g)\|_{L^q(\mathbb{R}^n)} \lesssim_{\alpha,n,p_1,p_2} \|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{5}$$

By (5) and the sublinearity of $\mathfrak{M}_{b,\alpha}$, we obtain

$$\mathfrak{M}_{b,\alpha} : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \text{ is continuous.} \tag{6}$$

The first one of main results can be formulated as follows:

THEOREM 1. *Let* $1 < p_1, p_2, p_1p_2/(p_1+p_2) < \infty$, $0 \leqslant \alpha < n(1/p_1+1/p_2)$, $1/q = 1/p_1+1/p_2-\alpha/n$ *and* $b \in \mathrm{Lip}(\mathbb{R}^n)$. *Then for* $i = 1,2$, *the map*

$$[\mathfrak{M}_\alpha, b]_i : W^{1,p_1}(\mathbb{R}^n) \times W^{1,p_2}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, if $f \in W^{1,p_1}(\mathbb{R}^n)$ *and* $g \in W^{1,p_2}(\mathbb{R}^n)$, *we have*

$$\begin{aligned} &|\nabla[\mathfrak{M}_\alpha, b]_i(f,g)(x)| \\ &\leqslant 2n\sqrt{n}\|b\|_{Lip(\mathbb{R}^n)}\mathfrak{M}_\alpha(f,g)(x) \\ &\quad + n(|b(x)| + \|b\|_{L^\infty(\mathbb{R}^n)})(\mathfrak{M}_\alpha(|\nabla f|,g)(x) + \mathfrak{M}_\alpha(f,|\nabla g|)(x)) \end{aligned} \tag{7}$$

for almost everywhere $x \in \mathbb{R}^n$. *As an application of* (7), *we have*

$$\|[\mathfrak{M}_\alpha, b]_i(f,g)\|_{W^{1,q}(\mathbb{R}^n)} \lesssim_{\alpha,n,p_1,p_2} \|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{W^{1,p_1}(\mathbb{R}^n)}\|g\|_{W^{1,p_2}(\mathbb{R}^n)}. \tag{8}$$

Inequalities (7) *and* (8) *hold for* $\mathfrak{M}_{b,\alpha}$.

The second motivation of this paper is to establish the bounds and continuity for the commutator of bilinear maximal operator and bilinear maximal commutator on the Triebel–Lizorkin spaces and Besov spaces. The rest of main results are the following.

THEOREM 2. *Let* $0 < s < 1$, $1 < p_1, p_2, p, q < \infty$, $1/p = 1/p_1 + 1/p_2$ *and* $b \in \mathrm{Lip}(\mathbb{R}^n)$. *Then for any* $i = 1, 2$, *the map*

$$[\mathfrak{M}, b]_i : F_s^{p_1,q}(\mathbb{R}^n) \times F_s^{p_2,q}(\mathbb{R}^n) \to F_s^{p,q}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, there exists a constant $C > 0$ *such that*

$$\|[\mathfrak{M}, b]_i(f, g)\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)} \tag{9}$$

holds for all $f \in F_s^{p_1,q}(\mathbb{R}^n)$ *and* $g \in F_s^{p_2,q}(\mathbb{R}^n)$. *The same conclusions hold for* $\mathfrak{M}_b$.

THEOREM 3. *Let* $0 < s < 1$, $1 < p_1, p_2, p, q < \infty$, $1/p = 1/p_1 + 1/p_2$ *and* $b \in \mathrm{Lip}(\mathbb{R}^n)$. *Then for any* $i = 1, 2$, *the map*

$$[\mathfrak{M}, b]_i : B_s^{p_1,q}(\mathbb{R}^n) \times B_s^{p_2,q}(\mathbb{R}^n) \to B_s^{p,q}(\mathbb{R}^n)$$

is bounded and continous. Moreover, there exists a constant $C > 0$ *such that*

$$\|[\mathfrak{M}, b]_i(f, g)\|_{B_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)} \tag{10}$$

holds for all $f \in B_s^{p_1,q}(\mathbb{R}^n)$ *and* $g \in B_s^{p_2,q}(\mathbb{R}^n)$. *The same conclusions hold for* $\mathfrak{M}_b$.

For $0 < s < 1$ and $1 < p < \infty$ we denote by $W^{s,p}(\mathbb{R}^n)$ the fractional Sobolev spaces defined by the Bessel potentials. It was pointed out in [7] that $F_s^{p,q}(\mathbb{R}^n) = W^{s,p}(\mathbb{R}^n)$ for any $s > 0$ and $1 < p < \infty$. As an application of Theorem 2, we have

COROLLARY 1. *Let* $0 < s < 1$, $1 < p_1, p_2, p < \infty$, $1/p = 1/p_1 + 1/p_2$ *and* $b \in \mathrm{Lip}(\mathbb{R}^n)$. *Then for any* $i = 1, 2$, *the map*

$$[\mathfrak{M}, b]_i : W^{s,p_1}(\mathbb{R}^n) \times W^{s,p_2}(\mathbb{R}^n) \to W^{s,p}(\mathbb{R}^n)$$

is bounded and continuous. Moreover, there exists a constant $C > 0$ *such that*

$$\|[\mathfrak{M}, b]_i(f, g)\|_{W^{s,p}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{W^{s,p_1}(\mathbb{R}^n)}\|g\|_{W^{s,p_2}(\mathbb{R}^n)}$$

holds for all $f \in W^{s,p_1}(\mathbb{R}^n)$ *and* $g \in W^{s,p_2}(\mathbb{R}^n)$. *The same conclusions hold for* $\mathfrak{M}_b$.

This paper is organized as follows.

In section 2 we prove Theorem 1. The proof of Theorem 1 for $[\mathfrak{M}_\alpha, b]_i$ follows from Theorem A and a characterization of the product of a Sobolev function and a Lipschitz function (see Lemma 1). The proof of Theorem 1 for $\mathfrak{M}_{b,\alpha}$ is based on some properties on Sobolev spaces and some refine analyses, which include the difference and derivative estimates for the bilinear fractional maximal commutators.

In section 3 we prove Theorem 2. The proof of Theorem 2 for $[\mathfrak{M}, b]_i$ is based on Theorem B(i) and a characterization of the product of a Triebel–Lizorkin function and a Lipschitz function (see Lemma 2). The proof of Theorem 2 for $\mathfrak{M}_b$ is based on some properties of Triebel–Lizorkin spaces and the difference estimates for the bilinear maximal commutators.

In section 4 we prove Theorem 3. The proof of Theorem 3 for $[\mathfrak{M},b]_i$ follows from Theorem B(ii) and a characterization of the product of a Besov function and a Lipschitz function (see Lemma 4). The proof of Theorem 3 for $\mathfrak{M}_b$ is based on some properties of Besov spaces and the difference estimates for the bilinear maximal commutators.

Throughout this paper, the letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables. For any $p \in (1,\infty)$, we denote by p' the conjugate index of p, i.e, $1/p+1/p'=1$. If there exists a constant $c>0$ depending only on ϑ such that $A \leqslant cB$, we then write $A \lesssim_{\vartheta} B$. For $r>1$ and a function f defined on $\mathbb{R}^n$, we define the operator M_r by $M_r(f)=(\mathscr{M}(f^r))^{1/r}$.

2. Proof of Theorem 1

In this section we prove Theorem 1. Throughout this section, let $f \in W^{1,p_1}(\mathbb{R}^n)$, $g \in W^{1,p_2}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$, where $1<p_1,p_2,p_1p_2/(p_1+p_2)<\infty$, $0 \leqslant \alpha < n(1/p_1+1/p_2)$ and $1/q=1/p_1+1/p_2-\alpha/n$. We divide the proof of Theorem 1 into two subsections.

2.1. Proof of Theorem 1 for $[\mathfrak{M}_\alpha,b]_i$ $(i=1,2)$

We start now the following lemma, which plays a key role in the proof of Theorem 1 for $[\mathfrak{M}_\alpha,b]_i$ $(i=1,2)$.

LEMMA 1. ([15]). *Let $1<p<\infty$. If $u \in W^{1,p}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$, then $bu \in W^{1,p}(\mathbb{R}^n)$. Moreover,*

$$\nabla(bu)=b\nabla u+u\nabla b,$$

almost everywhere in $\mathbb{R}^n$. In particular,

$$\|bu\|_{W^{1,p}(\mathbb{R}^n)} \leqslant \sqrt{n}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

We prove now Theorem 1 for $[\mathfrak{M},b]_i$ $(i=1,2)$.

Proof of Theorem 1 for $[\mathfrak{M}_\alpha,b]_i$ $(i=1,2)$. We only consider the case $i=1$, since another one is analogous. It follows from Theorem A that $\mathfrak{M}_\alpha(f,g) \in W^{1,q}(\mathbb{R}^n)$. Applying Lemma 1 and Theorem A, one obtains

$$\begin{aligned}\|b\mathfrak{M}_\alpha(f,g)\|_{W^{1,q}(\mathbb{R}^n)} &\lesssim_{\alpha,n,p_1,p_2} \|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|\mathfrak{M}_\alpha(f,g)\|_{W^{1,q}(\mathbb{R}^n)}\\ &\lesssim_{\alpha,n,p_1,p_2} \|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{W^{1,p_1}(\mathbb{R}^n)}\|g\|_{W^{1,p_2}(\mathbb{R}^n)},\end{aligned}$$

$$\begin{aligned}\|\mathfrak{M}_\alpha(bf,g)\|_{W^{1,q}(\mathbb{R}^n)} &\lesssim_{\alpha,n,p_1,p_2} \|bf\|_{W^{1,p_1}(\mathbb{R}^n)}\|g\|_{W^{1,p_2}(\mathbb{R}^n)}\\ &\lesssim_{\alpha,n,p_1,p_2} \|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{W^{1,p_1}(\mathbb{R}^n)}\|g\|_{W^{1,p_2}(\mathbb{R}^n)}.\end{aligned}$$

The above estimates show that

$$\begin{aligned}\|[\mathfrak{M}_\alpha,b]_1(f,g)\|_{W^{1,q}(\mathbb{R}^n)} &\leqslant \|b\mathfrak{M}_\alpha(f,g)\|_{W^{1,q}(\mathbb{R}^n)}+\|\mathfrak{M}_\alpha(bf,g)\|_{W^{1,q}(\mathbb{R}^n)}\\ &\lesssim_{\alpha,n,p_1,p_2} \|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{W^{1,p_1}(\mathbb{R}^n)}\|g\|_{W^{1,p_2}(\mathbb{R}^n)}.\end{aligned}$$

This proves (8) for $[\mathfrak{M}_{\alpha},b]_1$. The bound (7) follows from Theorem A and Lemma 1.

It remains to prove the continuity of $[\mathfrak{M}_{\alpha},b]_1$. Let $f_j \to f$ in $W^{1,p_1}(\mathbb{R}^n)$ and $g_j \to g$ in $W^{1,p_2}(\mathbb{R}^n)$ as $j\to\infty$. By Lemma 1, one has that $bf_j \to bf$ in $W^{1,p_1}(\mathbb{R}^n)$. This together with Theorem A implies that $\mathfrak{M}_{\alpha}(bf_j,g) \to \mathfrak{M}_{\alpha}(bf,g)$ in $W^{1,q}(\mathbb{R}^n)$ as $j\to\infty$. Moreover, we get from Theorem A that $\mathfrak{M}_{\alpha}(f_j,g_j)\to\mathfrak{M}_{\alpha}(f,g)$ in $W^{1,q}(\mathbb{R}^n)$ as $j\to\infty$. Hence, $[\mathfrak{M}_{\alpha},b]_1(f_j,g_j)\to[\mathfrak{M}_{\alpha},b]_1(f,g)$ in $W^{1,q}(\mathbb{R}^n)$ as $j\to\infty$. This completes the proof of Theorem 1 for $[\mathfrak{M}_{\alpha},b]_1$. □

2.2. Proof of Theorem 1 for $\mathfrak{M}_{b,\alpha}$

In this subsection we prove Theorem 1 for $\mathfrak{M}_{b,\alpha}$. Before presenting the proof, let us introduce some notation. Let $u\in L^p(\mathbb{R}^n)$ with $1<p<\infty$. For all $h\in\mathbb{R}\setminus\{0\}$, $y\in\mathbb{R}^n$ and $i=1,\dots,n$, we define the function $u_{h,i}$ by setting

$$u_{h,i}(x)=\frac{u(x+he_i)-u(x)}{h},\quad x\in\mathbb{R}^n.$$

It is well known that for $u_{h,i}\to D_iu$ in $L^p(\mathbb{R}^n)$ when $h\to 0$ if $u\in W^{1,p}(\mathbb{R}^n)$ for $1<p<\infty$. For $y\in\mathbb{R}^n$ and any arbitrary function u defined on $\mathbb{R}^n$, we define the first order difference of u by

$$\Delta_y u(x):=u_y(x)-u(x),$$

where $u_y(x)=u(x+y)$. We set

$$G(u;p):=\limsup_{|y|\to 0}\frac{\|\Delta_y u\|_{L^p(\mathbb{R}^n)}}{|y|}.$$

According to [6, Section 7.11], we have

$$u\in W^{1,q}(\mathbb{R}^n),\ \ 1<q<\infty \Longleftrightarrow u\in L^q(\mathbb{R}^n)\text{ and }G(u;q)<\infty. \tag{11}$$

In order to prove Theorem 1 for $\mathfrak{M}_{b,\alpha}$, we need the following remark.

REMARK 2. It was shown in [15] that if $b\in \mathrm{Lip}(\mathbb{R}^n)$, the weak partial derivatives D_ib, $i=1,\dots,n$, exist almost everywhere. Moreover, we have

$$D_ib(x)=\lim_{h\to 0}\frac{b(x+he_i)-b(x)}{h}$$

and

$$|D_ib(x)|\leqslant \|b\|_{Lip(\mathbb{R}^n)}$$

for almost every $x\in\mathbb{R}^n$. Here $e_i=(0,\dots,0,1,0,\dots,0)$ is the canonical i-th base vector in $\mathbb{R}^n$ for $i=1,\dots,n$.

Proof of Theorem 1 for $\mathfrak{M}_{b,\alpha}$. The proof of Theorem 1 for $\mathfrak{M}_{b,\alpha}$ will be divided into three steps:

Step 1: Proof of $\mathfrak{M}_{b,\alpha}(f,g)\in W^{1,q}(\mathbb{R}^n)$.

Fix $x, h \in \mathbb{R}^n$, by the sublinearity of $\mathfrak{M}_{b,\alpha}$ and the fact that $\mathfrak{M}_{b,\alpha}(f,g)(x+h) = \mathfrak{M}_{b_h,\alpha}(f_h,g_h)(x)$, we have

$$\begin{aligned}
&|\Delta_h \mathfrak{M}_{b,\alpha}(f,g)(x)| \\
&= |(\mathfrak{M}_{b,\alpha}(f,g))_h(x) - \mathfrak{M}_{b,\alpha}(f,g)(x)| \\
&\leqslant |\mathfrak{M}_{b_h,\alpha}(f_h,g_h)(x) - \mathfrak{M}_{b,\alpha}(f_h,g_h)(x) + \mathfrak{M}_{b,\alpha}(f_h,g_h)(x) - \mathfrak{M}_{b,\alpha}(f,g)(x)| \qquad (12) \\
&\leqslant |\Delta_h b(x)| \mathfrak{M}_\alpha(f_h,g_h)(x) + \mathfrak{M}_\alpha(|\Delta_h b| f_h, g_h)(x) + \mathfrak{M}_{b,\alpha}(\Delta_h f, \Delta_h g)(x) \\
&\quad + \mathfrak{M}_{b,\alpha}(\Delta_h f, g)(x) + \mathfrak{M}_{b,\alpha}(f, \Delta_h g)(x).
\end{aligned}$$

In view of Hölder's inequality, (2) and Remark 1, one has

$$\begin{aligned}
\| |\Delta_h b| \mathfrak{M}_\alpha(f_h,g_h)\|_{L^q(\mathbb{R}^n)} &\leqslant \|b\|_{Lip(\mathbb{R}^n)} |h| \|\mathfrak{M}_\alpha(f_h,g_h)\|_{L^q(\mathbb{R}^n)} \\
&\lesssim_{\alpha,n,p_1,p_2} \|b\|_{Lip(\mathbb{R}^n)} |h| \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}$$

$$\begin{aligned}
\|\mathfrak{M}_\alpha(|\Delta_h b| f_h, g_h)\|_{L^q(\mathbb{R}^n)} &\lesssim_{\alpha,n,p_1,p_2} \|\Delta_h b f_h\|_{L^{p_1}(\mathbb{R}^n)} \|g_h\|_{L^{p_2}(\mathbb{R}^n)} \\
&\lesssim_{\alpha,n,p_1,p_2} \|b\|_{Lip(\mathbb{R}^n)} |h| \|f_h\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} \\
&\lesssim_{\alpha,n,p_1,p_2} \|b\|_{Lip(\mathbb{R}^n)} |h| \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}$$

By (5) and Minkowski's inequality,

$$\begin{aligned}
&\|\mathfrak{M}_{b,\alpha}(\Delta_h f, \Delta_h g) + \mathfrak{M}_{b,\alpha}(\Delta_h f, g) + \mathfrak{M}_{b,\alpha}(f, \Delta_h g)\|_{L^q(\mathbb{R}^n)} \\
&\leqslant \|\mathfrak{M}_{b,\alpha}(\Delta_h f, \Delta_h g)\|_{L^q(\mathbb{R}^n)} + \|\mathfrak{M}_{b,\alpha}(\Delta_h f, g)\|_{L^q(\mathbb{R}^n)} + \|\mathfrak{M}_{b,\alpha}(f, \Delta_h g)\|_{L^q(\mathbb{R}^n)} \\
&\lesssim_{\alpha,n,p_1,p_2} \|b\|_{L^\infty(\mathbb{R}^n)} (\|\Delta_h f\|_{L^{p_1}(\mathbb{R}^n)} \|\Delta_h g\|_{L^{p_2}(\mathbb{R}^n)} + \|\Delta_h f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} \\
&\quad + \|f\|_{L^{p_1}(\mathbb{R}^n)} \|\Delta_h g\|_{L^{p_2}(\mathbb{R}^n)}) \\
&\lesssim_{\alpha,n,p_1,p_2} \|b\|_{L^\infty(\mathbb{R}^n)} (\|\Delta_h f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)} + \|f\|_{L^{p_1}(\mathbb{R}^n)} \|\Delta_h g\|_{L^{p_2}(\mathbb{R}^n)}).
\end{aligned}$$

Since $f \in W^{1,p_1}(\mathbb{R}^n)$ and $g \in W^{1,p_2}(\mathbb{R}^n)$, an application of (11) gives that

$$G(f;p_1) < \infty, \quad G(g;p_2) < \infty.$$

Above estimates together with (12) and Minkowski's inequality imply that

$$\begin{aligned}
&G(\mathfrak{M}_{b,\alpha}(f,g);q) \\
&= \limsup_{h\to 0} \frac{\|\Delta_h(\mathfrak{M}_{b,\alpha}(f,g))\|_{L^q(\mathbb{R}^n)}}{|h|} \\
&\leqslant \limsup_{h\to 0} \frac{1}{|h|} \| |\Delta_h b| \mathfrak{M}_\alpha(f_h,g_h)\|_{L^q(\mathbb{R}^n)} + \limsup_{h\to 0} \frac{1}{|h|} \|\mathfrak{M}_\alpha(|\Delta_h b| f_h, g_h)\|_{L^q(\mathbb{R}^n)} \\
&\quad + \limsup_{h\to 0} \frac{1}{|h|} \|\mathfrak{M}_{b,\alpha}(\Delta_h f, \Delta_h g) + \mathfrak{M}_{b,\alpha}(\Delta_h f, g) + \mathfrak{M}_{b,\alpha}(f, \Delta_h g)\|_{L^q(\mathbb{R}^n)} \\
&\lesssim_{\alpha,n,p_1,p_2} (\|b\|_{\mathrm{Lip}(\mathbb{R}^n)} + G(f;p_1) + G(g;p_2)) < \infty,
\end{aligned}$$

which combining with (5) and (11) leads to $\mathfrak{M}_{b,\alpha}(f,g) \in W^{1,q}(\mathbb{R}^n)$.

Step 2: Proof of (7) *for* $\mathfrak{M}_{b,\alpha}(f,g)$.

To prove (7) for $\mathfrak{M}_{b,\alpha}(f,g)$, it is enough to show that

$$\begin{aligned}
&|D_i(\mathfrak{M}_{b,\alpha}(f,g))(x)| \\
&\leqslant |D_ib(x)|\mathfrak{M}_\alpha(f,g)(x)+\mathfrak{M}_\alpha(|D_ib|f,g)(x)+|b(x)|\mathfrak{M}_\alpha(D_if,g)(x) \\
&\quad +\mathfrak{M}_\alpha(bD_if,g)(x)+|b(x)|\mathfrak{M}_\alpha(f,D_ig)(x)+\mathfrak{M}_\alpha(bf,D_ig)(x)
\end{aligned} \tag{13}$$

for any $i=1,\dots,n$ and almost every $x\in\mathbb{R}^n$.

Fix $i\in\{1,\dots,n\}$, we get by (12) that

$$\begin{aligned}
&|(\mathfrak{M}_{b,\alpha}(f,g))_{h,i}(x)| \\
&\leqslant |b_{h,i}(x)|\mathfrak{M}_\alpha(f_h,g_h)(x)+\mathfrak{M}_\alpha(|b_{h,i}|f_h,g_h)(x)+\mathfrak{M}_{b,\alpha}(f_{h,i},\Delta_hg)(x) \\
&\quad +\mathfrak{M}_{b,\alpha}(f_{h,i},g)(x)+\mathfrak{M}_{b,\alpha}(f,g_{h,i})(x).
\end{aligned} \tag{14}$$

Since $f\in W^{1,p_1}(\mathbb{R}^n)$, $g\in W^{1,p_2}(\mathbb{R}^n)$, $b\in \mathrm{Lip}(\mathbb{R}^n)$ and $\mathfrak{M}_{b,\alpha}(f,g)\in W^{1,q}(\mathbb{R}^n)$, we have that $f_{h,i}\to D_if$ in $L^{p_1}(\mathbb{R}^n)$ and $g_{h,i}\to D_ig$ in $L^{p_2}(\mathbb{R}^n)$ as $h\to 0$, $(\mathfrak{M}_{b,\alpha}(f,g))_{h,i}\to D_i\mathfrak{M}_{b,\alpha}(f,g)$ in $L^q(\mathbb{R}^n)$ as $h\to 0$. By Hölder's inequality, we have that $|b_{h,i}|f_h\to |D_ib|f$ in $L^{p_1}(\mathbb{R}^n)$ when $h\to 0$. Moreover, it is clear that $\|\Delta_hg\|_{L^{p_2}(\mathbb{R}^n)}\to 0$ as $h\to 0$. In view of (3), we see that $\mathfrak{M}_\alpha(f_h,g_h)\to\mathfrak{M}_\alpha(f,g)$ in $L^q(\mathbb{R}^n)$ as $h\to 0$ and $\mathfrak{M}_\alpha(|b_{h,i}|f_h,g_h)\to\mathfrak{M}_\alpha(|D_ib|f,g)$ in $L^q(\mathbb{R}^n)$ as $h\to 0$. In view of (6), we know that $\mathfrak{M}_{b,\alpha}(f_{h,i},\Delta_hg)\to 0$ in $L^q(\mathbb{R}^n)$ as $h\to 0$, $\mathfrak{M}_{b,\alpha}(f_{h,i},g)\to\mathfrak{M}_{b,\alpha}(D_if,g)$ in $L^q(\mathbb{R}^n)$ as $h\to 0$, $\mathfrak{M}_{b,\alpha}(f,g_{h,i})\to\mathfrak{M}_{b,\alpha}(f,D_ig)$ in $L^q(\mathbb{R}^n)$ as $h\to 0$. By Hölder's inequality, one has that $|b_{h,i}|\mathfrak{M}_\alpha(f_h,g_h)\to |D_ib|\mathfrak{M}_\alpha(f,g)$ in $L^q(\mathbb{R}^n)$ as $h\to 0$. Based on the above analyses, there exist a sequence $\{h_k\}_{k\geqslant 1}$ satisfying $h_k>0$ and $\lim_{k\to\infty}h_k=0$ and a measurable set E with $|\mathbb{R}^n\setminus E|=0$ such that for all $x\in E$,

(i) $(\mathfrak{M}_{b,\alpha}(f,g))_{h_k,i}(x)\to D_i(\mathfrak{M}_{b,\alpha}(f,g))(x)$ as $k\to\infty$;
(ii) $|b_{h_k,i}(x)|\mathfrak{M}_\alpha(f_{h_k},g_{h_k})(x)\to |D_ib(x)|\mathfrak{M}_\alpha(f,g)(x)$ as $k\to\infty$;
(iii) $\mathfrak{M}_\alpha(|b_{h_k,i}|f_{h_k},g_{h_k})(x)\to\mathfrak{M}_\alpha(|D_ib|f,g)(x)$ as $k\to\infty$;
(iv) $\mathfrak{M}_{b,\alpha}(f_{h_k,i},\Delta_{h_k}g)(x)\to 0$ as $k\to\infty$;
(v) $\mathfrak{M}_{b,\alpha}(f_{h_k,i},g)(x)\to\mathfrak{M}_{b,\alpha}(D_if,g)(x)$ as $k\to\infty$;
(vi) $\mathfrak{M}_{b,\alpha}(f,g_{h_k,i})(x)\to\mathfrak{M}_{b,\alpha}(f,D_ig)(x)$ as $k\to\infty$.

In view of (14) and the above estimates (i)–(vi), we have that for any $x\in E$,

$$\begin{aligned}
&|D_i(\mathfrak{M}_{b,\alpha}(f,g))(x)| \\
&=|\lim_{k\to\infty}(\mathfrak{M}_{b,\alpha}(f,g))_{h_k,i}(x)| \\
&\leqslant \lim_{k\to\infty}(|b_{h_k,i}(x)|\mathfrak{M}_\alpha(f_{h_k},g_{h_k})(x)+\mathfrak{M}_\alpha(|b_{h_k,i}|f_{h_k},g_{h_k})(x)+\mathfrak{M}_{b,\alpha}(f_{h_k,i},\Delta_{h_k}g)(x) \\
&\quad +\mathfrak{M}_{b,\alpha}(f_{h_k,i},g)(x)+\mathfrak{M}_{b,\alpha}(f,g_{h_k,i})(x)) \\
&=|D_ib(x)|\mathfrak{M}_\alpha(f,g)(x)+\mathfrak{M}_\alpha(|D_ib|f,g)(x)+\mathfrak{M}_{b,\alpha}(D_if,g)(x)+\mathfrak{M}_{b,\alpha}(f,D_ig)(x) \\
&\leqslant |D_ib(x)|\mathfrak{M}_\alpha(f,g)(x)+\mathfrak{M}_\alpha(|D_ib|f,g)(x)+|b(x)|\mathfrak{M}_\alpha(D_if,g)(x) \\
&\quad +\mathfrak{M}_\alpha(bD_if,g)(x)+|b(x)|\mathfrak{M}_\alpha(f,D_ig)(x)+\mathfrak{M}_\alpha(bf,D_ig)(x).
\end{aligned}$$

This proves (13).

Step 3: Proof of (8) *for* $\mathfrak{M}_{b,\alpha}$.

By (2), Minkowski's inequality, Hölder's inequality and (7) for $\mathfrak{M}_{b,\alpha}$, we have

$$\begin{aligned}\|\nabla\mathfrak{M}_{b,\alpha}(f,g)\|_{L^q(\mathbb{R}^n)} &\lesssim_n \|b\|_{Lip(\mathbb{R}^n)}\|\mathfrak{M}_\alpha(f,g)\|_{L^q(\mathbb{R}^n)} \\ &\quad + \|b\|_{L^\infty(\mathbb{R}^n)}(\|\mathfrak{M}_\alpha(|\nabla f|,g)\|_{L^q(\mathbb{R}^n)} + \|\mathfrak{M}_\alpha(f,|\nabla g|)\|_{L^q(\mathbb{R}^n)}) \\ &\lesssim_{\alpha,n,p_1,p_2} \|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{W^{1,p_1}(\mathbb{R}^n)}\|g\|_{W^{1,p_2}(\mathbb{R}^n)}.\end{aligned}$$

which together with (5) leads to (8) for $\mathfrak{M}_{b,\alpha}$. Then Theorem 1 is proved. □

3. Proof of Theorem 2

This section is devoted to proving Theorem 2. Throughout this section, let $f \in F_s^{p_1,q}(\mathbb{R}^n)$, $g \in F_s^{p_2,q}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$, where $1 < p_1,p_2,p,q < \infty$ and $1/p = 1/p_1 + 1/p_2$.

3.1. Proof of Theorem 2 for $[\mathfrak{M},b]_i$ $(i=1,2)$

In order to prove Theorem 2 for $[\mathfrak{M},b]_i$ $(i=1,2)$, we need the following result, which provides a characterization of the product of a Triebel–Lizorkin function and a Lipschitz function.

LEMMA 2. ([4]). *Let* $0 < s < 1$ *and* $1 < p,q < \infty$. *If* $u \in F_s^{p,q}(\mathbb{R}^n)$ *and* $b \in \mathrm{Lip}(\mathbb{R}^n)$*, then* $bu \in F_s^{p,q}(\mathbb{R}^n)$. *Moreover,*

$$\|bu\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|u\|_{F_s^{p,q}(\mathbb{R}^n)}.$$

Proof of Theorem 2 *for* $[\mathfrak{M},b]_i$ $(i=1,2)$. We consider the case $i=1$ since another one is analogous. It is not difficult to check that

$$\|[\mathfrak{M},b]_1(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant \|\mathfrak{M}(bf,g)\|_{F_s^{p,q}(\mathbb{R}^n)} + \|b\mathfrak{M}(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)}.$$

In view of Lemma 2, one has $bf \in F_s^{p_1,q}(\mathbb{R}^n)$. By Theorem B(i) and Lemma 2, we have

$$\begin{aligned}\|\mathfrak{M}(bf,g)\|_{F_s^{p,q}(\mathbb{R}^n)} &\leqslant C\|bf\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)} \\ &\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)};\end{aligned}$$

$$\begin{aligned}\|b\mathfrak{M}(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} &\leqslant \|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|\mathfrak{M}(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} \\ &\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.\end{aligned}$$

These estimates together imply that

$$\|[\mathfrak{M},b]_1(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.$$

This proves (9) for $[\mathfrak{M},b]_1$.

It remains to prove the continuity of $[\mathfrak{M},b]_1$. Let $f_j \to f$ in $F_s^{p_1,q}(\mathbb{R}^n)$ and $g_j \to g$ in $F_s^{p_2,q}(\mathbb{R}^n)$ as $j \to \infty$. By Lemma 2, one has that $bf_j \to bf$ in $F_s^{p_1,q}(\mathbb{R}^n)$. This together with Theorem B(i) implies that $\mathfrak{M}(bf_j,g_j) \to \mathfrak{M}(bf,g)$ in $F_s^{p,q}(\mathbb{R}^n)$ as $j \to \infty$. Invoking Theorem B(i), we have that $\mathfrak{M}(f_j,g_j) \to \mathfrak{M}(f,g)$ in $F_s^{p,q}(\mathbb{R}^n)$ as $j \to \infty$. Combining this with Lemma 2 gives that $b\mathfrak{M}(f_j,g_j) \to b\mathfrak{M}(f,g)$ in $F_s^{p,q}(\mathbb{R}^n)$ as $j \to \infty$. From the above facts we see that $[\mathfrak{M},b]_1(f_j,g_j) \to [\mathfrak{M},b]_1(f,g)$ in $F_s^{p,q}(\mathbb{R}^n)$ as $j \to \infty$. This completes the proof of Theorem 2 for $[\mathfrak{M},b]_1$. $\square$

3.2. Proof of Theorem 2 for $\mathfrak{M}_b$

In what follows, let $\mathfrak{R}_n = \{\zeta \in \mathbb{R}^n; 1/2 < |\zeta| \leqslant 1\}$. For a measurable function $g : \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n \to \mathbb{R}$, we set

$$\|g\|_{p,q,r,s} := \Big\| \Big(\sum_{k\in\mathbb{Z}} 2^{ksq} \Big(\int_{\mathfrak{R}_n} |g(x,k,\zeta)|^r d\zeta \Big)^{q/r} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}.$$

Denote by $\dot{F}_s^{p,q}(\mathbb{R}^n)$ the homogeneous Triebel–Lizorkin spaces. In [25], Yabuta observed that

$$\|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} \sim \|\Delta_{2^{-k}\zeta} f\|_{p,q,r,s}, \quad \text{for } 0<s<1,\ 1<p<\infty,\ \ 1<q\leqslant\infty,\ \ 1\leqslant r<\min\{p,q\}. \tag{15}$$

The following properties of Triebel–Lizorkin spaces are valid (see [5, 7, 24]):

$$\|f\|_{F_s^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{for } s>0,\ 1<p,q<\infty, \tag{16}$$

$$\|f\|_{F_{s_1}^{p,q}(\mathbb{R}^n)} \leqslant \|f\|_{F_{s_2}^{p,q}(\mathbb{R}^n)}, \quad \text{for } s_1 \leqslant s_2,\ 0<p,\ q<\infty, \tag{17}$$

$$\|f\|_{F_s^{p,q_2}(\mathbb{R}^n)} \leqslant \|f\|_{F_s^{p,q_1}(\mathbb{R}^n)}, \quad \text{for } s\in\mathbb{R},\ 0<p<\infty,\ 0<q_1\leqslant q_2<\infty. \tag{18}$$

To prove Theorem 2 for $\mathfrak{M}_b$, the following lemma is useful.

LEMMA 3. ([14]) *For a fixed $\delta > 1$, we have that for any $\delta < p,q,r < \infty$,*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} \|M_\delta(f_{k,\zeta})\|^q_{L^r(\mathfrak{R}_n)} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \lesssim_{p,q,r} \Big\| \Big(\sum_{k\in\mathbb{Z}} \|f_{k,\zeta}\|^q_{L^r(\mathfrak{R}_n)} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}.$$

Proof of Theorem 2 for $\mathfrak{M}_b$. The proof will be divided into two steps.

Step 1: Proof of the boundedness part. For any $\zeta \in \mathfrak{R}_n$, $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, we get from (12) that

$$\begin{aligned} &|\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f,g)(x)| \\ &\leqslant |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f_{2^{-k}\zeta},g_{2^{-k}\zeta})(x) + \mathfrak{M}(\Delta_{2^{-k}\zeta}bf_{2^{-k}\zeta},g_{2^{-k}\zeta})(x) \\ &\quad + \mathfrak{M}_b(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x) + \mathfrak{M}_b(\Delta_{2^{-k}\zeta}f,g)(x) + \mathfrak{M}_b(f,\Delta_{2^{-k}\zeta}g)(x). \end{aligned} \tag{19}$$

Let $t \in (1,\infty)$ be such that $t' \in (p_2/p, \min\{p_2, qp_2/p\})$. Clearly, $p_1 > p_1/p > t$ and $qp_1/p > t$. In view of (15), (19) and Minkowski's inequality, we write

$$\begin{aligned}
&\|\mathfrak{M}_b(f,g)\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f,g))|d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f_{2^{-k}\zeta},g_{2^{-k}\zeta})d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad +C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathfrak{M}(\Delta_{2^{-k}\zeta}bf_{2^{-k}\zeta},g_{2^{-k}\zeta})d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad +C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathfrak{M}_b(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad +C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathfrak{M}_b\left(\Delta_{2^{-k}\zeta}f,g\right)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad +C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathfrak{M}_b\left(f,\Delta_{2^{-k}\zeta}g\right)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&=: \sum_{i=1}^5 A_i.
\end{aligned} \tag{20}$$

By Hölder's inequality, one obtains

$$\mathfrak{M}(f,g)(x) \leqslant M_\tau(f)(x)M_{\tau'}(g)(x), \quad \forall \tau > 1. \tag{21}$$

Estimate for A_1. By the sublinearity of $\mathfrak{M}$, we have

$$\begin{aligned}
\mathfrak{M}(f_{2^{-k}\zeta},g_{2^{-k}\zeta})(x) \leqslant \mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x)\\
+\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x)+\mathfrak{M}(f,g)(x).
\end{aligned} \tag{22}$$

It follows that

$$\begin{aligned}
A_1 &\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad +C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad +C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad +C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f,g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&=: A_{1,1}+A_{1,2}+A_{1,3}+A_{1,4}.
\end{aligned} \tag{23}$$

By (21) and Hölder's inequality,

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}M_t(\Delta_{2^{-k}\zeta}f)M_{t'}(\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\|M_t(\Delta_{2^{-k}\zeta}f)\|^q_{L^{p_1/p}(\mathfrak{R}_n)}\|M_{t'}(\Delta_{2^{-k}\zeta}g)\|^q_{L^{p_2/p}(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_1}\|M_t(\Delta_{2^{-k}\zeta}f)\|_{L^{p_1/p}(\mathfrak{R}_n)})^{qp_1/p}\Big)^{p/(qp_1)}\Big\|_{L^{p_1}(\mathbb{R}^n)}\\
&\quad\times\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|M_{t'}(\Delta_{2^{-k}\zeta}g)\|_{L^{p_2/p}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\Big\|_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}\tag{24}$$

Notice that $p_1>p_1/p>t>1$ and $qp_1/p>t$. In view of (15)–(18) and Lemma 3,

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_1}\|M_t(\Delta_{2^{-k}\zeta}f)\|_{L^{p_1/p}(\mathfrak{R}_n)})^{qp_1/p}\Big)^{p/(qp_1)}\Big\|_{L^{p_1}(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_1}\|\Delta_{2^{-k}\zeta}f\|_{L^{p_1/p}(\mathfrak{R}_n)})^{qp_1/p}\Big)^{p/(qp_1)}\Big\|_{L^{p_1}(\mathbb{R}^n)}\\
&\leqslant C\|f\|_{\dot F^{p_1,qp_1/p}_{sp/p_1}(\mathbb{R}^n)}\leqslant C\|f\|_{F^{p_1,qp_1/p}_{sp/p_1}(\mathbb{R}^n)}\leqslant C\|f\|_{F^{p_1,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{25}$$

Clearly, $t'\in(p_2/p,\min\{p_2,qp_2/p\})$. Let $\alpha\in(t',\min\{p_2,qp_2/p\})$. In view of (15)–(18), Lemma 3 and Hölder's inequality, we obtain

$$\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|M_{t'}(\Delta_{2^{-k}\zeta}g)\|_{L^{p_2/p}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\Big\|_{L^{p_2}(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|M_{t'}(\Delta_{2^{-k}\zeta}g)\|_{L^{\alpha}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\Big\|_{L^{p_2}(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|\Delta_{2^{-k}\zeta}g\|_{L^{\alpha}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\Big\|_{L^{p_2}(\mathbb{R}^n)}\\
&\leqslant C\|g\|_{\dot F^{p_2,qp_2/p}_{sp/p_2}(\mathbb{R}^n)}\leqslant C\|g\|_{F^{p_2,qp_2/p}_{sp/p_2}(\mathbb{R}^n)}\leqslant C\|g\|_{F^{p_2,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{26}$$

It follows from (24)–(26) that

$$A_{1,1}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F^{p_1,q}_s(\mathbb{R}^n)}\|g\|_{F^{p_2,q}_s(\mathbb{R}^n)}.\tag{27}$$

By (15)–(18), (21), (25), Lemma 3 and Hölder's inequality, we have

$$\begin{aligned}
A_{1,2} &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}|\Delta_{2^{-k}\zeta}b|M_t(\Delta_{2^{-k}\zeta}f)M_{t'}(g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_1}\|M_t(\Delta_{2^{-k}\zeta}f)\|_{L^{p_1/p}(\mathfrak{R}_n)})^{qp_1/p}\Big)^{p/(qp_1)}\Big\|_{L^{p_1}(\mathbb{R}^n)}\\
&\quad\times\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|\Delta_{2^{-k}\zeta}bM_{t'}(g)\|_{L^{p_2/p}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\Big\|_{L^{p_2}(\mathbb{R}^n)}\\
&\leqslant C\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\\
&\quad\times\Big\|M_{t'}(g)\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|\Delta_{2^{-k}\zeta}b\|_{L^{p_2/p}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\Big\|_{L^{p_2}(\mathbb{R}^n)}
\end{aligned}\tag{28}$$

Note that

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|\Delta_{2^{-k}\zeta}b\|_{L^{p_2/p}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\\
&\leqslant\Big(\sum_{k=-\infty}^{0}(2^{ksp/p_2}\|\Delta_{2^{-k}\zeta}b\|_{L^{p_2/p}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\\
&\quad+\Big(\sum_{k=1}^{\infty}(2^{ksp/p_2}\|\Delta_{2^{-k}\zeta}b\|_{L^{p_2/p}(\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0}2^{ksq}\Big)^{p/(qp_2)}+C\|b\|_{Lip(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty}2^{kq(s-1)}\Big)^{p/(qp_2)}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}.
\end{aligned}\tag{29}$$

Combining (28) with (29) and (16) implies that

$$A_{1,2}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.\tag{30}$$

Similar arguments to (30) show that

$$A_{1,3}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.\tag{31}$$

An argument similar to (29) shows that for any $\tau\geqslant 1$,

$$\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\|\Delta_{2^{-k}\zeta}b\|^q_{L^\tau(\mathfrak{R}_n)}\Big)^{1/q}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}.\tag{32}$$

Combining (33) with (1) and (16) implies that

$$\begin{aligned}
A_{1,4}&\leqslant C\Big\|\mathfrak{M}(f,g)\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\|\Delta_{2^{-k}\zeta}b\|^q_{L^1(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|\mathfrak{M}(f,g)\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.
\end{aligned}\tag{33}$$

It follows from (23), (27), (30), (31) and (33) that

$$A_1 \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{34}$$

Estimate for A_2. By the sublinearity of $\mathfrak{M}$, we have

$$\begin{aligned}
&\mathfrak{M}(\Delta_{2^{-k}\zeta}bf_{2^{-k}\zeta},g_{2^{-k}\zeta})(x)\\
&\leqslant \mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}f,g)(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,\Delta_{2^{-k}\zeta}g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,g)(x).
\end{aligned} \tag{35}$$

It follows from (35) that

$$\begin{aligned}
A_2 &\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad+C\|b\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad+C\|b\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\quad+C\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&=: A_{2,1}+A_{2,2}+A_{2,3}+A_{2,4}.
\end{aligned} \tag{36}$$

For $A_{2,1}$, we get from (27) that

$$A_{2,1} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{37}$$

For $A_{2,2}$, we can choose $\delta>1$ such that $\delta<\min\{p_1,q\}$ and $\delta'<p_2$. Let $\beta\in(\delta,\min\{p_1,q\})$. In view of (15), (16), (21), Hölder's inequality and Lemma 3, we have

$$\begin{aligned}
A_{2,2} &\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}M_\delta(\Delta_{2^{-k}\zeta}f)M_{\delta'}(g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\|M_{\delta'}(g)\|_{L^{p_2}(\mathbb{R}^n)}\\
&\qquad\times\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\|M_\delta(\Delta_{2^{-k}\zeta}f)\|_{L^1(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^{p_1}(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\|M_\delta(\Delta_{2^{-k}\zeta}f)\|_{L^\beta(\mathfrak{R}_n)}^q\Big)^{1/q}\Big\|_{L^{p_1}(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}}(2^{ks}\|\Delta_{2^{-k}\zeta}f\|_{L^\beta(\mathfrak{R}_n)})^q\Big)^{1/q}\Big\|_{L^{p_1}(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\|f\|_{\dot F_s^{p_1,q}(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}.
\end{aligned} \tag{38}$$

Similar argument to (38) shows that

$$A_{2,3} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{39}$$

For $A_{2,4}$, let δ, β be given as in (38). By (16), (21), (32), Hölder's inequality and Lemma 3, we have

$$\begin{aligned} A_{2,4} &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} M_\delta(\Delta_{2^{-k}\zeta}bf)M_{\delta'}(g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C\|M_{\delta'}(g)\|_{L^{p_2}(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\|M_\delta(\Delta_{2^{-k}\zeta}bf)\|^q_{L^1(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^{p_1}(\mathbb{R}^n)} \\ &\leqslant C\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\|M_\delta(\Delta_{2^{-k}\zeta}bf)\|^q_{L^\beta(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^{p_1}(\mathbb{R}^n)} \\ &\leqslant C\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big\|f\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\|\Delta_{2^{-k}\zeta}b\|^q_{L^\beta(\mathfrak{R}_n)}\Big)^{1/q}\Big\|_{L^{p_1}(\mathbb{R}^n)} \\ &\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \end{aligned} \tag{40}$$

It follows from (36)–(40) that

$$A_2 \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{41}$$

Estimate for A_3. By (27), we have

$$\begin{aligned} A_3 \leqslant C\Big\|&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}(b(x)+\|b\|_{L^\infty(\mathbb{R}^n)}) \\ &\times \mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ \leqslant\,& C\|b\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ \leqslant\,& C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \end{aligned} \tag{42}$$

Estimate for A_4. In view of (38),

$$\begin{aligned} A_4 &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}(b(x)+\|b\|_{L^\infty(\mathbb{R}^n)})\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \end{aligned} \tag{43}$$

Estimate for A_5. Applying (39), we have

$$\begin{aligned} A_5 &\leqslant C\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}(b(x)+\|b\|_{L^\infty(\mathbb{R}^n)})\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big\|\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \\ &\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \end{aligned} \tag{44}$$

Finally, it follows from (20), (34) and (41)–(44) that

$$\|\mathfrak{M}_b(f,g)\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{45}$$

Combining (45) with (5) and (16) implies (9) for $\mathfrak{M}_b$.

Step 2: Proof of the continuity part.

Let $f_j \to f$ in $F_s^{p_1,q}(\mathbb{R}^n)$ and $g_j \to g$ in $F_s^{p_2,q}(\mathbb{R}^n)$ as $j \to \infty$. In view of (6), it suffices to prove that

$$\|\mathfrak{M}_b(f_j,g_j) - \mathfrak{M}_b(f,g)\|_{F_s^{p,q}(\mathbb{R}^n)} \to 0 \text{ as } j \to \infty. \tag{46}$$

Next we shall prove (46) by contradiction. If (46) is not true, without loss of generality we may assume that there exists $\gamma > 0$ such that

$$\|\mathfrak{M}_b(f_j,g_j) - \mathfrak{M}_b(f,g)\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} > \gamma, \text{ for all } j \geqslant 1. \tag{47}$$

Since $\mathfrak{M}_b(f_j,g_j) \to \mathfrak{M}_b(f,g)$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$, we may assume without loss of generality (by extracting a subsequence) that $|\mathfrak{M}_b(f_j,g_j)(x) - \mathfrak{M}_b(f,g)(x)| \to 0$ as $j \to \infty$ for almost every $x \in \mathbb{R}^n$. It follows that

$$\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_j,g_j) - \mathfrak{M}_b(f,g))(x) \to 0 \text{ as } j \to \infty \tag{48}$$

for every $(k,\zeta) \in \mathbb{Z} \times \mathfrak{R}_n$ and almost every $x \in \mathbb{R}^n$. By (12), we have that for $(x,k,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n$,

$$\begin{aligned}
&|\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f,g)(x)| \\
&\leqslant |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x) + |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x) \\
&\quad + |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x) + |\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f,g)(x) \\
&\quad + \mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x) + \mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}f,g)(x) \\
&\quad + \mathfrak{M}(\Delta_{2^{-k}\zeta}bf,\Delta_{2^{-k}\zeta}g)(x) + \mathfrak{M}(\Delta_{2^{-k}\zeta}bf,g)(x) \\
&\quad + |b(x)|\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x) + \mathfrak{M}(b\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x) \\
&\quad + |b(x)|\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x) + \mathfrak{M}(b\Delta_{2^{-k}\zeta}f,g)(x) \\
&\quad + |b(x)|\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x) + \mathfrak{M}(bf,\Delta_{2^{-k}\zeta}g)(x) \\
&=: \varphi_{f,g}(x,k,\zeta).
\end{aligned} \tag{49}$$

It is clear that

$$|\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f_j,g_j)(x)| \leqslant \varphi_{f_j,g_j}(x,k,\zeta).$$

Note that

$$
\begin{aligned}
&|\varphi_{f_j,g_j}(x,k,\zeta)-\varphi_{f,g}(x,k,\zeta)|\\
&\leqslant (|\Delta_{2^{-k}\zeta}b|+|b(x)|)\Big(\mathfrak{M}(\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}(g_j-g))(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}(g_j-g))(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}(f_j-f),g_j-g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}(f_j-f),g)(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g_j-g)(x)+\mathfrak{M}(f_j-f,\Delta_{2^{-k}\zeta}(g_j-g))(x)\\
&\quad+\mathfrak{M}(f_j-f,\Delta_{2^{-k}\zeta}g)(x)+\mathfrak{M}(f,\Delta_{2^{-k}\zeta}(g_j-g))(x)\Big)\\
&\quad+|\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f_j-f,g_j-g)(x)+|\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f_j-f,g)(x)\\
&\quad+|\Delta_{2^{-k}\zeta}b|\mathfrak{M}(f,g_j-g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}(g_j-g))(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}g)(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}(g_j-g)(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}(f_j-f),g_j-g)(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}(f_j-f),g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}f,g_j-g)(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}b(f_j-f),\Delta_{2^{-k}\zeta}(g_j-g))(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}b(f_j-f),\Delta_{2^{-k}\zeta}g)(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,\Delta_{2^{-k}\zeta}(g_j-g))(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}b(f_j-f),g_j-g)(x)\\
&\quad+\mathfrak{M}(\Delta_{2^{-k}\zeta}b(f_j-f),g)(x)+\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,g_j-g)(x)\\
&\quad+\mathfrak{M}(b\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}(g_j-g))(x)+\mathfrak{M}(b\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}g)(x)\\
&\quad+\mathfrak{M}(b\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}(g_j-g))(x)+\mathfrak{M}(b\Delta_{2^{-k}\zeta}(f_j-f),(g_j-g))(x)\\
&\quad+\mathfrak{M}(b\Delta_{2^{-k}\zeta}(f_j-f),g)(x)+\mathfrak{M}(b\Delta_{2^{-k}\zeta}f,g_j-g)(x)\\
&\quad+\mathfrak{M}(b(f_j-f),\Delta_{2^{-k}\zeta}(g_j-g))(x)+\mathfrak{M}(b(f_j-f),\Delta_{2^{-k}\zeta}g)(x)\\
&\quad+\mathfrak{M}(bf,\Delta_{2^{-k}\zeta}(g_j-g))(x)\\
&\quad=:\Phi_j(x,k,\zeta)
\end{aligned}
\tag{50}
$$

From (49) and (50) we see that

$$
\begin{aligned}
&|\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_j,g_j)-\mathfrak{M}_b(f,g))(x)|\\
&=|\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f_j,g_j)(x)-\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f,g)(x)|\\
&\leqslant \varphi_{f_j,g_j}(x,k,\zeta)+\varphi_{f,g}(x,k,\zeta)\leqslant \Phi_j(x,k,\zeta)+2\varphi_{f,g}(x,k,\zeta).
\end{aligned}
\tag{51}
$$

for all $(x,k,\zeta)\in\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n$. An argument similar to (45) gives that

$$
\begin{aligned}
\|\Phi_j\|_{p,q,1,s}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}(&\|f_j-f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g_j-g\|_{F_s^{p_2,q}(\mathbb{R}^n)}\\
&+\|f_j-f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}+\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g_j-g\|_{F_s^{p_2,q}(\mathbb{R}^n)}),
\end{aligned}
\tag{52}
$$

$$
\|2\varphi_{f,g}\|_{p,q,1,s}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{F_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{F_s^{p_2,q}(\mathbb{R}^n)}. \tag{53}
$$

In view of (52), we can find a subsequence $\{j_\ell\}_{\ell=1}^{\infty} \subset \{j\}_{j=1}^{\infty}$ such that

$$\sum_{\ell=1}^{\infty} \|\Phi_{j_\ell}\|_{p,q,1,s} < \infty. \tag{54}$$

It follows from (51) that

$$\begin{aligned} &|\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_{j_\ell}, g_{j_\ell}) - \mathfrak{M}_b(f,g))(x)| \\ &\leqslant \sum_{\ell=1}^{\infty} \Phi_{j_\ell}(x,k,\zeta) + 2\varphi_{f,g}(x,k,\zeta) =: \Gamma(x,k,\zeta), \end{aligned} \tag{55}$$

for all $(x,k,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n$. Then we get by (54), (55) and Minkowski's inequality that $\|\Gamma\|_{p,q,1,s} < \infty$. It follows that $\int_{\mathfrak{R}_n} \Gamma(x,k,\zeta)d\zeta < \infty$ for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$. Then we use (48), (55) and the dominated convergence theorem to get

$$\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_{j_\ell}, g_{j_\ell}) - \mathfrak{M}_b(f,g))(x)|d\zeta \to 0 \text{ as } \ell \to \infty \tag{56}$$

for every $k \in \mathbb{Z}$ and almost every $x \in \mathbb{R}^n$. By the fact that $\|\Gamma\|_{p,q,1,s} < \infty$ again

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} \Gamma(x,k,\zeta)d\zeta\Big)^q\Big)^{1/q} < \infty \tag{57}$$

for almost every $x \in \mathbb{R}^n$. In view of (55),

$$\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_{j_\ell}, g_{j_\ell}) - \mathfrak{M}_b(f,g))(x)|d\zeta \leqslant \int_{\mathfrak{R}_n} \Gamma(x,k,\zeta)d\zeta, \tag{58}$$

for all $(x,k,\zeta) \in \mathbb{R}^n \times \mathbb{Z} \times \mathfrak{R}_n$ and $\ell \geqslant 1$. By (56)–(58) and the dominated convergence theorem,

$$\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_{j_\ell}, g_{j_\ell}) - \mathfrak{M}_b(f,g))(x)|d\zeta\Big)^q\Big)^{1/q} \to 0 \text{ as } \ell \to \infty. \tag{59}$$

By (55) again and the fact that $\|\Gamma\|_{p,q,1,s} < \infty$,

$$\begin{aligned} &\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_{j_\ell}, g_{j_\ell}) - \mathfrak{M}_b(f,g))(x)|d\zeta\Big)^q\Big)^{1/q} \\ &\leqslant \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n} |\Gamma(x,k,\zeta)|d\zeta\Big)^q\Big)^{1/q} < \infty, \end{aligned} \tag{60}$$

for almost every $x \in \mathbb{R}^n$. By (59), (60) and the dominated convergence theorem,

$$\|\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_{j_\ell}, g_{j_\ell}) - \mathfrak{M}_b(f,g))\|_{p,q,1,s} \to 0 \text{ as } \ell \to \infty.$$

This leads to $\|\mathfrak{M}_b(f_{j_\ell}, g_{j_\ell}) - \mathfrak{M}_b(f,g)\|_{\dot{F}_s^{p,q}(\mathbb{R}^n)} \to 0$ as $\ell \to \infty$, which contradicts with (47). Theorem 2 in now proved. $\square$

4. Proof of Theorem 3

In this section we prove Theorem 3. Throughout this section let $f \in B_s^{p_1,q}(\mathbb{R}^n)$, $g \in B_s^{p_2,q}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$, where $1 < p_1, p_2, p, q < \infty$ and $1/p = 1/p_1 + 1/p_2$.

4.1. Proof of Theorem 3 for $[\mathfrak{M}, b]_i$ $(i = 1,2)$

To prove Theorem 3 for $[\mathfrak{M}, b]_i$ $(i = 1,2)$, we need the following Lemma.

LEMMA 4. ([4]). *Let $0 < s < 1$ and $1 < p,q < \infty$. If $u \in B_s^{p,q}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$, then $bu \in B_s^{p,q}(\mathbb{R}^n)$. Moreover,*

$$\|bu\|_{B_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|u\|_{B_s^{p,q}(\mathbb{R}^n)}.$$

Proof of Theorem 3 for $[\mathfrak{M}, b]_i$ $(i = 1,2)$. We only consider the case $i = 1$ since another one is analogous. One can easily check that

$$\|[\mathfrak{M}, b]_1(f,g)\|_{B_s^{p,q}(\mathbb{R}^n)} \leqslant \|\mathfrak{M}(bf,g)\|_{B_s^{p,q}(\mathbb{R}^n)} + \|b\mathfrak{M}(f,g)\|_{B_s^{p,q}(\mathbb{R}^n)}.$$

In view of Lemma 4 and Theorem B(ii), one has

$$\begin{aligned}\|\mathfrak{M}(bf,g)\|_{B_s^{p,q}(\mathbb{R}^n)} &\leqslant C\|bf\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}\\ &\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)};\end{aligned}$$

$$\begin{aligned}\|b\mathfrak{M}(f,g)\|_{B_s^{p,q}(\mathbb{R}^n)} &\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|\mathfrak{M}(f,g)\|_{B_s^{p,q}(\mathbb{R}^n)}\\ &\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}.\end{aligned}$$

Thus, we have

$$\|[\mathfrak{M}, b]_1(f,g)\|_{B_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}.$$

This proves (10) for $[\mathfrak{M}, b]_1$.

Next we prove the continuity of $[\mathfrak{M}, b]_1$. Let $f_j \to f$ in $B_s^{p_1,q}(\mathbb{R}^n)$ and $g_j \to g$ in $B_s^{p_2,q}(\mathbb{R}^n)$ as $j \to \infty$. By Lemma 4, one sees that $bf_j \to bf$ in $B_s^{p_1,q}(\mathbb{R}^n)$. This together with Theorem B(ii) implies that $\mathfrak{M}(bf_j, g_j) \to \mathfrak{M}(bf,g)$ in $B_s^{p,q}(\mathbb{R}^n)$ as $j \to \infty$. Theorem B(ii) also gives that $\mathfrak{M}(f_j,g_j) \to \mathfrak{M}(f,g)$ in $B_s^{p,q}(\mathbb{R}^n)$ as $j \to \infty$. This together with Lemma 4 implies that $b\mathfrak{M}(f_j,g_j) \to b\mathfrak{M}(f,g)$ in $B_s^{p,q}(\mathbb{R}^n)$ as $j \to \infty$. From the above facts we see that $[\mathfrak{M}, b]_1(f_j,g_j) \to [\mathfrak{M}, b]_1(f,g)$ in $B_s^{p,q}(\mathbb{R}^n)$ as $j \to \infty$. This completes the proof of Theorem 3 for $[\mathfrak{M}, b]_1$. □

4.2. Proof of Theorem 3 for $\mathfrak{M}_b$

We denote the homogeneous Besov spaces by $\dot{B}_s^{p,q}(\mathbb{R}^n)$. It is well known that

$$\|f\|_{B_s^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}, \quad \text{for } s > 0,\ 1 < p,q < \infty, \tag{61}$$

$$\|f\|_{B^{p,q}_{s_1}(\mathbb{R}^n)} \leqslant \|f\|_{B^{p,q}_{s_2}(\mathbb{R}^n)}, \quad \text{for } s_1 \leqslant s_2,\ 0<p,q<\infty. \tag{62}$$

$$\|f\|_{B^{p,q_2}_{s}(\mathbb{R}^n)} \leqslant \|f\|_{B^{p,q_1}_{s}(\mathbb{R}^n)}, \quad \text{for } s\in\mathbb{R},\ 0<q_1 \leqslant q_2<\infty,\ 0<p<\infty. \tag{63}$$

It was observed by Yabuta [25] that for $0<s<1$, $1\leqslant p<\infty$, $1\leqslant q\leqslant\infty$ and $1\leqslant r\leqslant p$,

$$\|f\|_{\dot{B}^{p,q}_{s}(\mathbb{R}^n)} \sim \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big\|\Big(\int_{\mathfrak{R}_n} |\Delta_{2^{-k}\zeta} f|^r d\zeta\Big)^{1/r}\Big\|^q_{L^p(\mathbb{R}^n)}\Big)^{1/q}. \tag{64}$$

For a measurable function $g:\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n\to\mathbb{R}$, we define

$$\|g\|_{p,q,s} := \Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |g(x,k,\zeta)|^p dx d\zeta\Big)^{q/p}\Big)^{1/q}.$$

By (64) and Fubini's theorem, one gets

$$\|f\|_{\dot{B}^{p,q}_{s}(\mathbb{R}^n)} \sim \|\Delta_{2^{-k}\zeta} f\|_{p,q,s}, \quad \text{for } 0<s<1, 1\leqslant p<\infty, 1\leqslant q\leqslant\infty. \tag{65}$$

Proof of Theorem 3 for $\mathfrak{M}_b$. We divide the proof of Theorem 3 for $\mathfrak{M}_b$ into two steps.

Step 1: Proof of the boundedness part.

By (65) and (19), we write

$$\begin{aligned}
&\|\mathfrak{M}_b(f,g)\|_{\dot{B}^{p,q}_{s}(\mathbb{R}^n)}\\
&\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f,g)(x)|^p dx d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} |\Delta_{2^{-k}\zeta} b\mathfrak{M}(f_{2^{-k}\zeta},g_{2^{-k}\zeta})(x)|^p dx d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad + C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}(\Delta_{2^{-k}\zeta} b f_{2^{-k}\zeta},g_{2^{-k}\zeta})(x))^p dx d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad + C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}_b(\Delta_{2^{-k}\zeta} f,\Delta_{2^{-k}\zeta} g)(x))^p dx d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad + C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}_b(\Delta_{2^{-k}\zeta} f,g)(x))^p dx d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad + C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}_b(f,\Delta_{2^{-k}\zeta} g)(x))^p dx d\zeta\Big)^{q/p}\Big)^{1/q}\\
&=: \sum_{i=1}^{5} B_i.
\end{aligned} \tag{66}$$

Estimate for B_1. By (22), we can write

$$\begin{aligned}
B_1 &\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}(\Delta_{2^{-k}\zeta} f,\Delta_{2^{-k}\zeta} g)(x))^p dx d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad + C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n} (\mathfrak{M}(\Delta_{2^{-k}\zeta} f,g)(x))^p dx d\zeta\Big)^{q/p}\Big)^{1/q}
\end{aligned} \tag{67}$$

$$+C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}$$
$$+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}b\mathfrak{M}(f,g)(x)|^p dxd\zeta\Big)^{q/p}\Big)^{1/q}.$$

By Hölder's inequality, (1) and (61)–(65), one has

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\|\Delta_{2^{-k}\zeta}f\|^p_{L^{p_1}(\mathbb{R}^n)}\|\Delta_{2^{-k}\zeta}g\|^p_{L^{p_2}(\mathbb{R}^n)}d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\|\Delta_{2^{-k}\zeta}f\|^q_{L^{p_1}(\mathbb{R}^n\times\mathfrak{R}_n)}\|\Delta_{2^{-k}\zeta}g\|^q_{L^{p_2}(\mathbb{R}^n\times\mathfrak{R}_n)}\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_1}\|\Delta_{2^{-k}\zeta}f\|_{L^{p_1}(\mathbb{R}^n\times\mathfrak{R}_n)})^{qp_1/p}\Big)^{p/(qp_1)}\\
&\quad\times\Big(\sum_{k\in\mathbb{Z}}(2^{ksp/p_2}\|\Delta_{2^{-k}\zeta}g\|_{L^{p_2}(\mathbb{R}^n\times\mathfrak{R}_n)})^{qp_2/p}\Big)^{p/(qp_2)}\\
&\leqslant C\|f\|_{\dot{B}^{p_1,qp_1/p}_{sp/p_1}(\mathbb{R}^n)}\|g\|_{\dot{B}^{p_2,qp_2/p}_{sp/p_2}(\mathbb{R}^n)}\\
&\leqslant C\|f\|_{B^{p_1,qp_1/p}_{sp/p_1}(\mathbb{R}^n)}\|g\|_{B^{p_2,qp_2/p}_{sp/p_2}(\mathbb{R}^n)}\\
&\leqslant C\|f\|_{B^{p_1,q}_s(\mathbb{R}^n)}\|g\|_{B^{p_2,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{68}$$

In view of (1), (61) and (65),

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\|\Delta_{2^{-k}\zeta}f\|^p_{L^{p_1}(\mathbb{R}^n)}d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\|\Delta_{2^{-k}\zeta}f\|^{p_1}_{L^{p_1}(\mathbb{R}^n)}d\zeta\Big)^{q/p_1}\Big)^{1/q}\\
&\leqslant C\|g\|_{L^{p_2}(\mathbb{R}^n)}\|f\|_{\dot{B}^{p_1,q}_s(\mathbb{R}^n)}\leqslant C\|f\|_{B^{p_1,q}_s(\mathbb{R}^n)}\|g\|_{B^{p_2,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{69}$$

An argument similar to (69) shows that

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|f\|_{B^{p_1,q}_s(\mathbb{R}^n)}\|g\|_{B^{p_2,q}_s(\mathbb{R}^n)}.
\end{aligned}\tag{70}$$

From (1) and (61) we see that

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}b\mathfrak{M}(f,g)(x)|^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k=-\infty}^{0}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}b\mathfrak{M}(f,g)(x)|^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad+C\Big(\sum_{k=1}^{\infty}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}|\Delta_{2^{-k}\zeta}b\mathfrak{M}(f,g)(x)|^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C|\mathfrak{R}_n|\|\mathfrak{M}(f,g)\|_{L^p(\mathbb{R}^n)}\\
&\quad\times\Big(\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0}2^{ksq}\Big)^{1/q}+\|b\|_{Lip(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty}2^{kq(s-1)}\Big)^{1/q}\Big)\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}.
\end{aligned}\tag{71}$$

It follows from (67)–(71) that

$$B_1\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}.\tag{72}$$

Estimate for B_2. Using (32) and (68)–(70), we have

$$\begin{aligned}
B_2&\leqslant C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}b\Delta_{2^{-k}\zeta}f,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad+C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad+C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}\\
&\quad+C\Big(\sum_{k\in\mathbb{Z}}2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}.
\end{aligned}\tag{73}$$

In view of (1) and (61), one has

$$
\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}bf,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\|\Delta_{2^{-k}\zeta}bf\|_{L^{p_1}(\mathbb{R}^n)}^p\|g\|_{L^{p_2}(\mathbb{R}^n)}^p d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\|\Delta_{2^{-k}\zeta}bf\|_{L^{p_1}(\mathbb{R}^n)}^p d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\|\Delta_{2^{-k}\zeta}bf\|_{L^{p_1}(\mathbb{R}^n)}^p d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\quad+C\|g\|_{L^{p_2}(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\|\Delta_{2^{-k}\zeta}bf\|_{L^{p_1}(\mathbb{R}^n)}^p d\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C|\mathfrak{R}_n|\|g\|_{L^{p_2}(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\\
&\quad\times\Big(\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k=-\infty}^{0} 2^{ksq}\Big)^{1/q}+\|b\|_{Lip(\mathbb{R}^n)}\Big(\sum_{k=1}^{\infty}2^{kq(s-1)}\Big)^{1/q}\Big)\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}.
\end{aligned}
\tag{74}
$$

Combining (74) with (73) implies that

$$
B_2\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}. \tag{75}
$$

Estimates for B_3, B_4 and B_5. We use (68), (69) and (70), respectively to obtain

$$
\begin{aligned}
B_3&\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(b(x)+\|b\|_{L^\infty(\mathbb{R}^n)})(\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)};
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
B_4&\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(b(x)+\|b\|_{L^\infty(\mathbb{R}^n)})(\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(\Delta_{2^{-k}\zeta}f,g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)};
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
B_5&\leqslant C\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(b(x)+\|b\|_{L^\infty(\mathbb{R}^n)})(\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|b\|_{L^\infty(\mathbb{R}^n)}\Big(\sum_{k\in\mathbb{Z}} 2^{ksq}\Big(\int_{\mathfrak{R}_n}\int_{\mathbb{R}^n}(\mathfrak{M}(f,\Delta_{2^{-k}\zeta}g)(x))^p dxd\zeta\Big)^{q/p}\Big)^{1/q}\\
&\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}.
\end{aligned}
\tag{78}
$$

Finally, we get from (66), (72) and (75)–(78) that

$$\|\mathfrak{M}_b(f,g)\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}. \tag{79}$$

Combining (79) with (5) and (61) implies (10) for $\mathfrak{M}_b$.

Step 2: Proof of the continuity part.

Let $f_j \to f$ in $B_s^{p_1,q}(\mathbb{R}^n)$ and $g_j \to g$ in $B_s^{p_2,q}(\mathbb{R}^n)$ as $j \to \infty$. In view of (6) and (61), it is enough to show that

$$\|\mathfrak{M}_b(f_j,g_j) - \mathfrak{M}_b(f,g)\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} \to 0 \ \text{ as } \ j \to \infty. \tag{80}$$

We now prove (80) by contradiction. Assume that (80) is not valid. We may assume without loss of generality that there exists $c > 0$ such that

$$\|\mathfrak{M}_b(f_j,g_j) - \mathfrak{M}_b(f,g)\|_{\dot{B}_s^{p,q}(\mathbb{R}^n)} > c, \ \text{ for all } j \geqslant 1. \tag{81}$$

Since $\mathfrak{M}_b(f_j,g_j) \to \mathfrak{M}_b(f,g)$ in $L^p(\mathbb{R}^n)$ as $j \to \infty$, by extracting a subsequence we may assume without loss of generality that $\mathfrak{M}_b(f_j,g_j)(x) - \mathfrak{M}_b(f,g)(x) \to 0$ as $j \to \infty$ for almost every $x \in \mathbb{R}^n$. Consequently it holds that

$$\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_j,g_j) - \mathfrak{M}_b(f,g))(x) \to 0 \ \text{ as } \ j \to \infty \tag{82}$$

for every $(k,\zeta) \in \mathbb{Z} \times \mathfrak{R}_n$ and almost every $x \in \mathbb{R}^n$. For convenience, we set

$$\begin{aligned}\psi_{f,g}(x,k,\zeta) := {} & |\Delta_{2^{-k}\zeta} b|\mathfrak{M}(f_{2^{-k}\zeta}, g_{2^{-k}\zeta})(x) \\ & + \mathfrak{M}(\Delta_{2^{-k}\zeta} b f_{2^{-k}\zeta}, g_{2^{-k}\zeta})(x) + \mathfrak{M}_b(\Delta_{2^{-k}\zeta} f, \Delta_{2^{-k}\zeta} g)(x) \\ & + \mathfrak{M}_b(\Delta_{2^{-k}\zeta} f, g)(x) + \mathfrak{M}_b(f, \Delta_{2^{-k}\zeta} g)(x).\end{aligned} \tag{83}$$

By (19), one has

$$|\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f,g)(x)| \leqslant \psi_{f,g}(x,k,\zeta) \tag{84}$$

for any $x \in \mathbb{R}^n$, $k \in \mathbb{Z}$ and $\zeta \in \mathfrak{R}_n$. Using (83) and the sublinearity of $\mathfrak{M}_b$ and $\mathfrak{M}$, one has

$$\begin{aligned}& |\psi_{f_j,g_j}(x,k,\zeta) - \psi_{f,g}(x,k,\zeta)| \\ & \leqslant |\Delta_{2^{-k}\zeta} b||\mathfrak{M}((f_j)_{2^{-k}\zeta},(g_j)_{2^{-k}\zeta})(x) - \mathfrak{M}(f_{2^{-k}\zeta}, g_{2^{-k}\zeta})(x)| \\ & \quad + |\mathfrak{M}(\Delta_{2^{-k}\zeta} b(f_j)_{2^{-k}\zeta},(g_j)_{2^{-k}\zeta})(x) - \mathfrak{M}(\Delta_{2^{-k}\zeta} b f_{2^{-k}\zeta}, g_{2^{-k}\zeta})(x)| \\ & \quad + |\mathfrak{M}_b(\Delta_{2^{-k}\zeta} f_j, \Delta_{2^{-k}\zeta} g_j)(x) - \mathfrak{M}_b(\Delta_{2^{-k}\zeta} f, \Delta_{2^{-k}\zeta} g)(x)| \\ & \quad + |\mathfrak{M}_b(\Delta_{2^{-k}\zeta} f_j, g_j)(x) - \mathfrak{M}_b(\Delta_{2^{-k}\zeta} f, g)(x)| \\ & \quad + |\mathfrak{M}_b(f_j, \Delta_{2^{-k}\zeta} g_j)(x) - \mathfrak{M}_b(f, \Delta_{2^{-k}\zeta} g)(x)| \\ & \leqslant |\Delta_{2^{-k}\zeta} b|\mathfrak{M}((f_j - f)_{2^{-k}\zeta},(g_j)_{2^{-k}\zeta})(x) + |\Delta_{2^{-k}\zeta} b|\mathfrak{M}(f_{2^{-k}\zeta},(g_j - g)_{2^{-k}\zeta})(x) \\ & \quad + \mathfrak{M}(\Delta_{2^{-k}\zeta} b(f_j - f)_{2^{-k}\zeta},(g_j)_{2^{-k}\zeta})(x) + \mathfrak{M}(\Delta_{2^{-k}\zeta} b f_{2^{-k}\zeta},(g_j - g)_{2^{-k}\zeta})(x)\end{aligned} \tag{85}$$

$$
\begin{aligned}
&+\mathfrak{M}_b(\Delta_{2^{-k}\zeta}(f_j-f),\Delta_{2^{-k}\zeta}g_j)(x)+\mathfrak{M}_b(\Delta_{2^{-k}\zeta}f,\Delta_{2^{-k}\zeta}(g_j-g))(x)\\
&+\mathfrak{M}_b(\Delta_{2^{-k}\zeta}(f_j-f),g_j)(x)+\mathfrak{M}_b(\Delta_{2^{-k}\zeta}f,g_j-g)(x)\\
&+\mathfrak{M}_b(f_j-f,\Delta_{2^{-k}\zeta}g_j)(x)+\mathfrak{M}_b(f,\Delta_{2^{-k}\zeta}(g_j-g))(x)\\
=:&\,\Psi_j(x,k,\zeta).
\end{aligned}
$$

Combining (85) with (84) implies that

$$
\begin{aligned}
&|\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f_j,g_j)(x)-\Delta_{2^{-k}\zeta}\mathfrak{M}_b(f,g)(x)|\\
&\leqslant \psi_{f_j,g_j}(x,k,\zeta)+\psi_{f,g}(x,k,\zeta)\leqslant \Psi_j(x,k,\zeta)+2\psi_{f,g}(x,k,\zeta)
\end{aligned}
\tag{86}
$$

for all $(x,k,\zeta)\in\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n$. By the arguments similar to those used in deriving (79),

$$
\|2\psi_{f,g}\|_{p,q,s}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)},\tag{87}
$$

$$
\begin{aligned}
\|\Psi_j\|_{p,q,s}\leqslant C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\big(&\|f_j-f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g_j-g\|_{B_s^{p_2,q}(\mathbb{R}^n)}\\
&+\|f_j-f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g\|_{B_s^{p_2,q}(\mathbb{R}^n)}+\|f\|_{B_s^{p_1,q}(\mathbb{R}^n)}\|g_j-g\|_{B_s^{p_2,q}(\mathbb{R}^n)}\big).
\end{aligned}
\tag{88}
$$

In view of (88), there exists a subsequence $\{j_i\}_{i=1}^{\infty}\subset\{j\}_{j=1}^{\infty}$ such that

$$
\sum_{i=1}^{\infty}\|\Psi_{j_i}\|_{p,q,s}<\infty.\tag{89}
$$

From (86) we have

$$
\begin{aligned}
&|\Delta_{2^{-k}\zeta}(\mathfrak{M}_b(f_{j_i},g_{j_i})-\mathfrak{M}_b(f,g))(x)|\\
&\leqslant\sum_{i=1}^{\infty}\Psi_{j_i}(x,k,\zeta)+2\psi_{f,g}(x,k,\zeta)=:\Theta(x,k,\zeta),
\end{aligned}
\tag{90}
$$

for all $(x,k,\zeta)\in\mathbb{R}^n\times\mathbb{Z}\times\mathfrak{R}_n$ and all $i\geqslant 1$. We get from (87)–(89) that $\|\Theta\|_{p,q,s}<\infty$. The rest of proof follows from (82), (90), the fact that $\|\Theta\|_{p,q,s}<\infty$ and the arguments similar to the proof of the continuity for $\mathfrak{M}_b$ in Theorem 2. We omit the details. □

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REFERENCES

[1] E. CARNEIRO AND D. MOREIRA, *On the regularity of maximal operators*, Proc. Amer. Math. Soc., **136**, 12 (2008), 4395–4404.

[2] P. CHEN, X. T. DUONG, J. LI AND Q. WU, *Compactness of Riesz transform commutator on stratified Lie groups*, J. Funct. Anal., **277**, (2019), 1639–1676.

[3] W. CHEN, Z. FU, L. GRAFAKOS AND Y. WU, *Fractional Fourier transforms on L^p and applications*, Appl. Comput. Harmonic Anal., **55**, (2021), 71–96.

[4] T. CHEN AND F. LIU, *Regularity of commutators of multilinear maximal operators with Lipschitz symbols*, Math. Inequal. Appl., **25**, 1 (2022), 109–134.

[5] M. FRAZIER, B. JAWERTH AND G. WEISS, *Littlewood–Paley theory and the study of function spaces*, CBMS Reg Conf Ser vol 79 Amer Math Soc, Providence, RI, 1991.
[6] D. GILBARG AND N. S. TRUDINGER, *Elliptic partial differential equations of second order*, 2nd edn., Springer-Verlag, Berlin, 1983.
[7] L. GRAFAKOS, *Classical and Modern Fourier Analysis*, Prentice Hall, Upper Saddle River, NJ, 2003.
[8] J. KINNUNEN, *The Hardy–Littlewood maximal function of a Sobolev function*, Israel J. Math., **100**, (1997), 117–124.
[9] J. KINNUNEN AND P. LINDQVIST, *The derivative of the maximal function*, J. Reine. Angew. Math., **503**, (1998), 161–167.
[10] J. KINNUNEN AND E. SAKSMAN, *Regularity of the fractional maximal function*, Bull. London Math. Soc., **35**, 4 (2003), 529–535.
[11] S. KORRY, *Boundedness of Hardy–Littlewood maximal operator in the framework of Lizorkin–Triebel spaces*, Rev. Mat. Complut., **15**, 2 (2002), 401–416.
[12] S. KORRY, *A class of bounded operators on Sobolev spaces*, Arch. Math., **82**, 1 (2004), 40–50.
[13] M. LACEY, *The bilinear maximal function map into L^p for $2/3 < p \leqslant 1$*, Ann. of Math., **151**, (2000), 35–57.
[14] F. LIU, S. LIU AND X. ZHANG, *Regularity properties of bilinear maximal function and its fractional variant*, Results Math., **75**, 88 (2020), 1–29.
[15] F. LIU AND G. WANG, *Regularity of commutators of maximal operators with Lipschitz symbols*, Taiwan. J. Math., **25**, 5 (2021), 1007–1039.
[16] F. LIU AND H. WU, *On the regularity of maximal operators supported by submanifolds*, J. Math. Anal. Appl., **453**, (2017), 144–158.
[17] F. LIU AND S. XI, *Sobolev regularity for commutators of the fractional maximal functions*, Banach J. Math. Anal., **15**, 5 (2021), 1–36.
[18] F. LIU, Q. XUE AND P. ZHANG, *Regularity and continuity of commutators of the Hardy–Littlewood maximal function*, Math. Nachr., **293**, 3 (2020), 491–509.
[19] H. LUIRO, *Continuity of the maximal operator in Sobolev spaces*, Proc. Amer. Math. Soc., **135**, 1 (2007), 243–251.
[20] H. LUIRO, *On the regularity of the Hardy–Littlewood maximal operator on subdomains of $\mathbb{R}^n$*, Proc. Edinburgh Math. Soc., **53**, 1 (2010), 211–237.
[21] C. PÉREZ AND R. H. TORRES, *Sharp maximal function estimates for multilinear singular integrals*, Contemp. Math., **320**, (2003), 323–331.
[22] S. SHI, Z. FU AND S. LU, *On the compactness of commutators of Hardy operators*, Pacific J. Math., **307**, (2020), 239–256.
[23] S. SHI AND J. XIAO, *Fractional capacities relative to bounded open Lipschitz sets complemented*, Calculus Var. Partial Differ. Equa., **56**, (2017), 1–22.
[24] H. TRIEBEL, *Theory of Function Spaces*, Monogr Math vol 78, Birkhäser Verlag, Basel, 1983.
[25] K. YABUTA, *Triebel–Lizorkin space boundedness of Marcinkiewicz integrals associated to surfaces*, Appl. Math. J. Chinese Univ. Ser. B, **30**, 4 (2015), 418–446.
[26] M. YANG, Z. FU AND J. SUN, *Existence and large time behavior to coupled chemotaxis fluid equations in Besov–Morrey spaces*, J. Differ. Equa., **266**, (2019), 5867–5894.

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