Mathematical
Inequalities
& Applications
Volume 25, Number 2 (2022), 601–610 doi:10.7153/mia-2022-25-36

q–HERMITE—HADAMARD INEQUALITIES FOR FUNCTIONS WITH CONVEX OR h–CONVEX q–DERIVATIVE

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(*Communicated by S. Varošanec*)

Abstract. In this work, using the definitions of convex functions and h-convex functions, new Hermite–Hadamard type inequalities are presented using the framework of q-calculus. We prove inequalities for the q_a- and q^b-definite integrals of functions which have a convex or general convex q_a- or q^b-derivative. These inequalities have consequences for q-integrals and classical integrals, while extending some results previously known from the literature.

1. Introduction

One of the most fruitful concepts in Mathematics is the convex function, not only because of its theoretical impact in various areas, but also because of the multiplicity of applications that have been developed in recent times.

A function $f:[a,b]\to\mathbb{R}$ is said to be convex if $f\big(tx+(1-t)y\big)\leqslant tf(x)+(1-t)f(y)$ holds for all $x,y\in[a,b]$, $x<y$ and $t\in[0,1]$. And it is said that function f is concave on $[a,b]$ if the above inequality is the opposite. During the paper, it is always assumed that $a<b$.

Readers interested in the aforementioned development, can consult e.g. paper [20], where a panorama, practically complete, of these branches is presented.

One of the most important inequalities, for convex functions, is the well-known Hermite–Hadamard inequality:

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx\leqslant\frac{f(a)+f(b)}{2} \tag{1}$$

holds for any function f convex on the interval $[a,b]$. This inequality was published by Hermite ([13]) in 1883 and, independently, by Hadamard in 1893 ([12]). This inequality gives a very useful boundedness to the mean value of a function, in this case convex functions.

More than one hundred years ago, the Reverend Frank Hilton Jackson defined a new derivative of a function at a point, without the use of the notion of limit, opening a new direction of work in classical calculus and number theory. He got q-analogs of

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Mathematics subject classification (2020): 26A51, 26D10, 26D15.

Keywords and phrases: Convex functions, h-convex functions, Hermite–Hadamard inequality, q-calculus.

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several known results from these areas. Interested readers can consult Jackson's works [15, 16], as well as [10, 18] on the basis of q-calculus.

The q-derivative of a real function $f(x)$ is defined for $q \in (0,1)$ as

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}, \quad x \neq 0 \tag{2}$$

and $D_q f(0) = f'(0)$ for functions f differentiable at $x = 0$. The q-derivative calculates the rise of $f(x)$ over the interval (qx, x). As such, it is numerically equal to the slope of the line going through points $(qx, f(qx))$ and $(x, f(x))$.

The Jackson integral of a real function f is defined by the series expansion

$$\int_0^x f(t)\, d_q t = (1-q)\, x \sum_{k=0}^{\infty} q^k f(q^k x) \tag{3}$$

provided that the series is finite, e.g. in case $|f(x)x^{\alpha}|$ is bounded on the interval $(0, A]$ for some $0 \leqslant \alpha < 1$ (see [10]).

The following expression, defines the q-number (see [18]) and will be used later:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}, \quad q \in (0,1), \quad n \in \mathbb{N},$$

e.g. $[2]_q = 1 + q$ and $[3]_q = 1 + q + q^2$.

A generalization of the definition of convex function is as follows.

DEFINITION 1. [21, 27] Let $h : [0,1] \to [0, \infty)$ and $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$. We say that f is an h-convex function, if for all $x, y \in [a,b]$ and $t \in (0,1)$, we have

$$f(tx + (1-t)y) \leqslant h(t) f(x) + h(1-t) f(y).$$

REMARK 1. Note that in papers [21, 27], non-negativity of function f is assumed, but in this paper, this condition is dropped to get more general results.

Special cases of the above definition are the following.

If $h(t) = t^s$, $s \in (0,1]$, then f is called an s-convex function ([7, 14]).

If $h(t) = t$, then f is a convex function.

Generalization of the q-derivative (2) was introduced in [25, 26] and [5] as follows.

DEFINITION 2. For a function $f : [a,b] \to \mathbb{R}$ and $q \in (0,1)$, the q_a-derivative of f at $x \in [a,b]$ is characterized by the expression

$${}_aD_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a. \tag{4}$$

The q^b-derivative of f at $x \in [a,b]$ is defined by

$${}^bD_q f(x) = \frac{f(x) - f(qx + (1-q)b)}{(1-q)(x-b)}, \quad x \neq b. \tag{5}$$

For $x = a$, we define ${}_aD_q f(a) = \lim_{x\to a} {}_aD_q f(x) = f'(a)$ if it exists and is finite. Analogously, for $x = b$, we define ${}^bD_q f(b) = \lim_{x\to b} {}^bD_q f(x) = f'(b)$ if it exists and is finite.

REMARK 2. Note that if $a = 0$ in (4), or, if $b = 0$ in (5), then we obtain the familiar q-derivative (2) of f at $x \in [a,b]$.

The q_a-definite integral defined in [25, 26] and the analogous q^b-definite integral defined in [5], are generalizations of the q-integral (3).

DEFINITION 3. For a function $f : [a,b] \to \mathbb{R}$ and $q \in (0,1)$, the q_a-definite integral of f is defined by the expression

$$\int_a^x f(t)\ {}_ad_qt = (1-q)(x-a)\sum_{k=0}^{\infty} q^k f(q^kx+(1-q^k)a), \quad x\in[a,b],$$

and similarly, the q^b-definite integral of f is

$$\int_x^b f(t)\ {}^bd_qt = (1-q)(b-x)\sum_{k=0}^{\infty} q^k f(q^kx+(1-q^k)b), \quad x\in[a,b],$$

provided that the series is finite, e.g. in case f is continuous on $[a,b]$.

EXAMPLE 1. It is well-known that

$$\int_a^b (mx+k)\ {}_ad_qx = (b-a)\left(\frac{m(qa+b)}{[2]_q}+k\right),$$
$$\int_a^b (mx+k)\ {}^bd_qx = (b-a)\left(\frac{m(a+bq)}{[2]_q}+k\right).$$

Some important properties of the q_a- and q^b-derivatives and integrals will be used later (see [5, 17, 25]):

THEOREM 1. *For continuous $f : [a,b] \to \mathbb{R}$ and $q \in (0,1)$, we have*

$$\int_a^x {}_aD_q f(t)\ {}_ad_qt = f(x)-f(a), \qquad \int_x^b {}^bD_q f(t)\ {}^bd_qt = f(b)-f(x).$$

Moreover, for continuous $f, g : [a,b] \to \mathbb{R}$, integration by parts says

$$\int_a^b g(qt+(1-q)a)\ {}_aD_q f(t)\ {}_ad_qt = \big[f(t)g(t)\big]_a^b - \int_a^b f(t)\ {}_aD_q g(t)\ {}_ad_qt,$$
$$\int_a^b g(qt+(1-q)b)\ {}^bD_q f(t)\ {}^bd_qt = \big[f(t)g(t)\big]_a^b - \int_a^b f(t)\ {}^bD_q g(t)\ {}^bd_qt.$$

Some theoretical preambles, on this subject, are the following. In [22], a q-analog of a classical integral identity is established, from this equality, various q-estimates are obtained for the Hermite–Hadamard inequality for convex and quasi-convex q-differentiable functions; in [3] some generalizations of the Hermite–Hadamard inequality are obtained within the framework of q-calculus for convex functions, but they incorporate the differentiability of the function, which can be omitted, as it was shown in [5].

THEOREM 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a convex function on* $[a,b]$ *and* $q\in(0,1)$. *Then we have*

$$f\left(\frac{qa+b}{[2]_q}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\leqslant\frac{qf(a)+f(b)}{[2]_q}\tag{6}$$

and

$$f\left(\frac{a+qb}{[2]_q}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)\ {}^bd_qx\leqslant\frac{f(a)+qf(b)}{[2]_q}.\tag{7}$$

Other results can be found in [1, 2, 4, 6, 8, 9, 17, 24, 28] and the references cited in them.

Considering the proof of [5, Theorem 18], one can deduce the next statement.

THEOREM 3. *Let* $f:[a,b]\to\mathbb{R}$ *be a continuous,* h*-convex function and* $q\in(0,1)$. *Then we have*

$$\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx\leqslant H_2f(a)+H_1f(b)\tag{8}$$

and

$$\frac{1}{b-a}\int_a^b f(x)\ {}^bd_qx\leqslant H_1f(a)+H_2f(b),\tag{9}$$

where $H_1=\int_0^1 h(t)\,d_qt$ *and* $H_2=\int_0^1 h(1-t)\,d_qt$, *provided that* H_1 *and* H_2 *exist and are finite.*

In this paper, we obtain new inequalities of Hermite–Hadamard type for functions which have q-derivatives (in the sense of (4) and (5)) h-convex or convex on finite intervals. As we will see, these results are extensions of some results known from the literature.

2. Main results

We start with a lemma that will be of benefit later.

LEMMA 1. *For continuous* $f:[a,b]\to\mathbb{R}$, *we have*

$$\int_a^b (x-a)\ {}_aD_qf(x)\,{}_ad_qx=\frac{1}{q}\left((b-a)f(b)-\int_a^b f(x)\ {}_ad_qx\right),$$

$$\int_a^b (x-a)\ {}_aD_qf\left(\tfrac{qa+x}{[2]_q}\right)\ {}_ad_qx=\frac{[2]_q}{q}\left((b-a)f\left(\tfrac{qa+b}{[2]_q}\right)-[2]_q\int_a^{\frac{qa+b}{[2]_q}} f(x)\ {}_ad_qx\right).$$

Proof. Considering integration by parts for $f(x)$ and $g(x) = x - a$, by Theorem 1, we can write

$$\int_a^b (qx + (1-q)a - a)\ {}_aD_q f(x)\ {}_ad_qx = \big[(x-a)f(x)\big]_a^b - \int_a^b f(x)\ {}_aD_q(x-a)\ {}_ad_qx$$

that is equivalent to

$$q\int_a^b (x-a)\ {}_aD_q f(x)\ {}_ad_qx = (b-a)f(b) - \int_a^b f(x)\ {}_ad_qx.$$

The first required equation is obtained.

Analogously, integration by parts for $f\left(\frac{qa+x}{[2]_q}\right)$ and $g(x) = x - a$ yields

$$\begin{aligned}
&\int_a^b (qx + (1-q)a - a)\ {}_aD_q f\left(\frac{qa+x}{[2]_q}\right)\ {}_ad_qx \\
&= \left[[2]_q(x-a)f\left(\frac{qa+x}{[2]_q}\right)\right]_a^b - [2]_q\int_a^b f\left(\frac{qa+x}{[2]_q}\right)\ {}_aD_q(x-a)\ {}_ad_qx \\
&= [2]_q\left((b-a)f\left(\frac{qa+b}{[2]_q}\right) - [2]_q\int_a^{\frac{qa+b}{[2]_q}} f(x)\ {}_ad_qx\right)
\end{aligned}$$

that is equivalent to the second required equation. □

Our first main result reads as follows.

THEOREM 4. *Let $f : [a,b] \to \mathbb{R}$ be such that ${}_aD_q f(x)$ is continuous, h-convex on $[a,b]$ and $f'(a)$ exists. The following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx \leqslant \frac{qf(a) + H_1 f(b)}{q + H_1} + \frac{qH_2(b-a)}{[2]_q(q+H_1)} f'(a).$$

Proof. Writing inequality (8) for ${}_aD_q f(x)$, substituting x in place of b and multiplying by $(x-a)$ yield

$$\int_a^x {}_aD_q f(t)\ {}_ad_qt \leqslant (x-a)(H_2\ {}_aD_q f(a) + H_1\ {}_aD_q f(x)).$$

By q_a-integrating both sides with respect to x on $[a,b]$, we have

$$\int_a^b \left(\int_a^x {}_aD_q f(t)\ {}_ad_qt\right) {}_ad_qx \leqslant \int_a^b (x-a)(H_2\ {}_aD_q f(a) + H_1\ {}_aD_q f(x))\ {}_ad_qx.$$

The left-hand side is equal to

$$\int_a^b \left(\int_a^x {}_aD_q f(t)\,{}_ad_qt\right)\ {}_ad_qx = \int_a^b f(x)\ {}_ad_qx - (b-a)f(a), \tag{10}$$

while, by Lemma 1, the right-hand side is

$$
\begin{aligned}
&\int_a^b (x-a)(H_2\, {}_aD_q f(a) + H_1\, {}_aD_q f(x))\, {}_ad_qx \\
&= \frac{(b-a)^2}{[2]_q} H_2\, {}_aD_q f(a) + \frac{H_1}{q}\left((b-a)f(b) - \int_a^b f(x)\, {}_ad_qx\right).
\end{aligned} \tag{11}
$$

Combining (10) and (11) yields the required result. □

The following results are consequences of considering convex functions instead of h-convex functions. In this case, $h(t)=t$, $H_1 = \frac{1}{[2]_q}$ and $H_2 = \frac{q}{[2]_q}$.

COROLLARY 1. *Let $f:[a,b]\to\mathbb{R}$ be such that ${}_aD_q f(x)$ is convex on $[a,b]$ and $f'(a)$ exists. The following inequality holds:*

$$
\frac{1}{b-a}\int_a^b f(x)\, {}_ad_qx \leqslant \frac{q[2]_q f(a) + f(b)}{[3]_q} + \frac{q^2(b-a)}{[2]_q[3]_q} f'(a). \tag{12}
$$

A lower estimation for the q_a-integral reads as follows.

THEOREM 5. *Let $f:[a,b]\to\mathbb{R}$ be such that ${}_aD_q f(x)$ is convex on $[a,b]$ and $f'(a)$ exists. Then*

$$
\frac{[2]_q^2 f\left(\frac{qa+b}{[2]_q}\right) - qf(a)}{[3]_q} - \frac{q(b-a)}{[2]_q[3]_q} f'(a) \leqslant \frac{1}{b-a}\int_a^b f(x)\, {}_ad_qx.
$$

Proof. Writing the first inequality in (6) for ${}_aD_q f(x)$, substituting x in place of b and multiplying by $(x-a)$ yield

$$
(x-a)\, {}_aD_q f\left(\frac{qa+x}{[2]_q}\right) \leqslant \int_a^x {}_aD_q f(t)\, {}_ad_qt.
$$

After q_a-integrating both sides with respect to x on $[a,b]$, we get

$$
\int_a^b (x-a)\, {}_aD_q f\left(\frac{qa+x}{[2]_q}\right) {}_ad_qx \leqslant \int_a^b \left(\int_a^x {}_aD_q f(t)\, {}_ad_qt\right) {}_ad_qx.
$$

The right-hand side is equal to (10), while, by Lemma 1 and (12), the left-hand side can be estimated as

$$
\begin{aligned}
&\int_a^b (x-a)\, {}_aD_q f\left(\frac{qa+x}{[2]_q}\right) {}_ad_qx \\
&= \frac{[2]_q}{q}\left((b-a)f\left(\frac{qa+b}{[2]_q}\right) - [2]_q \int_a^{\frac{qa+b}{[2]_q}} f(x)\, {}_ad_qx\right) \\
&\geqslant \frac{[2]_q}{q}(b-a)\left(f\left(\frac{qa+b}{[2]_q}\right) - \left(\frac{q[2]_q f(a) + f\left(\frac{qa+b}{[2]_q}\right)}{[3]_q} + \frac{q^2(b-a)}{[2]_q^2[3]_q} f'(a)\right)\right).
\end{aligned} \tag{13}
$$

Combining (10) and (13) yields the required result. □

We obtain analogous results for q^b-integrals. By considering inequalities (9) and (7) for ${}^bD_q f(x)$, we get the following results corresponding to Theorems 4 and 5.

THEOREM 6. *Let $f:[a,b]\to\mathbb{R}$ be such that ${}^bD_q f(x)$ is continuous, h-convex on $[a,b]$ and $f'(b)$ exists. The following inequality holds:*

$$\frac{qf(b)+H_1 f(a)}{q+H_1}-\frac{qH_2(b-a)}{[2]_q(q+H_1)}f'(b)\leqslant\frac{1}{b-a}\int_a^b f(x)\,{}^bd_q x.$$

THEOREM 7. *Let $f:[a,b]\to\mathbb{R}$ be such that ${}^bD_q f(x)$ is convex on $[a,b]$ and $f'(b)$ exists. Then*

$$\frac{f(a)+q[2]_q f(b)}{[3]_q}-\frac{q^2(b-a)}{[2]_q[3]_q}f'(b)\leqslant\frac{1}{b-a}\int_a^b f(x)\,{}^bd_q x$$
$$\leqslant\frac{[2]_q^2 f\left(\frac{a+qb}{[2]_q}\right)-qf(b)}{[3]_q}+\frac{q(b-a)}{[2]_q[3]_q}f'(b).$$

By taking $a=0$, Corollary 1 and Theorem 5 together imply a new version of the Hermite–Hadamard inequality (1) in the framework of classical q-calculus.

COROLLARY 2. *Let $f:[a,b]\to\mathbb{R}$ be such that $D_q f(x)$ is convex on $[0,b]$ and $f'(0)$ exists. Then*

$$\frac{[2]_q^2 f\left(\frac{b}{[2]_q}\right)-qf(0)}{[3]_q}-\frac{qb}{[2]_q[3]_q}f'(0)\leqslant\frac{1}{b}\int_0^b f(x)\,d_q x$$
$$\leqslant\frac{q[2]_q f(0)+f(b)}{[3]_q}+\frac{q^2 b}{[2]_q[3]_q}f'(0).$$

Combining Corollary 1 and Theorem 5, also taking the limit $q\to 1^-$, allows us to obtain an inequality for classical integrals, which can be also calculated directly from Corollary 2 and Corollary 4 of [19] (see also [11]).

COROLLARY 3. *Let $f:[a,b]\to\mathbb{R}$ be differentiable on $[a,b]$ such that $f'(x)$ is convex on $[a,b]$. The following inequality holds:*

$$\frac{4f\left(\frac{a+b}{2}\right)-f(a)}{3}-\frac{b-a}{6}f'(a)\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx$$
$$\leqslant\frac{2f(a)+f(b)}{3}+\frac{b-a}{6}f'(a).$$

REMARK 3. If in Corollary 3, we additionally assume that $f(x)$ is convex on $[a,b]$, then we have an improvement of the Hermite–Hadamard inequality (1), since

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \frac{4f\left(\frac{a+b}{2}\right)-f(a)}{3}-\frac{b-a}{6}f'(a)\leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \\ &\leqslant \frac{2f(a)+f(b)}{3}+\frac{b-a}{6}f'(a)\leqslant \frac{f(a)+f(b)}{2} \end{aligned}$$

by considering the inequality $f(x)+f'(x)(y-x)\leqslant f(y)$ for $x=a$, $y=\frac{a+b}{2}$ and $x=a$, $y=b$, respectively.

REMARK 4. If in the main results, we consider h-concave and concave functions, respectively, instead of h-convex and convex functions, then the opposite inequalities hold.

3. Conclusions

In this article, various generalizations of the Hermite–Hadamard inequality are obtained in the framework of the q-operators. These inequalities naturally extend some previously known results from the literature as we showed. The accuracy of our main results Theorems 4–7 can be highlighted through the function $f:[a,b]\to\mathbb{R}, f(x)=c_1x^2+c_2x+c_3$, in which case, the inequalities become equalities.

To illustrate the strength of our results, consider the following example. Let $f:\left[0,\frac{\pi}{2}\right]\to\mathbb{R}, f(x)=\sin x$. For this concave function f, Theorem 2 implies

$$\sin\frac{\pi}{2(1+q)}\geqslant\frac{2}{\pi}\int_0^{\pi/2}\sin x\,d_qx\geqslant\frac{1}{1+q}, \tag{14}$$

while Corollary 2 yields the following refinement of the previous inequality

$$\begin{aligned} &\frac{(1+q)^2\sin\frac{\pi}{2(1+q)}}{1+q+q^2}-\frac{q\pi}{2(1+q)(1+q+q^2)}\geqslant\frac{2}{\pi}\int_0^{\pi/2}\sin x\,d_qx \\ &\geqslant\frac{1}{1+q+q^2}+\frac{q^2\pi}{2(1+q)(1+q+q^2)}=\frac{1}{1+q}\cdot\frac{2+2q+\pi q^2}{2+2q+2q^2} \end{aligned} \tag{15}$$

by noticing the concavity of

$$D_q\sin x=\begin{cases}\frac{\sin(qx)-\sin x}{qx-x}, & x\neq 0,\\ 1, & x=0,\end{cases}$$

that can be shown by calculation. One can easily see that the upper bound in (15) is less than the one in (14), while the lower bound in (15) is greater than the one in (14).

It is clear that the problem of extending the lower estimation for the q_a-integral in Theorem 5 for h-convex functions, or, the generalization of our results for the case of more general definitions, such as the $(h-m)$-convex modified functions of [23], remains open.

Acknowledgement. The authors thank the anonymous referee for the valuable comments and suggestions regarding the earlier version of the paper.

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(Received October 27, 2021)

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