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ON q–MONOTONICITY OF α–BERNSTEIN OPERATORS

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Abstract. In this paper we show that α-Bernstein operators preserve q-monotonicity of all orders. We investigate the tensor product of two such operators and show that it preserves (q,s)-box convexity. Some Raşa type inequalities for the α-Bernstein operators are also derived.

1. Introduction

In [4], the following generalization of the Bernstein operators depending on a non-negative real parameter was derived. Given a function $f(x)$ on $[0,1]$, for each positive integer n and any fixed real α, the so called α-Bernstein operator for $f(x)$ is defined as

$$T_{n,\alpha}(f;x)=\sum_{i=0}^{n} p_{n,i}^{(\alpha)}(x)f\left(\frac{i}{n}\right), \tag{1}$$

where $p_{1,0}^{(\alpha)}(x)=1-x$, $p_{1,1}^{(\alpha)}(x)=x$ and

$$\begin{aligned} p_{n,i}^{(\alpha)}(x)=&\left[\binom{n-2}{i}(1-\alpha)x+\binom{n-2}{i-2}(1-\alpha)(1-x)+\binom{n}{i}\alpha x(1-x)\right]\\ &\times x^{i-1}(1-x)^{n-i-1}, \end{aligned}$$

for $n\geqslant 2$, $x\in[0,1]$. Here the binomial coefficients $\binom{k}{l}$ are given by

$$\binom{k}{l}=\begin{cases}\dfrac{k!}{l!(k-l)!}, & \text{if } 0\leqslant l\leqslant k\\ 0, & \text{else.}\end{cases}$$

When $\alpha=1$, the α-Bernstein operator reduces to the classical Bernstein operator

$$T_{n,1}(f;x)=B_n(f;x)=\sum_{i=0}^{n} b_{n,i}(x)f\left(\frac{i}{n}\right),\quad n\in\mathbb{N},$$

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where $b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}$, $i,\ n \in \mathbb{N}$.

In this paper we will consider only α-Bernstein operators with $\alpha \in [0,1]$. Under this assumption, the operators $T_{n,\alpha}$ are linear positive operators.

The rate of convergence and a Voronovskaja type theorem are given in [4]. The operators $T_{n,\alpha}$ preserve monotonicity and convexity ([4], Theorem 3.3 and Theorem 4.1.).

We observe that $p_{n,i}^{(\alpha)}(x)$ can be written in terms of the Bernstein basis as

$$p_{n,i}^{(\alpha)}(x) = (1-\alpha)(1-x)b_{n-2,i}(x) + (1-\alpha)x b_{n-2,i-2}(x) + \alpha b_{n,i}(x), \quad i,n \in \mathbb{N}. \tag{2}$$

It follows from (2) that the α-Bernstein operator $T_{n,\alpha}$ can be written in terms of the Bernstein operators

$$\begin{aligned} T_{n,\alpha}(f;x) = &(1-\alpha)(1-x)B_{n-2}\left(f\left(\frac{n-2}{n}t\right);x\right) \\ &+ (1-\alpha)xB_{n-2}\left(f\left(\frac{(n-2)t+2}{n}\right);x\right) + \alpha B_n\left(f\left(t\right);x\right), \end{aligned} \tag{3}$$

where $B_k\left(f(at+b);x\right)$ is the Bernstein polynomial of degree k, corresponding to the function $g(t) = f(at+b)$ evaluated at x.

We will use identity (3) to prove some new properties of the α-Bernstein operators $T_{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \in (0,1)$.

In [10], J. Mrowiec, T. Rajba and S. Wąsowicz solved for the first time the following problem, raised by I. Raşa ([12], Problem 2, p.164), related to the preservation of convexity by the Bernstein-Schnabl operators. In [8], Raşa's conjecture was studied for the case of Baskakov-Mastroianni operators.

PROBLEM. *Prove or disprove that*

$$\sum_{i,j=0}^{n} \left(b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y)\right) f\left(\frac{i+j}{2n}\right) \geqslant 0 \tag{4}$$

for each convex function $f \in C[0,1]$ *and for all* $x,y \in [0,1]$.

A simple proof of (4) was given by U. Abel in [1]. In [3], [6] and [7], inequality (4) was proved in a more general context.

Given $f \in C[0,1]$, we define

$$\Delta_h^1 f(x) := \Delta_h f(x) := \begin{cases} f(x+h) - f(x), & x,\ x+h \in [0,1] \\ 0, \ \text{otherwise} \end{cases}$$

and for $q \geqslant 1$

$$\Delta_h^{q+1} f(x) := \Delta_h^q \left(\Delta_h f(x)\right).$$

A function f defined on $[0,1]$ is called q-monotone if $\Delta_h^q f(x) \geqslant 0$, for all $h \geqslant 0$. In particular a 1–monotone function is non-decreasing and a 2–monotone one is convex. It is known (see, [9] for example) that the Bernstein polynomials preserve q-monotonicity for all orders $q \geqslant 1$. This property follows from the following identity, which will be used later in this paper:

$$(D^q B_n f)(x) = \binom{n}{q} \frac{q!}{n^q} \sum_{j=0}^{n-q} b_{n-q,j}(x) \left[\frac{j}{n}, \frac{j+1}{n}, \dots, \frac{j+q}{n}; f \right]. \tag{5}$$

Here, by $[x_0, \dots, x_q; f]$, we have denoted the divided difference of the function f on the distinct points $x_0, \dots, x_q \in [0,1]$, defined by the formulas

$$[x_0; f] = f(x_0)$$

$$[x_0, x_1, \dots, x_{q-1}, x_q; f] = \frac{[x_0, \dots, x_{q-1}; f] - [x_1, \dots, x_q; f]}{x_0 - x_q}$$

for $q \geqslant 1$. Given the divided difference $[x, x+h, \dots, x+qh; f]$ and $\Delta_h^q f(x)$, the following identity is well-known:

$$[x, x+h, \dots, x+qh; f] = \frac{1}{q!} \frac{1}{h^q} \Delta_h^q f(x) \tag{6}$$

U. Abel and D. Leviatan in [2] proved an analogous inequality of (4) for q-monotone functions. More precisely, they proved the following theorem.

THEOREM A. *Let $q, n \in \mathbb{N}$. If $f \in C[0,1]$ is a q-monotone function, then for all $x, y \in [0,1]$,*

$$sgn(x-y)^q \sum_{\nu_1, \dots, \nu_q = 0}^{n} \sum_{j=0}^{q} (-1)^{q-j} \binom{q}{j} \left(\prod_{i=1}^{j} b_{n,\nu_i}(x) \right) \left(\prod_{i=j+1}^{q} b_{n,\nu_i}(y) \right)$$
$$\times \int_0^1 f\left(\frac{\nu_1 + \ldots + \nu_q + \alpha t}{qn + \alpha} \right) dt \geqslant 0,$$

where $\alpha \in [0,1]$.

The aim of this paper is to show that the α-Bernstein operator, $T_{n,\alpha}$, preserves q-monotonicity of all orders, $q \geqslant 1$ and to extend Theorem A. Our main results are listed below, with the corresponding proofs given in Section 2.

THEOREM 1. *The α-Bernstein operator, $T_{n,\alpha}$, preserves q-monotonicity of all orders q, $q \in \mathbb{N}$.*

Before giving the next result, we recall the definition of box-convexity. A function $f \in C([0,1] \times [0,1])$ is called box-convex of order (q,s), [5], if for any distinct points $x_0, x_1, \dots, x_q \in [0,1]$ and any distinct points $y_0, y_1, \dots, y_s \in [0,1]$

$$\begin{bmatrix} x_0, x_1, \dots, x_q \\ y_0, y_1, \dots, y_s \end{bmatrix}; f \geqslant 0,$$

where

$$\begin{bmatrix} x_0, \ldots, x_q; f \\ y_0, \ldots, y_s \end{bmatrix} = [x_0, \ldots x_q; [y_0, \ldots, y_s; f(x, \cdot)]] = [y_0, \ldots, y_s; [x_0, \ldots, x_q; f(\cdot, y)]].$$

THEOREM 2. *Let $\alpha, \beta \in [0,1]$ be two fixed numbers and n, m be two natural numbers. If $T_{n,m,\alpha,\beta} : C([0,1]\times[0,1]) \to C([0,1]\times[0,1])$ is the tensorial product of $T_{n,\alpha}$ and $T_{m,\beta}$, i.e.*

$$T_{n,m,\alpha,\beta}(f)(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} p_{n,i}^{(\alpha)}(x) p_{m,j}^{(\beta)}(y) f\left(\frac{i}{n}, \frac{j}{m}\right), \tag{7}$$

then $T_{n,m,\alpha,\beta}$ preserves (q,s)–box convexity, for all $q, s \in \mathbb{N}$.

THEOREM 3. *Let $f \in C([0,1]\times[0,1])$ be a $(1,1)$–box convex function and x_1, t_1, y_1, $z_1 \in [0,1]$. Then*

$$\begin{aligned} sgn(x_1 - t_1)(y_1 - z_1) \sum_{i=0}^{n}\sum_{j=0}^{m} \left(p_{n,i}^{(\alpha)}(x_1) - p_{n,i}^{(\alpha)}(t_1)\right)\left(p_{m,j}^{(\beta)}(y_1) - p_{m,j}^{(\beta)}(z_1)\right) \\ \times A_{\frac{i}{n},\frac{j}{m}}(f) \geqslant 0, \end{aligned} \tag{8}$$

where $A_{\frac{i}{n},\frac{j}{m}}(f) = \int_0^1\int_0^1 f\left(\frac{i+au}{n+a}, \frac{j+bv}{m+b}\right) dudv$, $i = 0,1,\ldots,n$, $j = 0,1,\ldots,m$, a and b being two fixed positive numbers.

COROLLARY 4. *Let $f \in C[0,1]$ be a convex function and δ be a fixed positive number. Then*

$$\begin{aligned} sgn(x_1 - t_1)(y_1 - z_1) \sum_{i=0}^{n}\sum_{j=0}^{m} \left(p_{n,i}^{(\alpha)}(x_1) - p_{n,i}^{(\alpha)}(t_1)\right)\left(p_{m,j}^{(\beta)}(y_1) - p_{m,j}^{(\beta)}(z_1)\right) \\ \times \int_0^1 f\left(\frac{1}{2+\delta}\left(\frac{i}{n} + \frac{j}{m} + \delta t\right)\right) dt \geqslant 0. \end{aligned} \tag{9}$$

REMARK 5. *For $\alpha = \beta = 1$, $\delta = 0$, $x_1 = y_1 = x$ and $t_1 = z_1 = y$, $m = n$, we get inequality* (4).

2. Proofs

Proof of Theorem 1. Using Leibniz's rule in (3) we get

$$\begin{aligned} D^q T_{n,\alpha}(f;x) = (1-\alpha)\Bigg[& x D^q B_{n-2}\left(f\left(\frac{(n-2)t+2}{n}\right); x\right) \\ & + (1-x) D^q B_{n-2}\left(f\left(\frac{(n-2)t}{n}\right); x\right)\Bigg] + \alpha D^q B_n(f(t); x) \end{aligned} \tag{10}$$

$$+q\left[D^{q-1}B_{n-2}\left(f\left(\frac{(n-2)t+2}{n}\right);x\right)\right.$$
$$\left.-D^{q-1}B_{n-2}\left(f\left(\frac{(n-2)t}{n}\right);x\right)\right].$$

The first three terms in (10) are positive since Bernstein operators preserves q-monotonicity. For the last two terms, by (5) we have

$$D^{q-1}B_{n-2}\left(f\left(\frac{(n-2)t+2}{n}\right);x\right)-D^{q-1}B_{n-2}\left(f\left(\frac{(n-2)t}{n}\right);x\right) \tag{11}$$
$$=\binom{n-2}{q-1}\frac{(q-1)!}{n^{q-1}}\sum_{j=0}^{n-q-1}b_{n-q-1,j}(x)$$
$$\times\left\{\left[\frac{j+2}{n},\frac{j+3}{n},\ldots,\frac{j+q+1}{n};f(t)\right]-\left[\frac{j}{n},\frac{j+1}{n},\ldots,\frac{j+q-1}{n};f(t)\right]\right\}.$$

Using the recursive formula for divided differences, we obtain

$$\left[\frac{j+2}{n},\frac{j+3}{n},\ldots,\frac{j+q+1}{n};f(t)\right]-\left[\frac{j}{n},\frac{j+1}{n},\ldots,\frac{j+q-1}{n};f(t)\right] \tag{12}$$
$$=\left[\frac{j+2}{n},\frac{j+3}{n},\ldots,\frac{j+q+1}{n};f(t)\right]-\left[\frac{j+1}{n},\frac{j+2}{n},\ldots,\frac{j+q}{n};f(t)\right]$$
$$+\left[\frac{j+1}{n},\frac{j+2}{n},\ldots,\frac{j+q}{n};f(t)\right]-\left[\frac{j}{n},\frac{j+1}{n},\ldots,\frac{j+q-1}{n};f(t)\right]$$
$$=\frac{n}{q}\left\{\left[\frac{j+1}{n},\frac{j+2}{n},\ldots,\frac{j+q+1}{n};f(t)\right]+\left[\frac{j}{n},\frac{j+1}{n},\ldots,\frac{j+q}{n};f(t)\right]\right\}.$$

From (12), it follows that the last two terms are positive. This implies that $D^qT_{n,\alpha}(f;x)\geqslant 0$ and the proof is complete. □

For the proof of Theorem 2, we will use the following result due to T. Popoviciu, [11].

LEMMA 1. ([11], pp. 78, T. Popoviciu) *If $f\in C^{q+s}([0,1]\times[0,1])$ and the mixed derivative $\frac{\partial^{q+s}f}{\partial x^q\partial y^s}$ exists and is continuous, then f is (q,s)-box convex if and only if*

$$\frac{\partial^{q+s}f}{\partial x^q\partial y^s}\geqslant 0. \tag{13}$$

Proof of Theorem 2. We first note that the following indentity

$$T_{n,m,\alpha,\beta}(f)(x,y)=(1-\alpha)(1-\beta)L^{(1)}_{n-2,m-2}(f)(x,y)+(1-\alpha)\beta L^{(2)}_{n-2,m}(f)(x,y)$$
$$+\alpha(1-\beta)L^{(3)}_{n,m-2}(f)(x,y)+\alpha\beta B_{n,m}(f)(x,y),$$

where

$$\begin{aligned}
L^{(1)}_{n-2,m-2}(f)(x,y) &= \sum_{i=0}^{n}\sum_{j=0}^{m} u_{n-2,i}(x)u_{m-2,j}(y)f\left(\frac{i}{n},\frac{j}{m}\right),\\
L^{(2)}_{n-2,m}(f)(x,y) &= \sum_{i=0}^{n}\sum_{j=0}^{m} u_{n-2,i}(x)b_{m,j}(y)f\left(\frac{i}{n},\frac{j}{m}\right),\\
L^{(3)}_{n,m-2}(f)(x,y) &= \sum_{i=0}^{n}\sum_{j=0}^{m} b_{n,i}(x)u_{m-2,j}(y)f\left(\frac{i}{n},\frac{j}{m}\right)
\end{aligned}$$

and

$$u_{r,k}(t) = (1-t)b_{r,k}(t) + tb_{r,k-2}(t),$$

holds. We further have

$$\begin{aligned}
&\frac{\partial^{q+s}L^{(1)}_{n-2,m-2}(f)}{\partial x^q\partial y^s}(x,y)\\
&= \sum_{i=0}^{n}\sum_{j=0}^{m}\left[(1-x)D_x^q b_{n-2,i}(x) + xD_x^q b_{n-2,i-2}(x)\right]\\
&\quad\times\left[(1-y)D_y^s b_{m-2,j}(y) + yD_y^s b_{m-2,j-2}(y)\right] f\left(\frac{i}{n},\frac{j}{m}\right)\\
&\quad+qs\sum_{i=0}^{n}\sum_{j=0}^{m} D_x^{q-1}\left(b_{n-2,i-2}(x)-b_{n-2,i}(x)\right)D_y^{s-1}\left(b_{m-2,j-2}(y)-b_{m-2,j}(y)\right) f\left(\frac{i}{n},\frac{j}{m}\right)\\
&= \Sigma_I + qs\Sigma_{II},
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_I &= (1-x)(1-y)\frac{\partial^{q+s}B_{n-2,m-2}(f_1)}{\partial x^q\partial y^s}(x,y) + x(1-y)\frac{\partial^{q+s}B_{n-2,m-2}(f_2)}{\partial x^q\partial y^s}(x,y)\\
&\quad + y(1-x)\frac{\partial^{q+s}B_{n-2,m-2}(f_3)}{\partial x^q\partial y^s}(x,y) + xy\frac{\partial^{q+s}B_{n-2,m-2}(f_4)}{\partial x^q\partial y^s}(x,y),\\
\Sigma_{II} &= \sum_{i=0}^{n}\sum_{j=0}^{m} D_x^{q-1}\left(b_{n-2,i-2}(x)-b_{n-2,i}(x)\right)D_y^{s-1}\left(b_{m-2,j-2}(y)-b_{m-2,j}(y)\right) f\left(\frac{i}{n},\frac{j}{m}\right),
\end{aligned}$$

and f_1, f_2, f_3, f_4 are given by

$$\begin{aligned}
f_1(x,y) &= f\left(\frac{(n-2)x}{n},\frac{(m-2)y}{m}\right), \quad f_2(x,y) = f\left(\frac{(n-2)x+2}{n},\frac{(m-2)y}{m}\right),\\
f_3(x,y) &= f\left(\frac{(n-2)x}{n},\frac{(m-2)y+2}{m}\right), \quad f_4(x,y) = f\left(\frac{(n-2)x+2}{n},\frac{(m-2)y+2}{m}\right).
\end{aligned}$$

Since the functions $f_i, i=\overline{1,4}$ are (q,s)–box convex it follows that $\Sigma_I \geqslant 0$. From equations (11) and (12) we get successively

$$\begin{aligned}
\Sigma_{II} &= \frac{m}{s}\binom{m-2}{s-2}\frac{(s-1)!}{m^{s-1}}\sum_{i=0}^{n} D_x^{q-1}\left(b_{n-2,i-2}(x)-b_{n-2,i}(x)\right) \\
&\quad \times \sum_{j=0}^{m-s-1} b_{m-s-1,j}(y)\left\{\left[\frac{j+1}{m-2},\frac{j+2}{m-2},\dots,\frac{j+s+1}{m-2};f_1(x,y)\right]_y\right. \\
&\quad \left. +\left[\frac{j}{m-2},\frac{j+1}{m-2},\dots,\frac{j+s}{m-2};f_1(x,y)\right]_y\right\} \\
&= \frac{n}{q}\frac{m}{s}\binom{n-2}{q-1}\binom{m-2}{s-1}\frac{(q-1)!}{n^{q-1}}\frac{(s-1)!}{m^{s-1}}\sum_{i=0}^{n-q-1}\sum_{j=0}^{m-s-1} b_{n-q-1,i}(x)b_{m-s-1,j}(y) \\
&\quad \times\left(\begin{bmatrix}\frac{i+1}{n}, & \dots, & \frac{i+q+1}{n} \\ \frac{j+1}{m}, & \dots, & \frac{j+s+1}{m}\end{bmatrix};f\right] + \begin{bmatrix}\frac{i+1}{n}, & \dots, & \frac{i+q+1}{n} \\ \frac{j}{m}, & \dots, & \frac{j+s}{m}\end{bmatrix};f\right] + \begin{bmatrix}\frac{i}{n}, & \dots, & \frac{i+q}{n} \\ \frac{j+1}{m}, & \dots, & \frac{j+s+1}{m}\end{bmatrix};f\right] \\
&\quad + \begin{bmatrix}\frac{i}{n}, & \dots, & \frac{i+q}{n} \\ \frac{j}{m}, & \dots, & \frac{j+s}{m}\end{bmatrix};f\right]\Bigg).
\end{aligned}$$

This leads to $\Sigma_{II} \geqslant 0$. In a similar way one can prove that

$$\frac{\partial^{q+s}L_{n-2,m}^{(2)}(f)}{\partial x^q\partial y^s}(x,y) \geqslant 0$$

and

$$\frac{\partial^{q+s}L_{n,m-2}^{(3)}(f)}{\partial x^q\partial y^s}(x,y) \geqslant 0.$$

Therefore inequaltity (13) of Lemma 1 is satisfied by $T_{n,m,\alpha,\beta}(f)$ for any (q,s)-box convex function. This concludes our proof. $\square$

Proof of Theorem 3. Since for any continuous function $g:[0,1]\times[0,1]\to[0,\infty]$ we have

$$sgn(x_1-t_1)(y_1-z_1)\int_{t_1}^{x_1}\int_{z_1}^{y_1} g(x,y)dxdy \geqslant 0$$

it is sufficient to prove the theorem for the case $x_1 > t_1$ and $y_1 > z_1$.

We have

$$\begin{aligned}
&\sum_{i=0}^{n}\sum_{j=0}^{m}\left(p_{n,i}^{(\alpha)}(x_1)-p_{n,i}^{(\alpha)}(t_1)\right)\left(p_{m,j}^{(\beta)}(y_1)-p_{m,j}^{(\beta)}(z_1)\right)g\left(\frac{i}{n},\frac{j}{m}\right) \qquad (14)\\
&= \int_{t_1}^{x_1}\int_{z_1}^{y_1}\sum_{i=0}^{n}\sum_{j=0}^{m} p_{n,i}^{(\alpha)'}(u)p_{m,j}^{(\beta)'}(v)g\left(\frac{i}{n},\frac{j}{m}\right)dudv \\
&= \int_{t_1}^{x_1}\int_{z_1}^{y_1}\frac{\partial^2 T_{n,m,\alpha,\beta}(g)}{\partial x\partial y}(u,v)dudv,
\end{aligned}$$

for any $g \in C([0,1]\times[0,1])$. If g is $(1,1)$-box convex function, by virtue of Theorem 2, we obtain

$$\frac{\partial^2 T_{n,m,\alpha,\beta}(g)}{\partial x \partial y} \geqslant 0. \tag{15}$$

If f is a $(1,1)$–box convex function, then

$$g(x,y) = \int_0^1 \int_0^1 f\left(\frac{nx+au}{n+a}, \frac{mx+bv}{m+b}\right) du dv \tag{16}$$

is also a $(1,1)$–box convex function. Now, Theorem 3 follows from (15) with g given by (16). □

Proof of Corollary 4. If f is a convex function, then the function h, defined by

$$h(x,y) = \int_0^1 f\left(\frac{1}{2+\delta}(x+y+\delta t)\right) dt$$

is a $(1,1)$–box convex function on $[0,1]\times[0,1]$ for any $\delta \geqslant 0$. Now, (9) follows from (15) with $g := h$. □

We conclude this section by raising the following question.

PROBLEM. *Let q,s be two natural numbers, $q,s \geqslant 2$ and let $x_k, t_k \in [0,1], k = 1,\dots,q$ such that $x_k \neq t_k$ and $y_i, z_i \in [0,1], i = 1,\dots,s$ be such that $y_i \neq z_i$. If $g \in C([0,1]\times[0,1])$ is a (q,s)–box convex function, prove or disprove that*

$$\begin{aligned} & sgn\left(\prod_{k=1}^{q}(x_k - t_k)\right)\left(\prod_{i=1}^{s}(y_i - z_i)\right) \\ & \times \sum_{i_1,\dots,i_k=0}^{n} \sum_{j_1,\dots,j_s=0}^{m} \left(\prod_{k=1}^{q}\left(p_{n,i_k}^{(\alpha)}(x_k) - p_{n,i_k}^{(\alpha)}(t_k)\right)\right)\left(\prod_{r=1}^{s}\left(p_{m,j_r}^{(\beta)}(y_r) - p_{m,s_r}^{(\beta)}(z_r)\right)\right) \\ & \times g\left(\frac{i_1+\dots+i_q}{mq}, \frac{j_1+\dots+j_s}{ns}\right) \geqslant 0. \end{aligned} \tag{17}$$

REMARK 6. *For $\alpha = \beta = 1$ the assertion is true,* [6]. *For $\alpha = \beta = 1$, $s = 0$, $m = n$ and $g(x,y) = \int_0^1 f\left(\frac{nx+\alpha t}{qn+\alpha}\right) dt$, (17) is equivalent to the inequality from Theorem A,* [2].

3. Conclusions and future work

In this paper we prove that the α-Bernstein operators preserve q-monotonicity of all orders. We have also extended the result obtained by U. Abel and D. Leviatan in [2]. In the end of Section 2, we proposed an open problem related to (q,s)–box convex functions, that further extends the results from [2].

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