

## TRIANGULAR CESÀRO SUMMABILITY AND LEBESGUE POINTS OF TWO-DIMENSIONAL FOURIER SERIES

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*Abstract.* We prove that the triangular Cesàro means of two-dimensional functions  $f \in L_1(\mathbb{T}^2)$  converge to  $f$  at each strong  $(1, \omega)$ -Lebesgue point. Moreover, if  $f \in L_p(\mathbb{T}^2)$  with  $1 < p < \infty$ , then the Cesàro means converge to  $f$  at each  $(p, \omega)$ -Lebesgue point. This generalizes the well known classical Lebesgue's theorem.

### 1. Introduction

It is known that the Fejér means

$$\sigma_n f(x) := \sum_{k=-n}^n \left(1 - \frac{|k|}{n}\right) \widehat{f}(k) e^{ikx}$$

of a one-dimensional integrable function  $f$  converge to  $f(x)$  at each Lebesgue point (see Lebesgue [11]), where  $\widehat{f}(k)$  denotes the  $k$ th Fourier coefficient. The generalization of this theorem to Cesàro means is due to M. Riesz [15]. These means were considered in a great number of papers (see e.g. Gát [4, 5, 6], Goginava [7, 8, 9], Simon [16, 17], Nagy, Persson, Tephnadze and Wall [13, 14], Weisz [19, 20] and Zygmund [26]).

Here we investigate the triangular Cesàro means

$$\sigma_n^\alpha f(x) := \frac{1}{A_{n-1}^\alpha} \sum_{|k_1|+|k_2| \leq n} A_{n-1-|k_1|-|k_2|}^\alpha \widehat{f}(k) e^{i(k_1 x_1 + k_2 x_2)}$$

of two-dimensional functions. These means were also studied e.g. in Berens, Li and Xu [1, 2, 12, 25], Szili and Vértési [18]. In [19], we proved that  $\sigma_n^\alpha f \rightarrow f$  almost everywhere if  $f \in L_1(\mathbb{T}^2)$ .

In this paper, we partly characterize the set of this convergence, which is a generalization of the result of Lebesgue and Riesz just mentioned. We introduce two new types of Lebesgue points, the  $(p, \omega)$ - and strong  $(p, \omega)$ -Lebesgue points. The author proved in [21] that almost every point is a (strong)  $(p, \omega)$ -Lebesgue point of  $f \in L_p(\mathbb{T}^2)$  with

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$1 \leq p < \infty$ . We show that if the maximal function  $\mathcal{M}_1^\omega f(x)$  is finite and  $x$  is a strong  $(1, \omega)$ -Lebesgue point of  $f \in L_1(\mathbb{T}^2)$ , then

$$\lim_{n \rightarrow \infty} \sigma_n^\alpha f(x) = f(x).$$

For  $1 < p < \infty$ , we get a simpler result: the convergence holds for all  $(p, \omega)$ -Lebesgue points if  $f \in L_p(\mathbb{T}^2)$ . A more complicated version of the first result can be shown for higher dimensional functions with another method (see [23]). In [22], we verified similar results for triangular  $\theta$ -means of Fourier transforms. However, the  $\theta$ -summability does not contain the Cesàro summability and the proof for Fourier series is more complicated than for Fourier transforms, so new ideas are needed here. An analogous result for another type of summability, for the so called rectangular summability was proved in [24].

### 2. Maximal functions and Lebesgue points

We briefly write  $L_p(\mathbb{T}^2)$  instead of the  $L_p(\mathbb{T}^2, \lambda)$  space equipped with the norm

$$\|f\|_p := \left( \int_{\mathbb{T}^2} |f|^p d\lambda \right)^{1/p} \quad (1 \leq p < \infty),$$

where  $\mathbb{T} = [-\pi, \pi]$  is the torus and  $\lambda$  the Lebesgue measure.

For some  $\omega > 0$ ,  $1 \leq p < \infty$ ,  $x = (x_1, x_2)$  and  $f \in L_1(\mathbb{T}^2)$ , we introduce the next Hardy-Littlewood maximal functions:

$$\mathcal{M}_p^\omega f(x) = \sum_{j=1}^5 \mathcal{M}_p^{\omega, j} f(x),$$

where

$$\mathcal{M}_p^{\omega, 1} f(x) = \sup_{i_1, i_2 \in \mathbb{N}, h > 0} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x-t)|^p dt \right)^{1/p},$$

$$\mathcal{M}_p^{\omega, 2} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{t_1-2^{i_2} h}^{t_1+2^{i_2} h} |f(x-t)|^p dt_2 dt_1 \right)^{1/p},$$

$$\mathcal{M}_p^{\omega, 3} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-t_1-2^{i_2} h}^{-t_1+2^{i_2} h} |f(x-t)|^p dt_2 dt_1 \right)^{1/p},$$

$$\mathcal{M}_p^{\omega, 4} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_2} h}^{2^{i_2} h} \int_{t_2-2^{i_1} h}^{t_2+2^{i_1} h} |f(x-t)|^p dt_1 dt_2 \right)^{1/p}$$

and

$$\mathcal{M}_p^{\omega,5} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_2} h}^{2^{i_2} h} \int_{-2^{i_2-2^{i_1}} h}^{-2^{i_2}+2^{i_1} h} |f(x-t)|^p dt_1 dt_2 \right)^{1/p}.$$

Note that in  $\mathcal{M}_p^{\omega,1} f$ , we take the supremum over rectangles with sides parallel to the axes and in  $\mathcal{M}_p^{\omega,j} f$  ( $j = 2, 3, 4, 5$ ), over parallelograms with one side parallel to one of the axes and with the other side parallel to one of the diagonals of the square  $[0, \pi]^2$ . If  $\omega = 0$ , then  $\mathcal{M}_p^{\omega,1} f$  is exactly the strong Hardy-Littlewood maximal function, if  $\omega = 0$  and  $i_1 = i_2$ , then  $\mathcal{M}_p^{\omega,1} f$  is equal to the usual Hardy-Littlewood maximal function (see e.g. Feichtinger and Weisz [3]). The next two inequalities were proved in [21]. If  $\omega > 0$  and  $1 \leq p < \infty$ , then

$$\sup_{\rho > 0} \rho \lambda(\mathcal{M}_p^\omega f > \rho)^{1/p} \leq C \|f\|_p \quad (f \in L_p(\mathbb{T}^2)). \tag{1}$$

Moreover, if  $p < r \leq \infty$ , then

$$\|\mathcal{M}_p^\omega f\|_r \leq C_r \|f\|_r \quad (f \in L_r(\mathbb{T}^d)).$$

In this paper the constants  $C$  and  $C_p$  may vary from line to line.

Using the above maximal functions, we define the next operators:

$$U_{r,p}^{\omega,1} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0, 2^k h < r, k=1,2} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-2^{i_2} h}^{2^{i_2} h} |f(x-t) - f(x)|^p dt \right)^{1/p},$$

$$U_{r,p}^{\omega,2} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0, 2^k h < r, k=1,2} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{t_1-2^{i_2} h}^{t_1+2^{i_2} h} |f(x-t) - f(x)|^p dt_2 dt_1 \right)^{1/p},$$

$$U_{r,p}^{\omega,3} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0, 2^k h < r, k=1,2} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_1} h}^{2^{i_1} h} \int_{-t_1-2^{i_2} h}^{-t_1+2^{i_2} h} |f(x-t) - f(x)|^p dt_2 dt_1 \right)^{1/p},$$

$$U_{r,p}^{\omega,4} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0, 2^k h < r, k=1,2} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_2} h}^{2^{i_2} h} \int_{t_2-2^{i_1} h}^{t_2+2^{i_1} h} |f(x-t) - f(x)|^p dt_1 dt_2 \right)^{1/p},$$

$$U_{r,p}^{\omega,5} f(x) := \sup_{i_1, i_2 \in \mathbb{N}, h > 0, 2^{i_k} h < r, k=1,2} 2^{-\omega(i_1+i_2)} \left( \frac{1}{4 \cdot 2^{i_1+i_2} h^2} \int_{-2^{i_2} h}^{2^{i_2} h} \int_{-2^{i_1} h}^{-2^{i_1} h + 2^{i_2} h} |f(x-t) - f(x)|^p dt_1 dt_2 \right)^{1/p},$$

and

$$U_{r,p}^\omega f(x) = \sum_{j=1}^5 U_{r,p}^{\omega,j} f(x).$$

If  $p = 1$ , then we omit the index  $p$  and simply write  $\mathcal{M}^\omega, U_r^\omega, \dots$

For  $1 \leq p < \infty$  and  $\omega > 0$ , a point  $x \in \mathbb{T}^2$  is called a  $(p, \omega)$ -Lebesgue point of  $f \in L_p(\mathbb{T}^d)$  if

$$\lim_{r \rightarrow 0} U_{r,p}^{\omega,1} f(x) = 0.$$

If

$$\lim_{r \rightarrow 0} U_{p,r}^\omega f(x) = 0,$$

then  $x$  is called a strong  $(p, \omega)$ -Lebesgue point. If  $\omega = 0$ , then the  $(p, \omega)$ -Lebesgue points are the same as the well known strong Lebesgue points, if  $\omega = 0$  and  $i_1 = i_2$ , then the  $(p, \omega)$ -Lebesgue points give back the usual Lebesgue points, i.e.,

$$\lim_{h \rightarrow 0} \frac{1}{4h_1 h_2} \int_{-h_1}^{h_1} \int_{-h_2}^{h_2} |f(x-t) - f(x)| dt = 0$$

and

$$\lim_{h \rightarrow 0} \frac{1}{(2h)^2} \int_{-h}^h \int_{-h}^h |f(x-t) - f(x)| dt = 0$$

(see Feichtinger and Weisz [3]). Since  $U_{p,r}^{\omega_1} f \leq U_{p,r}^{\omega_2} f$ , every (strong)  $(p, \omega_2)$ -Lebesgue point is a (strong)  $(p, \omega_1)$ -Lebesgue point ( $0 < \omega_2 < \omega_1 < \infty$ ). If  $f$  is continuous at  $x$ , then  $x$  is a (strong)  $(p, \omega)$ -Lebesgue point of  $f$ . The next theorem was proved in [21].

**THEOREM 1.** *For  $1 \leq p < \infty$  and  $\omega > 0$ , almost every point  $x \in \mathbb{T}^2$  is a (strong)  $(p, \omega)$ -Lebesgue point of  $f \in L_p(\mathbb{T}^2)$ .*

### 3. The kernel functions

The  $k$ th Fourier coefficient of a two-dimensional integrable function  $f \in L_1(\mathbb{T}^2)$  is defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x) e^{-i(k_1 x_1 + k_2 x_2)} dx \quad (k \in \mathbb{Z}^2).$$

For  $f \in L_1(\mathbb{T}^2)$  and  $n \in \mathbb{N}$ , we define the  $n$ th triangular partial sum  $s_n f$  of the Fourier series of  $f$  by

$$s_n f(x) := \sum_{|k_1| + |k_2| \leq n} \widehat{f}(k) e^{i(k_1 x_1 + k_2 x_2)}$$

It is known (see e.g. Grafakos [10] or Weisz [20]) that for  $f \in L_p(\mathbb{T}^2)$ ,  $1 < p < \infty$ ,

$$\lim_{n \rightarrow \infty} s_n f = f \quad \text{in the } L_p(\mathbb{T}^2)\text{-norm and a.e.}$$

To extend this convergence to  $p = 1$ , we need to consider the Cesàro summation.

For  $\alpha \neq -1, -2, \dots$  and  $n \in \mathbb{N}$ , let

$$A_n^\alpha := \binom{n + \alpha}{n} = \frac{(\alpha + 1)(\alpha + 2) \cdots (\alpha + n)}{n!}.$$

Then  $A_0^\alpha = 1$ ,  $A_n^0 = 1$  and  $A_n^1 = n + 1$  ( $n \in \mathbb{N}$ ). Let  $f \in L_1(\mathbb{T}^d)$ ,  $n \in \mathbb{N}$  and  $\alpha \geq 0$ . In [19], we investigated the triangular Cesàro means  $\sigma_n^\alpha f$  given by

$$\sigma_n^\alpha f(x) := \frac{1}{A_{n-1}^\alpha} \sum_{|k_1|+|k_2| \leq n} A_{n-1-|k_1|-|k_2|}^\alpha \widehat{f}(k) e^{i(k_1 x_1 + k_2 x_2)}.$$

If  $\alpha = 0$ , we get back  $s_n f$ , if  $\alpha = 1$ , then the triangular Fejér means

$$\sigma_n^1 f(x) := \sum_{|k_1|+|k_2| \leq n} \left(1 - \frac{|k_1| + |k_2|}{n}\right) \widehat{f}(k) e^{i(k_1 x_1 + k_2 x_2)}.$$

It is easy to see that

$$\sigma_n^\alpha f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^2} f(x-t) K_n^\alpha(t) dt,$$

where

$$K_n^\alpha(t) := \frac{1}{A_{n-1}^\alpha} \sum_{|k_1|+|k_2| \leq n} A_{n-1-|k_1|-|k_2|}^\alpha e^{i(k_1 x_1 + k_2 x_2)}$$

are the triangular Cesàro kernels. Zygmund [26] verified the next lemma.

LEMMA 1. For  $\alpha > -1$  and  $h > 0$ , we have

$$\sigma_n^{\alpha+h} f = \frac{1}{A_{n-1}^{\alpha+h}} \sum_{k=1}^n A_{n-k}^{h-1} A_{k-1}^\alpha \sigma_k^\alpha f.$$

The next two lemmas were proved by the author in [19].

LEMMA 2. Suppose that  $0 < \alpha \leq 1$ ,  $0 \leq \beta \leq 1$  and  $\pi > x_1 > x_2 > 0$ . Then

$$|K_n^\alpha(x_1, x_2)| \leq Cn^2, \tag{2}$$

$$\begin{aligned} |K_n^\alpha(x_1, x_2)| &\leq C(x_1 - x_2)^{-1} (x_1 + x_2)^{-1} \mathbf{1}_{\{x_2 \leq \pi/2\}} \\ &\quad + C(x_1 - x_2)^{-1} (\pi - x_2)^{-1} \mathbf{1}_{\{x_2 > \pi/2\}}. \end{aligned} \tag{3}$$

If  $x_2 > 1/n$ , then

$$1_{\{x_2 \leq \pi/2\}} |K_n^\alpha(x_1, x_2)| \leq Cn^{-\alpha}(x_1 - x_2)^{-1-\beta} x_2^{\beta-\alpha-1} 1_{\{x_2 \leq \pi/2\}} \tag{4}$$

and

$$1_{\{x_2 \leq \pi/2\}} |K_n^\alpha(x_1, x_2)| \leq Cn^{1-\alpha} x_2^{-\alpha-1} 1_{\{x_2 \leq \pi/2\}}. \tag{5}$$

If  $\pi - x_2 > 1/n$ , then

$$1_{\{x_2 > \pi/2\}} |K_n^\alpha(x_1, x_2)| \leq Cn^{-\alpha}(x_1 - x_2)^{-1-\beta} (\pi - x_2)^{\beta-\alpha-1} 1_{\{x_2 > \pi/2\}} \tag{6}$$

and

$$1_{\{x_2 > \pi/2\}} |K_n^\alpha(x_1, x_2)| \leq Cn^{1-\alpha} (\pi - x_2)^{-\alpha-1} 1_{\{x_2 > \pi/2\}}. \tag{7}$$

LEMMA 3. If  $0 < \alpha \leq 1$ , then

$$\sup_{n \in \mathbb{N}} \int_{\mathbb{T}^2} |K_n^\alpha| d\lambda \leq C.$$

#### 4. Convergence of the Cesàro means at Lebesgue points

Now we are ready to prove our main theorem.

THEOREM 2. Suppose that  $0 < \alpha < \infty$ ,  $0 < \omega < \min(\alpha, 1)/2$  and  $\mathcal{M}^\omega f(x)$  is finite. If  $f \in L_1(\mathbb{T}^2)$  is periodic with respect to  $\pi$  and  $x$  is a strong  $(1, \omega)$ -Lebesgue point of  $f$ , then

$$\lim_{n \rightarrow \infty} \sigma_n^\alpha f(x) = f(x).$$

*Proof.* By Lemma 1, it is enough to prove the theorem for  $0 < \alpha \leq 1$ . Since

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^2} K_n^\alpha(t) dt = 1,$$

we have

$$|\sigma_n^\alpha f(x) - f(x)| \leq \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{T}^2} |f(x-t) - f(x)| |K_n^\alpha(t)| dt.$$

Let  $0 < \omega < \alpha/2$  and fix a number  $r < 1$  such that  $U_r^\omega f(x_1, x_2) < \varepsilon$ . Suppose that  $2/n < r/2$ . Let us introduce the sets

$$S_{r/2} := \left[-\frac{r}{2}, \frac{r}{2}\right] \times \left[-\frac{r}{2}, \frac{r}{2}\right], \quad S'_{r/2} := \left[\pi - \frac{r}{2}, \pi + \frac{r}{2}\right] \times \left[\pi - \frac{r}{2}, \pi + \frac{r}{2}\right]$$

and

- $A_1 := \{(x_1, x_2) : 0 < x_1 \leq 2/n, 0 < x_2 < x_1 < \pi, x_2 \leq \pi/2\},$
- $A_2 := \{(x_1, x_2) : 2/n < x_1 < \pi, 0 < x_2 \leq 1/n, x_2 \leq \pi/2\},$
- $A_3 := \{(x_1, x_2) : 2/n < x_1 < \pi, 1/n < x_2 \leq x_1/2, x_2 \leq \pi/2\},$
- $A_4 := \{(x_1, x_2) : 2/n < x_1 < \pi, x_1/2 < x_2 \leq x_1 - 1/n, x_2 \leq \pi/2\},$
- $A_5 := \{(x_1, x_2) : 2/n < x_1 < \pi, x_1 - 1/n < x_2 < x_1, x_2 \leq \pi/2\}$
- $A_6 := \{(x_1, x_2) : x_2 > \pi/2, \pi - 2/n \leq x_2 < \pi, 0 < x_2 < x_1 < \pi\},$
- $A_7 := \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, \pi - 1/n < x_1 < \pi\},$
- $A_8 := \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, (\pi + x_2)/2 < x_1 \leq \pi - 1/n\},$
- $A_9 := \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, x_2 + 1/n < x_1 \leq (\pi + x_2)/2\},$
- $A_{10} := \{(x_1, x_2) : \pi/2 < x_2 < \pi - 2/n, x_2 < x_1 \leq x_2 + 1/n\}$

(see Figure 1). It is enough to integrate on the set  $\{(t_1, t_2) : 0 < t_2 < t_1 < \pi\},$

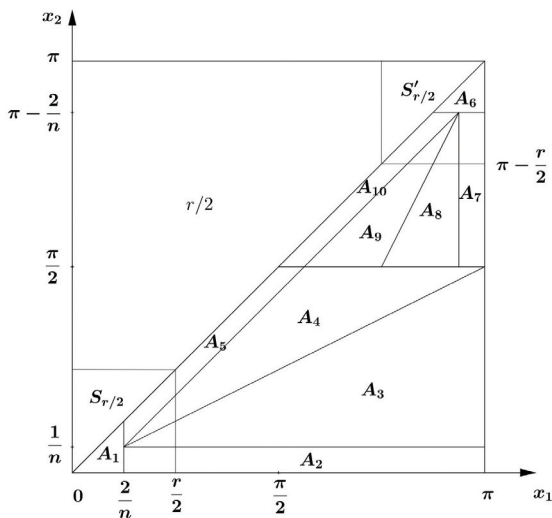


Figure 1: The sets  $A_i$ .

in other words, on

$$\bigcup_{i=1}^5 (A_i \cap S_{r/2}), \quad \bigcup_{i=1}^5 (A_i \cap S_{r/2}^c), \quad \bigcup_{i=6}^{10} (A_i \cap S'_{r/2}), \quad \bigcup_{i=6}^{10} (A_i \cap (S'_{r/2})^c).$$

Since  $A_1 \subset S_{r/2}$  and  $A_6 \subset S'_{r/2}$ , inequality (2) implies

$$\begin{aligned} \int_{A_1} |f(x-t) - f(x)| |K_n^\alpha(t)| dt &\leq Cn^2 \int_0^{2/n} \int_0^{2/n} |f(x-t) - f(x)| dt_2 dt_1 \\ &\leq CU_r^{\omega,1} f(x) < C\varepsilon \end{aligned}$$

and

$$\begin{aligned} \int_{A_6} |f(x-t) - f(x)| |K_n^\alpha(t)| dt &\leq Cn^2 \int_{\pi-2/n}^\pi \int_{\pi-2/n}^\pi |f(x-t) - f(x)| dt_2 dt_1 \\ &\leq Cn^2 \int_{-2/n}^0 \int_{-2/n}^0 |f(x-u-\pi) - f(x)| du_2 du_1 \\ &\leq CU_r^{\omega,1} f(x) < C\mathcal{E}. \end{aligned}$$

Let us denote by  $r_0$  the largest number  $i$ , for which  $r/2 \leq 2^{i+1}/n < r$ . By (3),

$$\begin{aligned} &\int_{A_2 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ &\leq C \sum_{i=1}^{r_0} \left(\frac{2^i}{n} - \frac{1}{n}\right)^{-1} \left(\frac{2^i}{n}\right)^{-1} \int_{2^i/n}^{2^{i+1}/n} \int_0^{1/n} |f(x-t) - f(x)| dt_2 dt_1 \\ &\leq C \sum_{i=1}^{r_0} 2^{(\omega-1)i} 2^{-\omega i} \left(\frac{n^2}{2^i}\right) \int_{2^i/n}^{2^{i+1}/n} \int_0^{1/n} |f(x-t) - f(x)| dt_2 dt_1 \\ &\leq C \sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,1} f(x) < C\mathcal{E}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\int_{A_7 \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ &\leq C \sum_{i=1}^{r_0} \left(\frac{2^i}{n} - \frac{1}{n}\right)^{-1} \left(\frac{2^i}{n}\right)^{-1} \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{\pi-1/n}^\pi |f(x-t) - f(x)| dt_1 dt_2 \\ &\leq C \sum_{i=1}^{r_0} 2^{(\omega-1)i} 2^{-\omega i} \left(\frac{n^2}{2^i}\right) \int_{-2^{i+1}/n}^{-2^i/n} \int_{-1/n}^0 |f(x-t-\pi) - f(x)| dt_1 dt_2 \\ &\leq C \sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,1} f(x_1, x_2) < C\mathcal{E}. \end{aligned}$$

It follows in the same way that

$$\begin{aligned} &\int_{A_2 \cap S_{r/2}^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt + \int_{A_7 \cap (S'_{r/2})^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ &\leq C \sum_{i=r_0}^\infty 2^{(\omega-1)i} \mathcal{M}^{\omega,1} f(x_1, x_2) + C \sum_{i=r_0}^\infty 2^{-i} |f(x_1, x_2)| \\ &\leq C2^{(\omega-1)r_0} \mathcal{M}^{\omega,1} f(x_1, x_2) + C2^{-r_0} |f(x_1, x_2)| \\ &\leq C(nr)^{\omega-1} \mathcal{M}^{\omega,1} f(x_1, x_2) + C(nr)^{-1} |f(x_1, x_2)| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . Observe that  $t_1 - t_2 > t_1/2$  and  $t_1 - t_2 > t_2$  on  $A_3$ . We use (4) to estimate the kernel as follows:

$$|K_n^\alpha(t)| \leq Cn^{-\alpha} (t_1 - t_2)^{-1-\alpha/2} (t_1 - t_2)^{-\beta+\alpha/2} t_2^{\beta-\alpha-1} \leq Cn^{-\alpha} t_1^{-1-\alpha/2} t_2^{-1-\alpha/2}$$



if  $\beta > \alpha/2$ . Hence

$$\begin{aligned} & \int_{A_3 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n}\right)^{-1-\alpha/2} \left(\frac{2^j}{n}\right)^{-1-\alpha/2} \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x-t) - f(x)| dt_2 dt_1 \\ & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}}\right) \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x-t) - f(x)| dt_2 dt_1 \\ & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,1} f(x_1, x_2) < C\varepsilon. \end{aligned}$$

Similarly,  $t_1 - t_2 > (\pi - t_2)/2 \geq (\pi - t_1)/2$  on  $A_8$ . By inequality (6),

$$\begin{aligned} |K_n^\alpha(t)| & \leq Cn^{-\alpha} (t_1 - t_2)^{-1-\alpha/2} (t_1 - t_2)^{-\beta+\alpha/2} (\pi - t_2)^{\beta-\alpha-1} \\ & \leq Cn^{-\alpha} (\pi - t_1)^{-1-\alpha/2} (\pi - t_2)^{-1-\alpha/2} \end{aligned}$$

if  $\beta > \alpha/2$ , which implies

$$\begin{aligned} & \int_{A_8 \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n}\right)^{-1-\alpha/2} \left(\frac{2^j}{n}\right)^{-1-\alpha/2} \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x-t) - f(x)| dt_1 dt_2 \\ & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}}\right) \int_{-2^{i+1}/n}^{-2^i/n} \int_{-2^{j+1}/n}^{-2^j/n} |f(x-t-\pi) - f(x)| dt_1 dt_2 \\ & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,1} f(x_1, x_2) < C\varepsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_{A_3 \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt + \int_{A_8 \cap (S'_{r/2})^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C \sum_{i=r_0}^{\infty} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} \mathcal{M}^{\omega,1} f(x) + C \sum_{i=r_0}^{\infty} \sum_{j=0}^{i-1} 2^{-\alpha(i+j)/2} |f(x)| \\ & \leq C2^{(\omega-\alpha/2)r_0} \mathcal{M}^{\omega,1} f(x) + C2^{-\alpha r_0/2} |f(x)| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . It is easy to see that  $t_2 > t_1/2$  on  $A_4$ . We apply (4) with  $\beta = \alpha/2$  and conclude

$$|K_n^\alpha(t)| \leq Cn^{-\alpha} (t_1 - t_2)^{-1-\alpha/2} t_2^{-1-\alpha/2} \leq Cn^{-\alpha} t_1^{-1-\alpha/2} (t_1 - t_2)^{-1-\alpha/2}$$

and so

$$\begin{aligned}
 & \int_{A_4 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
 & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n}\right)^{-1-\alpha/2} \left(\frac{2^j}{n}\right)^{-1-\alpha/2} \int_{2^i/n}^{2^{i+1}/n} \int_{t_1-2^{j+1}/n}^{t_1-2^j/n} |f(x-t) - f(x)| dt_2 dt_1 \\
 & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}}\right) \int_{2^i/n}^{2^{i+1}/n} \int_{t_1-2^{j+1}/n}^{t_1-2^j/n} |f(x-t) - f(x)| dt_2 dt_1 \\
 & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,2} f(x_1, x_2) < C\varepsilon.
 \end{aligned}$$

The inequality

$$|K_n^\alpha(t)| \leq Cn^{-\alpha}(t_1 - t_2)^{-1-\alpha/2} (\pi - t_2)^{-1-\alpha/2}$$

follows from (6) with  $\beta = \alpha/2$ . Hence

$$\begin{aligned}
 & \int_{A_9 \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
 & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} n^{-\alpha} \left(\frac{2^i}{n}\right)^{-1-\alpha/2} \left(\frac{2^j}{n}\right)^{-1-\alpha/2} \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{t_2+2^j/n}^{t_2+2^{j+1}/n} |f(x-t) - f(x)| dt_1 dt_2 \\
 & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} 2^{-\omega(i+j)} \left(\frac{n^2}{2^{i+j}}\right) \\
 & \quad \int_{-2^{i+1}/n}^{-2^i/n} \int_{t_2+2^j/n}^{t_2+2^{j+1}/n} |f(x-t-\pi) - f(x)| dt_1 dt_2 \\
 & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} U_r^{\omega,4} f(x_1, x_2) < C\varepsilon.
 \end{aligned}$$

We conclude in the same way that

$$\begin{aligned}
 & \int_{A_4 \cap S_{r/2}^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt + \int_{A_9 \cap (S'_{r/2})^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\
 & \leq C \sum_{i=r_0}^{\infty} \sum_{j=0}^{i-1} 2^{(\omega-\alpha/2)(i+j)} (\mathcal{M}^{\omega,2} f(x) + \mathcal{M}^{\omega,4} f(x)) + C \sum_{i=r_0}^{\infty} \sum_{j=0}^{i-1} 2^{-\alpha(i+j)/2} |f(x)| \\
 & \leq C2^{(\omega-\alpha/2)r_0} \mathcal{M}^\omega f(x) + C2^{-\alpha r_0/2} |f(x)| \rightarrow 0
 \end{aligned}$$

as  $n \rightarrow \infty$ . On  $A_5$ , we get

$$|K_n^\alpha(t)| \leq Cn^{1-\alpha} t_1^{-\alpha-1}$$

from (5). Then

$$\begin{aligned} & \int_{A_5 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C \sum_{i=1}^{r_0} n^{1-\alpha} \left(\frac{2^i}{n}\right)^{-\alpha-1} \int_{2^{i/n}}^{2^{i+1}/n} \int_{t_1-1/n}^{t_1} |f(x-t) - f(x)| dt_1 dt_2 \\ & \leq C \sum_{i=1}^{r_0} 2^{(\omega-\alpha)i} 2^{-\omega i} \left(\frac{n^2}{2^i}\right) \int_{2^{i/n}}^{2^{i+1}/n} \int_{t_1-1/n}^{t_1} |f(x-t) - f(x)| dt_1 dt_2 \\ & \leq C \sum_{i=1}^{r_0} 2^{(\omega-1)i} U_r^{\omega,2} f(x) < C\varepsilon. \end{aligned}$$

Similarly, by (7),

$$\begin{aligned} & \int_{A_{10} \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C \sum_{i=1}^{r_0} n^{1-\alpha} \left(\frac{2^i}{n}\right)^{-1-\alpha} \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{t_2}^{t_2+1/n} |f(x-t) - f(x)| dt_1 dt_2 \\ & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha)i} 2^{-\omega i} \left(\frac{n^2}{2^i}\right) \int_{-2^{i+1}/n}^{-2^i/n} \int_{t_2}^{t_2+1/n} |f(x-t-\pi) - f(x)| dt_1 dt_2 \\ & \leq C \sum_{i=1}^{r_0} \sum_{j=0}^{i-1} 2^{(\omega-\alpha)i} U_r^{\omega,4} f(x_1, x_2) < C\varepsilon. \end{aligned}$$

Finally,

$$\begin{aligned} & \int_{A_5 \cap S_{r/2}^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt + \int_{A_{10} \cap (S'_{r/2})^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C \sum_{i=r_0}^{\infty} 2^{(\omega-\alpha)i} (\mathcal{M}^{\omega,2} f(x) + \mathcal{M}^{\omega,4} f(x)) + C \sum_{i=r_0}^{\infty} 2^{-\alpha i} |f(x)| \\ & \leq C 2^{(\omega-\alpha)r_0} \mathcal{M}^\omega f(x) + C 2^{-\alpha r_0} |f(x)| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , which finishes the proof.  $\square$

Theorems 1, 2 and (1) imply

**COROLLARY 1.** *If  $0 < \alpha < \infty$  and  $f \in L_1(\mathbb{T}^2)$  is periodic with respect to  $\pi$ , then*

$$\lim_{n \rightarrow \infty} \sigma_n^\alpha f = f \quad a.e.$$

This corollary was also obtained in Weisz [19]. For  $1 < p < \infty$ , we can prove the theorem for  $(p, \omega)$ -Lebesgue points.

**THEOREM 3.** *Suppose that  $0 < \alpha < \infty$ ,  $1/(\min(\alpha, 1)) < p < \infty$ ,  $1/p + 1/q = 1$ ,  $0 < \omega < (1 + q \min(\alpha, 1) - q)/2q$  and  $\mathcal{M}_p^{\omega, 1} f(x)$  is finite. If  $f \in L_p(\mathbb{T}^2)$  is periodic with respect to  $\pi$  and  $x$  is a  $(p, \omega)$ -Lebesgue point of  $f$ , then*

$$\lim_{n \rightarrow \infty} \sigma_n^\alpha f(x) = f(x).$$

*Proof.* We prove the theorem again for  $0 < \alpha \leq 1$ . Note that  $1/\alpha < p < \infty$  implies  $1 < q < 1/(1 - \alpha)$  and so  $1 + \alpha q - q > 0$ . Moreover,  $(1 + \alpha q - q)/(2q) < \alpha/2$ . Fix a number  $r < 1$  such that  $U_{r,p}^{\omega, 1} f(x_1, x_2) < \varepsilon$ . We have seen in in Theorem 2 that

$$\int_{A_i} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \rightarrow 0,$$

for  $i = 1, 2, 3, 6, 7, 8$ , as  $n \rightarrow \infty$  and  $\omega < \alpha/2$ . So we have to estimate the integrals on  $A_4, A_5, A_9$  and  $A_{10}$ .

Since  $t_2 > t_1/2$  on  $A_4$ , (4) with  $\beta = 0$  implies

$$|K_n^\alpha(t)| \leq Cn^{-\alpha}(t_1 - t_2)^{-1}t_1^{-\alpha-1}.$$

By Hölder’s inequality,

$$\begin{aligned} & \int_{A_4 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq \sum_{i=1}^{r_0} \sum_{j=i-1}^i \int_{2^j/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x-t) - f(x)| |K_n^\alpha(t)| 1_{A_4}(t) dt_2 dt_1 \\ & \leq \sum_{i=1}^{r_0} \sum_{j=i-1}^i \left( \int_{2^j/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x-t) - f(x)|^p dt_2 dt_1 \right)^{1/p} \\ & \quad \left( \int_{2^j/n}^{2^{i+1}/n} \int_{2^{i-1}/n}^{t_1-1/n} n^{-\alpha q} (t_1 - t_2)^{-q} t_1^{-q(1+\alpha)} 1_{A_4}(t) dt_2 dt_1 \right)^{1/q}. \end{aligned}$$

We compute the last integral as follows:

$$\begin{aligned} & \int_{2^j/n}^{2^{i+1}/n} \int_{2^{i-1}/n}^{t_1-1/n} n^{-\alpha q} (t_1 - t_2)^{-q} t_1^{-q(1+\alpha)} 1_{A_4}(t) dt_2 dt_1 \\ & \leq Cn^{-\alpha q} \left(\frac{1}{n}\right)^{1-q} \int_{2^j/n}^{2^{i+1}/n} t_1^{-q(1+\alpha)} dt_1 \\ & \leq Cn^{-\alpha q} \left(\frac{1}{n}\right)^{1-q} \left(\frac{2^i}{n}\right)^{1-q(1+\alpha)} \\ & \leq C \left(\frac{n}{2^i}\right)^{2q-2} 2^{-i(1+\alpha q-q)} \end{aligned}$$

and so

$$\begin{aligned} & \int_{A_4 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} \\ & \quad 2^{-\omega(i+j)} \left( \frac{n^2}{2^{i+j}} \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x-t) - f(x)|^p dt_2 dt_1 \right)^{1/p} \\ & \leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} U_{r,p}^{\omega,1} f(x) < C_p \varepsilon. \end{aligned}$$

By (6) with  $\beta = 0$ ,

$$\begin{aligned} & \int_{A_9 \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq \sum_{i=1}^{r_0} \sum_{j=i-1}^i \left( \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x-t) - f(x)|^p dt_1 dt_2 \right)^{1/p} \\ & \quad \left( \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{t_2+1/n}^{\pi-2^{i-1}/n} n^{-\alpha q} (t_1 - t_2)^{-q} (\pi - t_2)^{-q(1+\alpha)} 1_{A_4}(t) dt_1 dt_2 \right)^{1/q}. \end{aligned}$$

The last integral is equal to

$$\begin{aligned} & \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{t_2+1/n}^{\pi-2^{i-1}/n} n^{-\alpha q} (t_1 - t_2)^{-q} (\pi - t_2)^{-q(1+\alpha)} 1_{A_4}(t) dt_1 dt_2 \\ & \leq C n^{-\alpha q} \left(\frac{1}{n}\right)^{1-q} \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} (\pi - t_2)^{-q(1+\alpha)} dt_2 \\ & \leq C n^{-\alpha q} \left(\frac{1}{n}\right)^{1-q} \left(\frac{2^i}{n}\right)^{1-q(1+\alpha)} \leq C \left(\frac{n}{2^i}\right)^{2q-2} 2^{-i(1+\alpha q-q)}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \int_{A_9 \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} 2^{-\omega(i+j)} \\ & \quad \left( \frac{n^2}{2^{i+j}} \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x-t) - f(x)|^p dt_2 dt_1 \right)^{1/p} \\ & \leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} U_{r,p}^{\omega,1} f(x) < C_p \varepsilon. \end{aligned}$$

Similarly, we can see that

$$\begin{aligned} & \int_{A_4 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt + \int_{A_9 \cap (S_{r/2})^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C_p \sum_{i=r_0}^\infty \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} \mathcal{M}_p^{\omega,1} f(x) \\ & \quad + C_p \sum_{i=r_0}^\infty \sum_{j=i-1}^i 2^{-(1+\alpha q-q)(i+j)/2q} |f(x)| \\ & \leq C_p 2^{r_0(2\omega-(1+\alpha q-q)/q)} \mathcal{M}_p^{\omega,1} f(x) + C_p 2^{-r_0(1+\alpha q-q)/q} |f(x)| \\ & \leq C(nr)^{2\omega-(1+\alpha q-q)/q} \mathcal{M}_p^{\omega,1} f(x) + C(nr)^{-(1+\alpha q-q)} |f(x)| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . The fact that  $t_2 > t_1/2$  on  $A_5$  and (5) imply

$$\begin{aligned} & \int_{A_5 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq \sum_{i=1}^{r_0} \sum_{j=i-1}^i \left( \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x-t) - f(x)|^p dt_2 dt_1 \right)^{1/p} \\ & \quad \left( \int_{2^i/n}^{2^{i+1}/n} \int_{t_1-1/n}^{t_1} n^{q(1-\alpha)} t_1^{-q(\alpha+1)} dt_2 dt_1 \right)^{1/q}. \end{aligned}$$

For the last integral, we have

$$\begin{aligned} \int_{2^i/n}^{2^{i+1}/n} \int_{t_1-1/n}^{t_1} n^{q(1-\alpha)} t_1^{-q(\alpha+1)} dt_2 dt_1 & \leq n^{q(1-\alpha)-1} \left( \frac{2^i}{n} \right)^{1-q(\alpha+1)} \\ & \leq C \left( \frac{n}{2^i} \right)^{2q-2} 2^{-i(1+\alpha q-q)}. \end{aligned}$$

Then

$$\begin{aligned} & \int_{A_5 \cap S_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq \sum_{i=1}^{r_0} \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} 2^{-\omega(i+j)} \\ & \quad \left( \frac{n^2}{2^{2i+j}} \int_{2^i/n}^{2^{i+1}/n} \int_{2^j/n}^{2^{j+1}/n} |f(x-t) - f(x)|^p dt_2 dt_1 \right)^{1/p} \\ & \leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} U_{r,p}^{\omega,1} f(x) < C\epsilon. \end{aligned}$$

By (7),

$$\begin{aligned} & \int_{A_{10} \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq \sum_{i=1}^{r_0} \sum_{j=i-1}^i \left( \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x-t) - f(x)|^p dt_1 dt_2 \right)^{1/p} \\ & \quad \left( \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{t_2}^{t_2+1/n} n^{q(1-\alpha)} (\pi - t_2)^{-q(\alpha+1)} dt_1 dt_2 \right)^{1/q}. \end{aligned}$$

Then

$$\begin{aligned} \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{t_2}^{t_2+1/n} n^{q(1-\alpha)} (\pi - t_2)^{-q(\alpha+1)} dt_1 dt_2 & \leq n^{-1} n^{q(1-\alpha)} \left(\frac{2^i}{n}\right)^{1-q(\alpha+1)} \\ & \leq C \left(\frac{n}{2^i}\right)^{2q-2} 2^{-i(1+\alpha q-q)} \end{aligned}$$

and

$$\begin{aligned} & \int_{A_{10} \cap S'_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq \sum_{i=1}^{r_0} \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} 2^{-\omega(i+j)} \\ & \quad \left( \frac{n^2}{2^{i+j}} \int_{\pi-2^{i+1}/n}^{\pi-2^i/n} \int_{\pi-2^{j+1}/n}^{\pi-2^j/n} |f(x-t) - f(x)|^p dt_2 dt_1 \right)^{1/p} \\ & \leq C_p \sum_{i=1}^{r_0} \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} U_{r,p}^{\omega,1} f(x) < C\varepsilon. \end{aligned}$$

Finally,

$$\begin{aligned} & \int_{A_5 \cap S^c_{r/2}} |f(x-t) - f(x)| |K_n^\alpha(t)| dt + \int_{A_{10} \cap (S'_{r/2})^c} |f(x-t) - f(x)| |K_n^\alpha(t)| dt \\ & \leq C_p \sum_{i=r_0}^\infty \sum_{j=i-1}^i 2^{(\omega-(1+\alpha q-q)/2q)(i+j)} \mathcal{M}_p^{\omega,1} f(x) \\ & \quad + C_p \sum_{i=r_0}^\infty \sum_{j=i-1}^i 2^{-(1+\alpha q-q)(i+j)/2q} |f(x)| \\ & \leq C_p 2^{r_0(2\omega-(1+\alpha q-q)/q)} \mathcal{M}_p^{\omega,1} f(x) + C_p 2^{-r_0(1+\alpha q-q)/q} |f(x)| \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ . The proof of the theorem is complete.  $\square$

Note that for the Fejér means, i.e., if  $\alpha = 1$  or even  $\alpha \geq 1$ , in Theorem 3 we have  $1 < p < \infty$  and  $0 < \omega < 1/2q$ .

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