Mathematical
Inequalities
& Applications
Volume 25, Number 3 (2022), 647–657 doi:10.7153/mia-2022-25-40

LOG–CONVEXITY OF GENERALIZED KANTOROVICH FUNCTION

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(*Communicated by M. Praljak*)

Abstract. We aim to derive some important properties about the generalized Kantorovich constant $K(h,p) := \frac{h^p-h}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h}\right)^p$ for $h>0$ and $p \in \mathbb{R}$. In particular, we point out that $K(h,p)$ is a log-convex function for p. As applications, we show the monotonicity of $K(h,p)^{\frac{1}{p}}$.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$ with the identity operator $I_{\mathscr{H}}$, and let $\mathbb{B}^+(\mathscr{H})$ be the set of all positive operators in $\mathbb{B}(\mathscr{H})$. Let $P[\mathbb{B}(\mathscr{H}),\mathbb{B}(\mathscr{K})]$ denote a set of all normalized positive linear maps $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$ such that $A \in \mathbb{B}^+(\mathscr{H}) \to \Phi(A) \in \mathbb{B}^+(\mathscr{K})$ with $\Phi(I_{\mathscr{H}}) = I_{\mathscr{K}}$.

Greub and Rheinboldt [11] gave the following Kantorovich operator inequality: If a positive operator A fulfills the condition $0 < mI_{\mathscr{H}} \leqslant A \leqslant MI_{\mathscr{H}}$ for some scalars $m \leqslant M$, then

$$\langle x,x\rangle \leqslant \langle Ax,x\rangle \left\langle A^{-1}x,x\right\rangle \leqslant \frac{(M+m)^2}{4Mm}\langle x,x\rangle \tag{1.1}$$

for all $x \in \mathscr{H}$. The constant $\frac{(M+m)^2}{4Mm}$ in (1.1) is called the *Kantorovich constant*. Kantorovich represented (1.1) as a sequence of positive real numbers in [13, p.142].

Mond and Pečarić [18] generalized (1.1) as follows: Let Φ be a normalized positive linear map in $P[\mathbb{B}(\mathscr{H}),\mathbb{B}(\mathscr{K})]$. If A is a positive operator on $\mathscr{H}$ satisfying $0 < mI_{\mathscr{H}} \leqslant A \leqslant MI_{\mathscr{H}}$ for some scalars $m \leqslant M$, then

$$\Phi(A^{-1}) \leqslant \frac{(M+m)^2}{4Mm}\Phi(A)^{-1}, \tag{1.2}$$

also see [10].

On the other hand, Furuta gave complementary inequalities to the Hölder-McCarthy inequality as an extension of the Kantorovich type one as follows [5], [6], [7], [9], [10]

Mathematics subject classification (2020): Primary 47A63; Secondary 26D07.
Keywords and phrases: Generalized Kantorovich constant, Klein inequality, Kantorovich inequality and log-convex function.

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and [2]: Let A be a positive operator on $\mathscr{H}$ such that $0 < mI_{\mathscr{H}} \leqslant A \leqslant MI_{\mathscr{H}}$ for some scalars $m < M$. Let $h = \frac{M}{m}$. If $p \notin [0,1]$, then

$$\langle Ax, x\rangle^p \leqslant \langle A^p x, x\rangle \leqslant K(h,p)\,\langle Ax, x\rangle^p \tag{1.3}$$

for every unit vector $x \in \mathscr{H}$, where the *generalized Kantorovich constant* $K(h,p)$ is defined by

$$K(h,p) := \frac{h^p - h}{(p-1)(h-1)}\left(\frac{p-1}{p}\frac{h^p-1}{h^p-h}\right)^p \quad \text{for all } p \in \mathbb{R} \tag{1.4}$$

with $K(h,1) = 1$. (See [5] in detail.) If $p \in [0,1]$, then the reverse inequality is valid in (1.3).

In addition, we cite the following ratio inequality ([10] and [17]), where the estimation is given by the generalized Kantorovich constant $K(h,p)$. This inequality is an extension of the result of C.-K.Li and R.Mathias [16] considered for the matrix case: Let Φ be a normalized positive linear map in $P[\mathbb{B}(\mathscr{H}),\mathbb{B}(\mathscr{K})]$. If A is a positive operator on $\mathscr{H}$ satisfying $0 < mI_{\mathscr{H}} \leqslant A \leqslant MI_{\mathscr{H}}$ for some scalars $m < M$, then for any $p \in \mathbb{R}\backslash\{0\}$

$$\gamma_2 \Phi(A)^p \leqslant \Phi(A^p) \leqslant \gamma_1 \Phi(A)^p \tag{1.5}$$

where

$$\gamma_1 := \begin{cases} K(h,p) & \text{if } p > 1 \text{ or } p < 0, \\ 1 & \text{if } 0 < p \leqslant 1. \end{cases}$$

$$\gamma_2 := \begin{cases} K(h,p)^{-1} & \text{if } p > 2 \text{ or } p < -1, \\ 1 & \text{if } -1 \leqslant p < 0 \text{ or } 1 \leqslant p \leqslant 2, \\ K(h,p) & \text{if } 0 < p < 1, \end{cases}$$

The ineqality (1.5) can be considered as an extension of inequalities (1.2) and (1.3) for power functions, and the generalized Kantorovich constant $K(h,p)$ plays an important role in estimations of these inequalities.

Here we remark that Nakamura [20] gave a simple proof of (1.1) by using a convexity of $f(t) = t^{-1}$. The inequality (1.3) is generalized to arbitrary general convex functions in [19]. The formulated method which is reduced to solving a single variable maximization or minimization problem by using the concavity of a real valued function is called the *Mond-Pečarić method* [10]. Using it we can derive several operator inequalities for the difference and ratio including inequalities (1.1)–(1.3) and (1.5), see [5] and [10].

In this paper, we show some meaningful properties of the generalized Kantorovich constant $K(h,p)$. One of them, the symmetric property $K(h,p) = K(h,1-p)$ given by Furuta [5], [9] is noteworthy. In connection with this, we give some properties of the *generalized Kantorovich function* $K(h,p)$ for $p \in \mathbb{R}$. In particular, we prove a basic and important fact that $K(h,p)$ is log-convex on p, and so convex. For this, an inequality due to Takahasi et al. [23] and some properties of the Klein inequality play an essential role. One of applications, we show that $K(h,p)^{\frac{1}{p}}$ is strictly increasing for $p \in \mathbb{R}$ which is an extension of [14, Lemma 3.8].

2. Some properties of the Klein function and its applications

The following inequality is known as the *Klein inequality* [15] and [21]: For a positive real number $c > 0$ with $c \neq 1$

$$\log c < c - 1 \qquad (\text{or} \quad c \log c > c - 1). \tag{2.1}$$

We define the *Klein function*: For a fixed $c > 0$

$$\mathscr{K}_c(t) := c^t \log c - (c-1) \quad \text{for } t \geqslant 0. \tag{2.2}$$

It has the following properties:

THEOREM 2.1. *Let $c > 0$. Then the Klein function $\mathscr{K}_c(t)$ for $t \geqslant 0$ has the following properties*:

(1) *$\mathscr{K}_c(t)$ is a strictly increasing function for $t \geqslant 0$ with*

$$\mathscr{K}_c(0)\,(= \log c - (c-1)) \leqslant 0 \leqslant \mathscr{K}_c(1)\ (= c\log c - (c-1)).$$

In the above inequalities, equalities hold only the case of $c = 1$.

(2) *The equation $\mathscr{K}_c(t) = 0$ has a unique solution $t = t_0 \begin{cases} \in (0, \frac{1}{2}) \text{ if } 0 < c < 1 \\ \in (\frac{1}{2}, 1) \text{ if } c > 1. \end{cases}$*

(3) *The inequality $\mathscr{K}_c(1) + \mathscr{K}_c(0) (= c\log c + \log c - 2(c-1)) > 0$ holds for $c > 1$. If $0 < c < 1$, then the reverse inequality holds. Consequently, the inequality $(\mathscr{K}_c(1) + \mathscr{K}_c(0)) \log c > 0$ holds for $c > 0$ with $c \neq 1$.*

Proof. It is easy to see (1) from $\frac{d}{dt}\mathscr{K}_c(t) = c^t (\log c)^2 > 0$ and (2.1).

The property (2) is given by $\mathscr{K}_1(\frac{1}{2}) = 0$, $\frac{d}{dc}\mathscr{K}_c(\frac{1}{2}) = c^{-\frac{1}{2}}\left(\log c^{\frac{1}{2}} - (c^{\frac{1}{2}} - 1)\right) < 0$ and (1).

Since $\mathscr{K}_1(1) + \mathscr{K}_1(0) = 0$ and $\frac{d}{dc}(\mathscr{K}_c(1) + \mathscr{K}_c(0)) = \log c + \frac{1}{c} - 1 = (\frac{1}{c} - 1) - \log\frac{1}{c} > 0$, we have (3). $\square$

Next we recall the generalized logarithmic function $\ln_p(c) := \frac{c^p - 1}{p}$. S.-E. Takahasi et al [23] treat the following function related to the generalized logarithmic function for $M \geqslant m > 0$

$$\sigma_p(m,M) = \frac{p}{p-1}\frac{mM^p - Mm^p}{M^p - m^p}\left(= \frac{\ln_{p-1}(h)}{\ln_p(h)}M\right) \quad \text{for any real number } p(\neq 0, 1)$$

where $h = \frac{M}{m}$. They proved that the function $p \mapsto \sigma_p(m,M)$ is strictly monotone decreasing. In the following lemma, we give a simplified proof.

LEMMA 2.2. *Let c be a positive real number with $c \neq 1$. Then the following properties hold*:

(1) *The following inequality holds for* $p \notin (0,1)$

$$(c-1)\log c \leqslant \frac{c^p-1}{p}\cdot\frac{c^{1-p}-1}{1-p}\left(=\frac{\ln_p(h)}{\ln_{1-p}(h)}M\right). \tag{2.3}$$

If $p \in [0,1]$*, then the reverse inequality of* (2.3) *holds. The equality is attained if and only if* $p \to 0,1$*.*

(2) *The function* $f(p) := \frac{1-c^{-p}}{p}\cdot\frac{p-1}{1-c^{-(p-1)}}\left(=\frac{\ln_{-p}(h)}{\ln_{1-p}(h)}\right)$ *is strictly monotone increasing for* $p \in \mathbb{R}$*.*

(3) *Let* $c > 1$*. Then the function* $g(p) := \frac{p}{1-c^{-p}} - \frac{p-1}{1-c^{-(p-1)}}\Big(=\ln_{-p}(h)^{-1} - \ln_{1-p}(h)^{-1}\Big)$ *is strictly monotone increasing for* $p \in \mathbb{R}$*.*

If $0 < c < 1$*, then* $g(p)$ *is strictly monotone decreasing for* $p \in \mathbb{R}$*.*

Proof. (1) We easily see that $\frac{c^p-1}{p}\cdot\frac{c^{1-p}-1}{1-p}$ converges $(c-1)\log c$ for $p \to 0,1$. So we may prove that for $p \in \mathbb{R}\backslash\{0,1\}$, the function $f_0(x) := p(1-p)(x-1)\log x - (x^p-1)(x^{1-p}-1)$ is positive for any positive real number x ($x \neq 1$). Then we have

$$\begin{aligned}\frac{d}{dx}f_0(x) &= p(1-p)(\log x - x^{-1}+1) - \{px^{p-1}(x^{1-p}-1)+(1-p)(x^p-1)x^{-p}\}\\ &= p(1-p)(\log x - x^{-1}+1) + px^{p-1}+(1-p)x^{-p}-1,\end{aligned}$$

and moreover

$$\begin{aligned}\frac{d^2}{dx^2}f_0(x) &= p(1-p)(x^{-1}+x^{-2}-x^{p-2}-x^{-p-1})\\ &= p(1-p)x^{-\frac{3}{2}}\left((x^{\frac{1}{2}}+x^{-\frac{1}{2}})-(x^{p-\frac{1}{2}}+x^{-p+\frac{1}{2}})\right) > 0,\end{aligned}$$

because

$$(x^{\frac{1}{2}}+x^{-\frac{1}{2}})-(x^{p-\frac{1}{2}}+x^{-p+\frac{1}{2}})\begin{cases} < 0 & \text{if } p \notin [0,1]\\ > 0 & \text{if } p \in (0,1).\end{cases}$$

Moreover, it follows from $\lim_{x\to 1} f_0(x) = \lim_{x\to 1}\frac{d}{dx}f_0(x) = \lim_{x\to 1}\frac{d^2}{d^2x}f_0(x) = 0$ that $f_0(x) \geqslant 0$. So the desired property (2.3) holds.

(2) Since $\lim_{p\to 0} f(p) = \frac{\log c}{c-1}$ and $\lim_{p\to 1} f(p) = \frac{c-1}{c\log c}$, the function $f(p)$ is continuous on $\mathbb{R}$. So we may show this property only for $p \in \mathbb{R}\backslash\{0,1\}$.

It follows from (1) that

$$\begin{aligned}\frac{d}{dp}\log f(p) &= \frac{1}{(p-1)p} + \frac{(1-c^{1-p})-(c-c^{1-p})}{(1-c^{-p})(1-c^{1-p})}c^{-p}\log c\\ &= \frac{1}{(c^p-1)(c^{1-p}-1)}\left(-\frac{(c^p-1)(c^{1-p}-1)}{p(1-p)}+(c-1)\log c\right) > 0.\end{aligned}$$

In the above last inequality, we remark that $(c^p-1)(c^{1-p}-1)\begin{cases} <0 & (p\notin[0,1]) \\ >0 & (p\in(0,1)). \end{cases}$ Moreover, we have $\frac{d}{dp}f(p)=f(p)\cdot\frac{d}{dp}\log f(p)>0$ by $f(p)>0$, and so the property (2) holds.

(3) Define a function $G(t)$ by

$$G(t):=\begin{cases} \frac{t\log t}{(t-1)\log c} & (t\neq 1) \\ (\log c)^{-1} & (t=1) \end{cases} \qquad \text{for} \quad t>0.$$

Then we have for $t\neq 1$

$$\frac{d}{dt}G(t)=\frac{(t-1)-\log t}{(t-1)^2\log c}, \quad \text{and} \quad \frac{d^2}{dt^2}G(t)=\frac{1-t^2+2t\log t}{t(t-1)^3\log c}.$$

Here we put $t=c^p(>0)$. Then it follows from $G(c^p)=\frac{p}{1-c^{-p}}$ that

$$g(p)=G(c^p)-G(c^{p-1}) \quad \text{and} \quad \frac{d}{dp}g(p)=\frac{d}{dp}G(c^p)-\frac{d}{dp}G(c^{p-1}).$$

Moreover we have $\frac{dt}{dp}=t\log c$ and so

$$\begin{aligned}\frac{d^2}{dp^2}G(c^p)&=\frac{d}{dt}\left(\frac{d}{dt}G(t)\cdot\frac{dt}{dp}\right)\cdot\frac{dt}{dp}\\ &=\frac{d^2}{dt^2}G(t)\cdot(t\log c)^2+\frac{d}{dt}G(t)\cdot t(\log c)^2\\ &=\frac{t\cdot(t\log t+\log t-2(t-1))\log t}{(t-1)^3\cdot p}.\end{aligned}$$

Hence we have $\frac{d^2}{dp^2}G(c^p)\begin{cases} >0\ (c>1) \\ <0\ (0<c<1) \end{cases}$ by $(t-1)^3\cdot p\begin{cases} >0 & (c>1) \\ <0 & (0<c<1) \end{cases}$ and Theorem 2.1 (3). If $c>1$ (resp. $0<c<1$), then $\frac{d}{dp}G(c^p)$ is an increasing function (resp. a decreasing function) for p. So the property (3) is given by

$$\frac{d}{dp}g(p)\begin{cases} >0\ (c>1) \\ <0\ (0<c<1). \end{cases} \qquad \square$$

3. Log-convexity of the generalized Kantorovich function

We recall several *properties* of the generalized Kantorovich function $K(h,p)$ as follows [3], [5], [8], [9] and [10]: Let $h>0$ be given. Then

(K-1) $K(h,p)=K(\frac{1}{h},p)$ for all $p\in\mathbb{R}$.

(K-2) $K(h,p)=K(h,1-p)$ (i.e., $K(h,\frac{1}{2}+p)=K(h,\frac{1}{2}-p)$) for all $p\in\mathbb{R}$, that is, $K(h,p)$ is symmetric with respect to $p=\frac{1}{2}$.

(K-3) $K(h,0) = K(h,1) = 1$ and $K(1,p) = 1$ for all $p \in \mathbb{R}$, where $K(h,0) = \lim_{p\to 0} K(h,p)$, $K(h,1) = \lim_{p\to 1} K(h,p)$ and $K(1,p) = \lim_{h\to 1} K(h,p)$.

(K-4) $K(h,p)$ is increasing for $p > \frac{1}{2}$ and decreasing for $p < \frac{1}{2}$, and

$$\min_{p\in\mathbb{R}} K(h,p) = K\left(h,\frac{1}{2}\right) = \frac{2h^{1/4}}{h^{1/2}+1} \in (0,1].$$

(K-5) $K\left(h^r,\frac{p}{r}\right)^{\frac{1}{p}} = K\left(h^p,\frac{r}{p}\right)^{-\frac{1}{r}}$ for $rp \neq 0$.
In particular, if $r = 1$, then $K(h,p)^{\frac{1}{p}} = K\left(h^p,\frac{1}{p}\right)^{-1}$ for $p \neq 0$.

(K-6) $K(h,p) < h^{p-1}$ for all $h > 1$ and $p > 1$.

Here it follows from (K-2), (K-3) and (K-4) that $K(h,p) > 0$ for any $p \in \mathbb{R}$ and

$$K(h,p)\begin{cases} \geqslant 1 & \text{if } p \notin (0,1) \\ < 1 & \text{if } p \in (0,1) \end{cases} \quad \text{(e.g. [5], [9]).}$$

Moreover, we mention the following properties [12]:

(K-7) Let $h > 1$. If $p > 1$ (resp. $0 < p < 1$), then $K(h^t,p)^{\frac{1}{t}}$ is increasing (resp. decreasing) for $t > 0$ ([4]), and $1 < K(h^t,p)^{\frac{1}{t}} < h^{p-1}$ for all $t > 0$.

(K-8) $\lim_{t\to 0} K(h^t,p)^{\frac{1}{t}} = 1 \quad$ for all $p \in \mathbb{R}$.

Here we provide a proof of (K-8) for the sake of convenience:

Proof of (K-8). We may assume that $h > 1$ and $t \downarrow 0$ by (K-1). By L'Hospital's rule, we have

$$\lim_{t\downarrow 0} \frac{h^t - h^{tp}}{h^t - 1} = \lim_{t\downarrow 0} \frac{h^t \log h - h^{tp}\log h^p}{h^t \log h} = 1 - p \quad \text{and} \quad \lim_{t\downarrow 0} \frac{h^{tp}-1}{h^t-1} = p \tag{3.1}$$

and moreover

$$\lim_{t\downarrow 0} \frac{d}{dt}\frac{h^t - h^{tp}}{h^t - 1} = \frac{p(1-p)}{2}\log h \quad \text{and} \quad \lim_{t\downarrow 0} \frac{d}{dt}\frac{h^{tp}-1}{h^t-1} = \frac{p(p-1)}{2}\log h. \tag{3.2}$$

As a result, applying L'Hospital's rule by (3.1) and moreover using (3.2), we obtain

$$\begin{aligned}\lim_{t\downarrow 0}\log K(h^t,p)^{\frac{1}{t}} &= \lim_{t\downarrow 0}\log\left\{\frac{h^{tp}-h^t}{(p-1)(h^t-1)}\left(\frac{p-1}{p}\frac{h^{tp}-1}{h^{tp}-h^t}\right)^p\right\}^{\frac{1}{t}} \\ &= \lim_{t\downarrow 0}\log\left\{\left(\frac{1}{1-p}\frac{h^t-h^{tp}}{h^t-1}\right)^{\frac{1-p}{t}}\left(\frac{1}{p}\frac{h^{tp}-1}{h^t-1}\right)^{\frac{p}{t}}\right\} \\ &= \lim_{t\downarrow 0}\frac{1}{t}\left\{(1-p)\log\left(\frac{1}{1-p}\frac{h^t-h^{tp}}{h^t-1}\right)+p\log\left(\frac{1}{p}\frac{h^{tp}-1}{h^t-1}\right)\right\} \\ &= (1-p)\frac{\frac{p(1-p)}{2}\log h}{1-p}+p\frac{\frac{p(p-1)}{2}\log h}{p} \\ &= 0.\end{aligned}$$

In the above equality, we remark that $\frac{h^t-h^{tp}}{1-p}$ and $\frac{h^{tp}-1}{p}$ are positive. Hence we have the desired equality (K-8). □

The following result represents the relation of the convex function and its secant line (cf. [10, Corollary 2.10]):

(K-9) Let $0<m<M$ with $h:=\frac{M}{m}>1$. For $p>1$, the convex function t^p $(t>0)$ has a secant line $\alpha_p t+\beta_p$ at $t=m,M$, where $\alpha_p:=\frac{M^p-m^p}{M-m}$ and $\beta_p:=\frac{Mm^p-mM^p}{M-m}$. Then it follows that

$$\max_{m\leqslant t\leqslant M}\frac{\alpha_p t+\beta_p}{t^p}=K(h,p).$$

Here we treat the Specht ratio $S(h)=\dfrac{h^{\frac{1}{h-1}}}{e\log h^{\frac{1}{h-1}}}$ [22] which is the best constant of the reverse arithmetic-geometric mean inequality. It has the following property (e.g. [3], [8]) related to the generalized Kantorovich function $K(h,p)$:

(K-10) $$\lim_{p\to 1}\frac{\partial}{\partial p}\log K(h,p)=\lim_{p\to 1}\frac{\frac{\partial}{\partial p}K(h,p)}{K(h,p)}=\lim_{p\to 1}\frac{\partial}{\partial p}K(h,p)=\log S(h).$$

We mention some important properties of the generalized Kantorovich function $K(h,p)$ as our main result:

THEOREM 3.1. *Let $h>0$ and $p\in\mathbb{R}$. The generalized Kantorovich function $K(h,p)$ has the following properties:*

(K-11) $\log K(h,p)$ *is a convex function for* $p\in\mathbb{R}$.

Consequently,

(K-12) $K(h,p)$ *is a convex function for* $p\in\mathbb{R}$.

Proof. (K-11) From the properties (K-1), (K-2), (K-3) and (K-4), we may show the properties (K-11) (and (K-12)) for the case of $h \geqslant 1$ and $p \geqslant \frac{1}{2}$. In particular, we only prove for the case $p > 1$. The case $\frac{1}{2} \leqslant p < 1$ is given by a similar method. Here we remark that $\lim_{p\to 1} \frac{\partial}{\partial p} \log K(h,p)$ exists by (K-10).

The generalized Kantorovich function is represented as follows:

$$K(h,p) = \frac{(p-1)^{p-1}}{p^p} \frac{(h^p-1)^p}{(h^p-h)^{p-1}(h-1)},$$

and so

$$\begin{aligned}\log K(h,p) = &\log(h^p-h) - \log(h-1) - \log(p-1) \\ &+ p\left(\log(h^p-1) - \log(h^p-h) + \log(p-1) - \log p\right).\end{aligned}$$

Moreover, we have

$$\begin{aligned}\frac{\partial}{\partial p}\log K(h,p) &= \left(\frac{(1-p)h^p}{h^p-h} + \frac{ph^p}{h^p-1}\right)\log h + \log\frac{(p-1)(h^p-1)}{p(h^p-h)} \qquad (3.3)\\ &= \left(\frac{p}{1-h^{-p}} - \frac{p-1}{1-h^{-(p-1)}}\right)\log h + \log\frac{(p-1)(1-h^{-p})}{p(1-h^{-(p-1)})}.\end{aligned}$$

By (2) and (3) in Lemma 2.2, the function $\frac{\partial}{\partial p}\log K(h,p)$ is strictly monotone increasing, and so we hold the property (K-11).

(K-12) This property is satisfied by (K-11). □

Kian et al. obtained the following result [14, Lemma 3.8] by using the property (K-12):

LEMMA KMS. *Let $h \geqslant 1$. Then the generalized Kantorovich function $K(h,p)$ has the following property*:

$$K(h,-p) \leqslant K(h,-1)^p \quad \textit{for } p \in (0,1).$$

If $p \notin (0,1)$, then the reverse inequality of above holds.

The above lemma is equivalent to the following result: For $p \leqslant 1$

$$K(h,-p)^{-\frac{1}{p}} \geqslant K(h,-1)^{-1}.$$

If $p \geqslant 1$, then the reverse inequality of above holds.

From this view point, we improve it as follows:

COROLLARY 3.2. *Let $h > 0$. The generalized Kantorovich function $K(h,p)$ has the following monotone property*:

(K-13) $K(h,p)^{\frac{1}{p}}$ *is strictly increasing for $p \in \mathbb{R}$.*

Proof. First of all, we have

$$\frac{\partial}{\partial p}\log K(h,p)^{\frac{1}{p}} = \frac{\partial}{\partial p}\frac{\log K(h,p)}{p} = \frac{p\frac{\partial}{\partial p}\log K(h,p) - \log K(h,p)}{p^2}. \tag{3.4}$$

Next, we consider the tangent line $\ell(p)$ of $\log K(h,p)$ at any $p = p_0 \in \mathbb{R}$. It is represented as follows:

$$\ell(p) = \frac{\partial}{\partial p}\log K(h,p)\Big|_{p=p_0}(p-p_0) + \log K(h,p_0).$$

By (K-11), we have

$$\log K(h,p) \geqslant \ell(p) = \frac{\partial}{\partial p}\log K(h,p)\Big|_{p=p_0}(p-p_0) + \log K(h,p_0). \tag{3.5}$$

If $p=0$, then the inequation (3.5) implies

$$0 \geqslant -p_0\frac{\partial}{\partial p}\log K(h,p)\Big|_{p=p_0} + \log K(h,p_0).$$

So the equation (3.4) is positive, and hence we have the property (K-13). □

As another application, we have the following corollary:

COROLLARY 3.3. *The generalized Kantorovich function $K(h,p)$ for $p \in \mathbb{R}$ has the following properties*:

(K-14) *For a fixed $h > 0$, the following equation holds*:

$$\lim_{p\to\infty}\frac{\partial}{\partial p}\log K(h,p) = \lim_{p\to\infty}\frac{\frac{\partial}{\partial p}K(h,p)}{K(h,p)} = \log h.$$

Moreover, $\left|\frac{\partial}{\partial p}\log K(h,p)\right| < |\log h|$. *Consequently, there is a unique solution $p = p_0 \in \mathbb{R}$ such that* $\frac{\partial}{\partial p}K(h,p)\Big|_{p=p_0} = \log h_0$ *for any $h_0 \in I_h$, where I_h is the open interval determined by $\frac{1}{h}$ and h.*

(K-15) *Let $h \geqslant 1$ and $h_0 > 0$. Then the equation $K(h,p) = h_0^{p-1}$ has the following solutions $p \in \mathbb{R}$*:

$$p := \begin{cases} 1, p_0(\in(-\infty,1)) & \text{if } h^{-1} < h_0 < S(h) \\ 1, p_0(\in(1,\infty)) & \text{if } S(h) < h_0 < h \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, suppose that $h_0 \in (h^{-1},h)\setminus\{S(h)\}$. Let I_0 be the closed interval determined by 1 and p_0 with $K(h,p_0) = h_0^{p_0-1}$. Then the following inequality holds

$$K(h,p) - h_0^{p-1}\begin{cases} \leqslant 0 & \text{if } p \in I_0 \\ \geqslant 0 & \text{otherwise.} \end{cases}$$

Proof. (K-14) By (3.3) and $\lim_{p\to\infty} ph^{-p} = 0$, we have

$$\begin{aligned}\lim_{p\to\infty}\frac{\partial}{\partial p}\log K(h,p) &= \lim_{p\to\infty}\left\{\left(\frac{p}{1-h^{-p}} - \frac{p-1}{1-h^{-(p-1)}}\right)\log h + \log\frac{(p-1)(1-h^{-p})}{p(1-h^{-(p-1)})}\right\}\\ &= \lim_{p\to\infty}\frac{(1-h)\cdot ph^{-p} - h^{-p}+1}{(1-h^{-p})(1-h^{-(p-1)})}\log h\\ &= \log h.\end{aligned}$$

(K-15) We consider

$$\log K(h,p) = (p-1)\log h_0 \tag{3.6}$$

instead of $K(h,p) = h_0^{p-1}$. Then the equation (3.6) has a solution $p = 1$.

We consider the tangent line $\ell(p)$ at $p = 1$ with respect to the function $\log K(h,p)$ for p. Then we have $\ell(p) = (p-1)\log S(h)$ by (K-3) and (K-10).

Let $h^{-1} < h_0 < h$. Then the equation (3.6) has only the following solution p where

$$p := \begin{cases} 1, p_0(\in(-\infty,1)) & \text{if } h^{-1} < h_0 < S(h)\\ 1, p_0(\in(1,\infty)) & \text{if } S(h) < h_0 < h\\ 1 & \text{if } h_0 = S(h)\end{cases}$$

by (K-2), (K-10), (K-11) and (K-14).

Next, let $h_0 \leqslant h^{-1}$ or $h \leqslant h_0$. Then we see that the equation (3.6) has a solution $p = 1$ only. □

4. Concluding remarks

In Theorem 3.1, we see that $K(h,p)$ and $\log K(h,p)$ are convex functions for $p \in \mathbb{R}$. In this section, we give some remarks of the function $K(h,p)$ for $h > 0$.

It is known that $K(h,2)$ is convex for $h > 1$ as in [1]. But, $\log K(h,2)$ is not a convex function for $h > 1$. As a matter of fact, since

$$\log K\left(\frac{2+4}{2},2\right) = \log\frac{4}{3} \quad \text{and} \quad \frac{\log K(2,2)+\log K(4,2)}{2} = \frac{\log\frac{9}{8}+\log\frac{25}{16}}{2} = \log\frac{15}{8\sqrt{2}},$$

we have $\log K(\frac{2+4}{2},2) > \frac{\log K(2,2)+\log K(4,2)}{2}$, i.e., $\log K(h,2)$ is not a convex function.

In addition, that there exists $p = p_0 \in \mathbb{R}$ such that $K(h,p_0)$ is not convex for $h > 1$. Indeed, since

$$K\left(\frac{3+7}{2},\frac{3}{2}\right) \risingdotseq 1.25726 \quad \text{and} \quad \frac{K(3,\frac{3}{2})+K(7,\frac{3}{2})}{2} \risingdotseq \frac{1.11626+1.38604}{2} = 1.25115,$$

we have $K(\frac{3+7}{2},\frac{3}{2}) > \frac{K(3,\frac{3}{2})+K(7,\frac{3}{2})}{2}$.

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(Received June 2, 2021)

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