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SPACEABILITY ON SOME CLASSES OF BANACH SPACES

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Abstract. In this paper, we study spaceability of subsets of generalized Orlicz and Lebesgue spaces associated to a Banach function space. Also, we give some sufficient conditions for spaceability of subsets of a general Banach space which improves an important result on this topic. As an application, it is shown that the set of all bounded linear operators which are not positive semidefinite on a separable Hilbert space is spaceable.

1. Introduction

A subset of a topological vector space is called spaceable if its union with the singleton $\{0\}$ contains a closed infinite-dimensional linear subspace. This concept was introduced in [11, 1] and so far has been considered by many researchers. As a useful tool, L. Bernal-González and M.O. Cabrera in [5, Theorem 2.2] give some sufficient conditions for spaceability of the complement of a cone in a Banach function space. This result covers some important ones proved in [8, 9]. By this tool, in [21, 22] it is shown that the set $\mathscr{M}_q^p(\mathbb{R}^n)\setminus\bigcup_{q<r\leqslant p}\mathscr{M}_r^p(\mathbb{R}^n)$ is spaceable in the Morrey space $\mathscr{M}_q^p(\mathbb{R}^n)$, if $0<q<p<\infty$. Also, technically it is also proved that $\mathrm{w}\mathscr{M}_q^p(\mathbb{R}^n)\setminus\mathscr{M}_q^p(\mathbb{R}^n)$ is spaceable in the weak Morrey space $\mathrm{w}\mathscr{M}_q^p(\mathbb{R}^n)$. In [14, Theorem 3.3] D. Kitson and R. M. Timoney present another nice sufficient condition for a set to be spaceable in a Fréchet space. This topic has been studied in the context of some special sequence and function spaces in several papers (see [3, 4, 7, 8, 9, 12, 24] for example).

In this paper, we focus on generalized Orlicz and Lebesgue spaces X^Φ and X^p associated to a Banach function space X, where Φ is a Young function and $p\geqslant 1$. These structures were studied in [10, 13, 19, 23] and contains usual Orlicz and Lebesgue spaces. Let X be a solid Banach function space on the measure space $(\Omega,\mathscr{A},\mu)$ and $\mathscr{A}_0:=\{E\in\mathscr{A}: 0<\mu(E) \text{ and } \chi_E\in X\}$. Inspired by [8, 9] and as an extension of [5, Theorem 3.3] we prove that if $\inf\{\|\chi_E\|_X : E\in\mathscr{A}_0\}=0$ and $\sup\{\|\chi_E\|_X : E\in\mathscr{A}_\infty\}<\infty$, where $\mathscr{A}_\infty:=\{E\in\mathscr{A}:\chi_E\in X\}$, then for each $p\geqslant 1$, $X^p\setminus\bigcup_{p<q}X^q$ is spaceable in X^p. This result is concluded from the technical Lemma 1 which is a generalization of [16, Theorem 14.22]. In the sequel, we provide a necessary condition for inclusion

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of two generalized Orlicz spaces (as a generalization of [20, Theorem 3, page 155]), and then prove that if the Young function Φ_2 is not stronger than the other one Φ_1, then $X^{\Phi_2} \setminus X^{\Phi_1}$ is spaceable in X^{Φ_2}. Finally, we give an abstract improvement of [5, Theorem 2.2]. To emphasize the capacity of the obtained result, we apply it to show that if X is a solid Banach function space on Ω and $\inf\{\|\chi_E\|_X : E \in \mathscr{A}_0\} = 0$, then for each $1 \leqslant p,q < r$, the set $\{(f,g) \in X^p \times X^q : fg \notin X^r\}$ is spaceable in $X^p \times X^q$. As another application, we prove that the set of all bounded linear operators which are not positive semidefinite on a separable Hilbert space is spaceable. Moreover, it is shown that if K is a two sided ideal cone in $B(\mathscr{H})$ and there exists a sequence of mutually disjoint subsets $\{J_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ satisfying the condition $P_{J_n}KP_{J_n} \neq P_{J_n}B(\mathscr{H})P_{J_n}$ for all $n \in \mathbb{N}$, then $B(\mathscr{H}) \setminus K$ is spaceable in $B(\mathscr{H})$, where $\mathscr{H}$ is a separable Hilbert space with an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$, and P_{J_n} is the orthogonal projection on the closed linear span of $\{e_j\}_{j\in J_n}$.

2. Preliminaries

In sequel, $(\Omega, \mathscr{A}, \mu)$ is always a σ-finite measure space, and $\mathscr{M}_0(\Omega)$ is set of all $\mathscr{A}$-measurable complex-valued functions on Ω.

A linear subspace X of $\mathscr{M}_0(\Omega)$ equipped with a given norm $\|\cdot\|_X$ is called a *Banach function space on* Ω if $(X, \|\cdot\|_X)$ is a Banach space. It is called *solid* if for each $f \in X$ and $g \in \mathscr{M}_0(\Omega)$ we have $g \in X$ and $\|g\|_X \leqslant \|f\|_X$ whenever $|g| \leqslant |f|$ a.e.

A convex function $\Phi : [0,\infty) \to [0,\infty)$ is called a *Young function* if $\Phi(0) = \lim_{x\to 0} \Phi(x) = 0$ and $\lim_{x\to\infty} \Phi(x) = \infty$.

Let X be a Banach function space on Ω. For each $f \in \mathscr{M}_0(\Omega)$ we put

$$\|f\|_\Phi := \inf\left\{\lambda > 0 : \Phi\Big(\frac{|f|}{\lambda}\Big) \in X,\ \left\|\Phi\Big(\frac{|f|}{\lambda}\Big)\right\|_X \leqslant 1\right\}. \tag{1}$$

Then, the set of all $f \in \mathscr{M}_0(\Omega)$ with $\|f\|_\Phi < \infty$ is denoted by X^Φ.

As in [10, Theorem 4.11], $(X^\Phi, \|\cdot\|_\Phi)$ is a Banach function space on Ω (two functions in X^Φ which are equal almost everywhere are considered the same). For each $p \geqslant 1$, the function Φ_0 defined by $\Phi_0(x) := x^p$ for all $x \geqslant 0$ is a Young function. Then, we denote $X^p := X^{\Phi_0}$ and $\|\cdot\|_p := \|\cdot\|_{\Phi_0}$. In particular, if $X := L^1(\mu)$, then $X^\Phi = L^\Phi(\mu)$ and $X^p = L^p(\mu)$, the classical Orlicz and Lebesgue spaces.

3. Main results

A subset S of a Banach space Y is called *spaceable* in Y if $S \cup \{0\}$ contains a closed infinite-dimensional subspace of Y. In this section, first we study the spaceability of special subsets of X^p. As in [23], for each function f in $M_0(\Omega)$ we denote $E_f := \{x \in \Omega : f(x) \neq 0\}$, the set-theoretical support of f.

REMARK 1. Recall from [5] that a Banach function space $(\mathscr{E}, \|\cdot\|)$ on Ω is a *PCS-space* if for each sequence (f_n) with $f_n \to f$ in $\mathscr{E}$, there is a subsequence (f_{n_k})

of (f_n) such that $f_{n_k} \to f$ a.e. This property plays a key role in the subject *spaceability*. For instance, see Theorem 1 below as a main result on this topic. A Banach function space X on a σ-finite measure space $(\Omega, \mathscr{A}, \mu)$ is a PCS-space if and only if the embedding of X into $\mathscr{M}_0(\Omega)$ is continuous, where $\mathscr{M}_0(\Omega)$ is equipped with the topology of convergence in measure on finite measure subsets. If X is a solid quasi-Banach function space on a σ-finite measure space then the embedding X in $\mathscr{M}_0(\Omega)$ is always continuous, see [17, Proposition 2.2 (i)] for the finite measure case.

Next, we recall a result which was proved in [5, Theorem 2.2]. In this paper (as in [5]) a subset B of a vector space is called a *cone* if for each scalar c, $cB \subseteq B$.

THEOREM 1. *Let $(\mathscr{E}, \|\cdot\|)$ be a Banach function space on Ω and B be a nonempty subset of $\mathscr{E}$ such that:*

1. *$\mathscr{E}$ is a PCS-space;*
2. *there is a constant $k > 0$ such that $\|f+g\| \geqslant k\|f\|$ for all $f, g \in \mathscr{E}$ with $E_f \cap E_g = \varnothing$;*
3. *B is a cone;*
4. *if $f, g \in \mathscr{E}$ such that $f+g \in B$ and $E_f \cap E_g = \varnothing$ then $f, g \in B$;*
5. *there is a sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathscr{E} \setminus B$ such that for each distinct $m, n \in \mathbb{N}$, $E_{f_n} \cap E_{f_m} = \varnothing$.*

Then, $\mathscr{E} \setminus B$ is spaceable in $\mathscr{E}$.

For the sequel, we need the next result which is a generalization of [16, Theorem 14.22]. The main idea for the proof comes from [16, Theorem 14.22] but details are different. The item (a) in this theorem is a more general version of the relation (α) in [5]. Denote

$$\mathscr{A}_0 := \{E \in \mathscr{A} : 0 < \mu(E) \text{ and } \chi_E \in X\}.$$

LEMMA 1. *Let X be a solid Banach function space. Then the following properties are equivalent:*

(a) $\inf\{\|\chi_E\|_X : E \in \mathscr{A}_0\} = 0$.

(b) There exists a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{A}_0$ such that $A_n \cap A_m = \varnothing$ for all distinct $m, n \in \mathbb{N}$ and

$$0 < \|\chi_{A_n}\|_X \leqslant \frac{1}{2^n}, \quad (n \in \mathbb{N}).$$

Proof. $(b) \Rightarrow (a)$: Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\mathscr{A}$ which satisfies in (b). Let $1 \leqslant p < q < \infty$. We claim that $X^p \nsubseteq X^q$. By the assumptions, we can write

$$\sum_{n=1}^{\infty} \left\| n \|\chi_{A_n}\|_X^{\frac{-p}{q}} \cdot \chi_{A_n} \right\|_X = \sum_{n=1}^{\infty} n \|\chi_{A_n}\|_X^{1-\frac{p}{q}} \leqslant \sum_{n=1}^{\infty} n (2^{\frac{p}{q}-1})^n < \infty. \tag{2}$$

Set

$$f := \sum_{n=1}^{\infty} n^{\frac{1}{p}} \|\chi_{A_n}\|_X^{\frac{-1}{q}} \cdot \chi_{A_n} \quad \text{and} \qquad S_N := \sum_{n=1}^{N} n^{\frac{1}{p}} \|\chi_{A_n}\|_X^{\frac{-1}{q}} \cdot \chi_{A_n}$$

for all $N \in \mathbb{N}$. By the relation (2) the sequence $\{S_N^p\}_{N=1}^{\infty}$ is Cauchy in X and so it converges to some $g \in X$ in the norm topology, because X is complete. Now, by Remark 1, there exists a subsequence of $\{S_N^p\}_{N=1}^{\infty}$ that converges to g a.e. Therefore

$$g = \sum_{n=1}^{\infty} n \|\chi_{A_n}\|_X^{\frac{-p}{q}} \cdot \chi_{A_n} = \left(\sum_{n=1}^{\infty} n^{\frac{1}{p}} \|\chi_{A_n}\|_X^{\frac{-1}{q}} \cdot \chi_{A_n} \right)^p = f^p \quad \text{a.e.}$$

This implies that $|f|^p \in X$, and so $f \in X^p$. On the other hand, in contrast, let $f \in X^q$. Then, since X is solid we have

$$\begin{aligned} \|f\|_q &= \| |f|^q \|_X \\ &= \left\| \left(\sum_{n=1}^{\infty} n^{\frac{1}{p}} \|\chi_{A_n}\|_X^{\frac{-1}{q}} \cdot \chi_{A_n} \right)^q \right\|_X \\ &\geqslant \left\| k^{\frac{q}{p}} \|\chi_{A_k}\|_X^{-1} \cdot \chi_{A_k} \right\|_X = k^{\frac{q}{p}} \end{aligned}$$

for all $k \in \mathbb{N}$, and this implies that $\|f\|_q = \infty$, a contradiction. Hence, $f \in X^p \setminus X^q$. Now, thanks to [23, Theorem 2.1] we have $\inf\{\|\chi_E\|_X : E \in \mathscr{A}_0\} = 0$.

$(a) \Rightarrow (b)$: Let $\inf\{\|\chi_E\|_X : E \in \mathscr{A}_0\} = 0$. For each $A \in \mathscr{A}$ put

$$\mathscr{K}(A) := \inf\{\|\chi_B\|_X : B \in \mathscr{A}_0, B \subseteq A\}.$$

Clearly, we have

1. if $A_1, A_2 \in \mathscr{A}$ and $A_1 \subseteq A_2$, then $\mathscr{K}(A_2) \leqslant \mathscr{K}(A_1)$, and
2. for each $C, B \in \mathscr{A}$ with $B \subseteq C$, if $\mathscr{K}(B), \mathscr{K}(C \setminus B) > 0$, then $\mathscr{K}(C) > 0$.

Note that (2) holds since for each $E \in \mathscr{A}_0$, if $E \subseteq C$, then $\|\chi_E\|_X \geqslant \min\{\mathscr{K}(B), \mathscr{K}(C \setminus B)\}$.

For each $A \in \mathscr{A}$ we put

$$\mathscr{K}'(A) := \sup\{\|\chi_B\|_X : B \in \mathscr{A}_0, B \subseteq A\}.$$

Similar to the proof of [16, Theorem 14.22] with different details, one can prove that

- if $C \in \mathscr{A}$ and $\mathscr{K}(C) = 0$, then for each $\varepsilon > 0$ there exists $A \in \mathscr{A}_0$ such that $A \subseteq C$, $0 < \|\chi_A\|_X < \min\{\varepsilon, \mathscr{K}'(C)\}$ and $\mathscr{K}(C \setminus A) = 0$.

Indeed, let $\mathscr{K}(C) = 0$. Then, there exists a set $B \subseteq C$ such that $0 < \|\chi_B\|_X < \min\{\varepsilon, \mathscr{K}'(C)\}$. If $\mathscr{K}(C \setminus B) = 0$ we set $A := B$. If $\mathscr{K}(C \setminus B) > 0$, by (2) we have $\mathscr{K}(B) = 0$, and so there is a set $D \in \mathscr{A}_0$ such that $D \subseteq B$ and $0 < \|\chi_D\|_X < \|\chi_B\|_X$. In this situation, because of (2) we have $\mathscr{K}(D) = 0$ or $\mathscr{K}(B \setminus D) = 0$, and then by (1) it would be enough to set $A := B \setminus D$ or $A := D$, respectively.

Now, since $\inf\{\|\chi_E\|_X : E \in \mathscr{A}_0\} = 0$, we have $\mathscr{K}(\Omega) = 0$. So, there exists $A_1 \in \mathscr{A}_0$ such that $0 < \|\chi_{A_1}\|_X < \min\{\frac{1}{2}, \mathscr{K}'(\Omega)\}$ and $\mathscr{K}(\Omega \setminus A_1) = 0$. Setting $C := \Omega \setminus A_1$ in the above fact, there exists $A_2 \in \mathscr{A}_0$ such that $A_2 \subsetneq \Omega \setminus A_1$,

$$0 < \|\chi_{A_2}\|_X < \min\left\{\frac{1}{2^2}, \mathscr{K}'(\Omega \setminus A_1)\right\},$$

and $\mathscr{K}(\Omega \setminus (A_1 \cup A_2)) = \mathscr{K}((\Omega \setminus A_1) \setminus A_2) = 0$. By continuing this method, the desired sequence in (b) is obtained. $\square$

Now, we can give one of the main results of this section.

THEOREM 2. *Let X be a solid Banach function space and* $\inf\{\|\chi_E\|_X : E \in \mathscr{A}_0\} = 0$. *Also, assume that* $\sup\{\|\chi_E\|_X : E \in \mathscr{A}_\infty\} < \infty$, *where* $\mathscr{A}_\infty := \{E \in \mathscr{A} : \chi_E \in X\}$. *Then, for each $p \geqslant 1$, the set $X^p_{\text{r-strict}} := X^p \setminus \bigcup_{p<q} X^q$ is spaceable in X^p.*

Proof. We shall show that the conditions of Theorem 1 hold with $\mathscr{E} := X^p$ and $B = \bigcup_{p<q} X^q$. Note that since $\Omega \in \mathscr{A}_0$, for each $q > p$ we have $X^q \subset X^p$ thanks to [23, Theorem 2.4]. Clearly, B is a cone because each X^q is a linear space, and X^p is a PCS-space by Remark 1. Also, the condition (2) in Theorem 1 holds since X is solid. For the condition (4), let $f, g \in X^p$ with $E_f \cap E_g = \varnothing$ and $f + g \in B$. Then, there exists $q > p$ such that $f + g \in X^q$. We have

$$|f|^q, |g|^q \leqslant |f|^q + |g|^q = |f + g|^q \in X,$$

and this implies that $f, g \in X^q \subset B$. At the end, we show that the condition (5) in Theorem 1 hold. The main idea for the proof of this part comes from [5, Theorem 3.3]. Since $\inf\{\|\chi_E\|_X : E \in \mathscr{A}_0\} = 0$, by Lemma 1 there exists a sequence $\{A_n\}_{n=1}^\infty$ in $\mathscr{A}$ with pairwise disjoint terms such that $0 < \|\chi_{A_n}\|_X \leqslant \frac{1}{2^n}$ for all $n \in \mathbb{N}$. As in [5, Theorem 3.3], for each $n \in \mathbb{N}$, we choose a strictly increasing sequence $\{p_{n,k}\}_{k=1}^\infty$ of natural numbers such that $k \leqslant p_{n,k}$ for all $n, k \in \mathbb{N}$ and the elements of family $\{\{p_{n,k}\}_{k=1}^\infty : n \in \mathbb{N}\}$ are mutually disjoint. For each $n, k, m \in \mathbb{N}$ we put

$$\alpha_{n,k} := \frac{1}{\left(k(\log(1+k))^2 \|\chi_{A_{p_{n,k}}}\|_X\right)^{\frac{1}{p}}} \quad \text{and} \quad S_{n,m} := \sum_{k=1}^m \alpha_{n,k} \chi_{A_{p_{n,k}}}.$$

Since $\sum_{k=1}^\infty \|\alpha_{n,k}^p \chi_{A_{p_{n,k}}}\|_X = \sum_{k=1}^\infty \frac{1}{k \log(1+k)^2} < \infty$, the sequence $\{|S_{n,m}|^p\}_{m=1}^\infty$ is Cauchy and so convergent in X for all $n \in \mathbb{N}$. Now, we have

$$\lim_{m \to \infty} |S_{n,m}|^p = \sum_{k=1}^\infty \alpha_{n,k}^p \chi_{A_{p_{n,k}}}$$

in X because X is a PCS-space (see Remark 1). In particular, we have $f_n^p = \sum_{k=1}^\infty \alpha_{n,k}^p \chi_{A_{p_{n,k}}} \in X$, where

$$f_n := \sum_{k=1}^\infty \alpha_{n,k} \chi_{A_{p_{n,k}}} \qquad (n \in \mathbb{N}).$$

In fact, we have $\{f_n\}_{n=1}^{\infty} \subseteq X^p$ with $E_{f_n} \cap E_{f_m} = \varnothing$ for all distinct $m,n \in \mathbb{N}$. On the other hand, for each $q > p$, if $f_n \in X^q$, then

$$\begin{aligned}\|f_n\|_{X^q} &= \big(\||f_n|^q\|_X\big)^{\frac{1}{q}}\\ &= \left\|\sum_{k=1}^{\infty} \alpha_{n,k}^q \chi_{A_{p_{n,k}}}\right\|_X^{\frac{1}{q}}\\ &\geqslant \left\|\alpha_{n,k}^q \chi_{A_{p_{n,k}}}\right\|_X^{\frac{1}{q}}\\ &\geqslant \left(\frac{2^{(\frac{q}{p}-1)\,p_{n,k}}}{k^{\frac{q}{p}}\log(1+k)^{\frac{2q}{p}}}\right)^{\frac{1}{q}}\\ &\geqslant \left(2^{(\frac{q}{p}-1)k}\right)^{\frac{1}{q}}.\end{aligned}$$

So, since $k \in \mathbb{N}$ is arbitrary, we have $\|f_n\|_{X^q} = \infty$, a contradiction. Therefore, $\{f_n\}_{n=1}^{\infty} \subseteq X^p_{\mathrm{r-strict}}$ and the proof is complete. □

Next, an extension of the main part of [20, Theorem 3, page 155] is proved.

Motivated by the definition of a diffuse set for a measure (see [20, page 46]), we initiate the following concept. For each $E \subseteq \Omega$, denote $\mathscr{A}_E := \{A \subseteq E : A \in \mathscr{A}\}$.

DEFINITION 1. A set $E \in \mathscr{A}$ is called *diffuse* for a Banach function space X if $\chi_E \in X$ and for each $Y \in \mathscr{A}_E$ and each α with $0 \leqslant \alpha \leqslant \|\chi_Y\|_X$ there exists some $F \in \mathscr{A}_Y$ such that $\|\chi_F\|_X = \alpha$.

The main idea for proof of the next result comes from [20, Theorem 3, page 155], but the details are different because the situation is more general. Recall from [20, Definition 1, page 15] that a Young function Φ_2 is *stronger* than the other one Φ_1 (and we write $\Phi_1 \prec \Phi_2$) if there are constants $a > 0$ and $x_0 \geqslant 0$ such that for each $x \geqslant x_0$, $\Phi_1(x) \leqslant \Phi_2(ax)$.

THEOREM 3. *Let Φ_1, Φ_2 be two strictly increasing continuous Young functions. If there exists a diffuse set $E \in \mathscr{A}_\infty$ for X with $\mu(E) > 0$, then the inclusion $X^{\Phi_2} \subseteq X^{\Phi_1}$ implies that $\Phi_1 \prec \Phi_2$.*

Proof. Let the assumptions hold and $X^{\Phi_2} \subseteq X^{\Phi_1}$. In contrast, assume that $\Phi_1 \not\prec \Phi_2$. Then, there exists an increasing sequence $\{a_n\}_{n=1}^{\infty}$ in $(0,\infty)$ such that $\lim_{n\to\infty} a_n = \infty$ and

$$\Phi_1(a_n) > n2^n\Phi_2(n^2 a_n) \qquad (n \in \mathbb{N}). \tag{3}$$

Since $\sum_{n=1}^{\infty} \frac{\Phi_2(a_1)\|\chi_E\|_X}{2^n\Phi_2(n^2 a_n)} < \|\chi_E\|_X$, there exists $E_0 \in \mathscr{A}_E$ such that

$$\|\chi_{E_0}\|_X = \sum_{n=1}^{\infty} \frac{\Phi_2(a_1)\|\chi_E\|_X}{2^n\Phi_2(n^2 a_n)},$$

because E is a diffuse set for X. Inductively, one can find a pairwise disjoint sequence $\{E_n\}_{n=1}^{\infty}$ in $\mathscr{A}_{E_0}$ such that

$$\|\chi_{E_n}\|_X = \frac{\Phi_2(a_1)\|\chi_E\|_X}{2^n\Phi_2(n^2a_n)} \qquad (n \in \mathbb{N}). \tag{4}$$

So, setting $f := \sum_{n=1}^{\infty} na_n\chi_{E_n}$ we have

$$\begin{aligned}\sum_{n=1}^{\infty}\Phi_2(n^2a_n)\|\chi_{E_n}\|_X &= \sum_{n=1}^{\infty}\frac{\Phi_2(a_1)\|\chi_E\|_X}{2^n}\\ &= \Phi_2(a_1)\|\chi_E\|_X < \infty.\end{aligned}$$

This implies that the sequence

$$\left(\sum_{n=1}^{k}\Phi_2(n^2a_n)\chi_{E_n}\right)_k$$

is Cauchy and so convergent in X. But, by Remark 1, the convergence point is $\Phi_2(f) = \sum_{n=1}^{\infty}\Phi_2(n^2a_n)\chi_{E_n}$. So, $f \in X^{\Phi_2}$.

On the other hand, let $\alpha > 0$ be arbitrary. In contrast, let $\Phi_1(\alpha f) \in X$. Fix a number $m \in \mathbb{N}$ such that $\frac{1}{n} < \alpha$ for all $n \geqslant m$. Then, thanks to the relations (3) and (4) we have

$$\begin{aligned}\|\Phi_1(\alpha f)\|_X &= \left\|\sum_{n=1}^{\infty}\Phi_1(\alpha na_n)\chi_{E_n}\right\|_X\\ &\geqslant \Phi_1(\alpha ka_k)\|\chi_{E_k}\|_X\\ &\geqslant \Phi_1(a_k)\|\chi_{E_k}\|_X\\ &\geqslant k\Phi_2(a_1)\|\chi_E\|_X.\end{aligned}$$

for all $k \geqslant m$, and so $\|\Phi_1(\alpha f)\|_X = \infty$, a contradiction. This shows that $f \notin X^{\Phi_1}$, and the proof is complete. □

COROLLARY 1. *Under the assumptions of Theorem 3, if $\Phi_1 \not\prec \Phi_2$, then $X^{\Phi_2} \setminus X^{\Phi_1}$ is spaceable in X^{Φ_2}.*

Proof. Let for each $n \in \mathbb{N}$, N_n be a strictly increasing sequence of natural numbers and $\{N_n\}_{n=1}^{\infty}$ be a partition of $\mathbb{N}$. Then, similarly to the proof of Theorem 3 it would be routine to construct a sequence (f_n) in $X^{\Phi_2} \setminus X^{\Phi_1}$ such that for each distinct $m,n \in \mathbb{N}$, $E_{f_n} \cap E_{f_m} = \varnothing$. Now, the statement follows easily from Theorem 1. □

4. Some new sufficient conditions with applications

In this section, first we give an abstract version of Theorem 1 and then present some applications regarding Cartesian product of X^p spaces and also the space of bounded linear operators on a Hilbert space. For this we need to initiate the next concept.

DEFINITION 2. Let $\mathscr{E}$ be a topological vector space. We say that a relation $\sim$ on $\mathscr{E}$ has property (D) if the following conditions hold:

1. If (x_n) is a sequence in $\mathscr{E}$ such that $x_n \sim x_m$ for all distinct index m,n, then for each disjoint finite subsets A,B of $\mathbb{N}$ we have
$$\sum_{n\in A} \alpha_n x_n \sim \sum_{m\in B} \beta_m x_m,$$
where α_n and β_m's are arbitrary scalars.
2. If a sequence (x_n) converges to x in $\mathscr{E}$ and for some $y \in \mathscr{E}$, $x_n \sim y$ for all $n \in \mathbb{N}$, then $x \sim y$.

We recall that a sequence (x_n) in a Banach space $\mathscr{E}$ is called a *basic sequence* if for each x in $\overline{\text{span}}\{x_1,x_2,\ldots\}$, the closed linear span of $\{x_1,x_2,\ldots\}$, there are unique scalars $\alpha_1,\alpha_2,\ldots$ such that $x = \lim_n \sum_{k=1}^n \alpha_k x_k$ in $\mathscr{E}$. Note that, by [2, Proposition 1, Chapter II], (x_n) is a basic sequence if and only if there is a constant $k > 0$ such that for each m,n with $m \geqslant n$ and each scalars $\alpha_1,\ldots,\alpha_m$, $\left\|\sum_{j=1}^n \alpha_j x_j\right\| \leqslant k \left\|\sum_{j=1}^m \alpha_j x_j\right\|$.

THEOREM 4. *Let $(\mathscr{E}, \|\cdot\|)$ be a Banach space, $\sim$ be a relation on $\mathscr{E}$ with property (D), and K be a nonempty subset of $\mathscr{E}$. Assume that:*

1. *there is a constant $k > 0$ such that $\|x+y\| \geqslant k\|x\|$ for all $x,y \in \mathscr{E}$ with $x \sim y$;*
2. *K is a cone;*
3. *if $x,y \in \mathscr{E}$ such that $x+y \in K$ and $x \sim y$ then $x,y \in K$;*
4. *there is an infinite sequence $\{x_n\}_{n=1}^\infty \subseteq \mathscr{E} \setminus K$ such that for each distinct $m,n \in \mathbb{N}$, $x_m \sim x_n$.*

Then, $\mathscr{E} \setminus K$ is spaceable in $\mathscr{E}$.

Proof. The main idea of the proof comes from [5, Theorem 2.2]. Indeed, applying condition (D) in Definition 2 and thanks to [2, Proposition 1, Chapter II] one can see that the sequence (x_n) in assumption (4) is a basic sequence, and this shows that (x_n) is linearly independent. Let $0 \neq x \in \overline{\text{span}}\{x_1,x_2,\ldots\}$. Then, from the definition of basic sequences, there exist unique scalars $\alpha_1,\alpha_2,\ldots$ such that $x = \sum_{n=1}^\infty \alpha_n x_n$. Put $N := \min\{n \in \mathbb{N} : \alpha_n \neq 0\}$. So, $x = \alpha_N x_N + y$, where $y := \lim_{m\to\infty} \sum_{n=N+1}^m \alpha_n x_n$. Again, applying both conditions in Definition 2 we have $x_N \sim y$. In contrast, if $x \in K$, then by the assumptions (3) and (2) we have $x_N \in K$, a contradiction. Therefore, $(\mathscr{E} \setminus K) \cup \{0\}$ contains the closed infinite-dimensional space $\overline{\text{span}}\{x_1,x_2,\ldots\}$, and this completes the proof. $\square$

REMARK 2. We mention that this theorem is a generalization of [5, Theorem 2.2] (Theorem 1). Just note that for each Banach function space X, the relation $\sim$ defined by
$$f \sim g \text{ if and only if } E_f \cap E_g = \varnothing$$
for all $f,g \in X$, has the property (D).

Applying Theorem 4, we give the next result which could not be concluded from [5, Theorem 2.2].

THEOREM 5. *Let X be a solid Banach function space on Ω and assume that* $\inf\{\|\chi_E\|_X : E \in \mathscr{A}_0\} = 0$. *Then, for each $1 \leqslant p,q < r$, the set $\{(f,g) \in X^p \times X^q : fg \notin X^r\}$ is spaceable in $X^p \times X^q$.*

Proof. Let $1 \leqslant p,q < r$. By Lemma 1, there is a sequence $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{A}_0$ with disjoint terms such that $0 < \|\chi_{A_n}\|_X \leqslant \frac{1}{2^n}$ for all $n \in \mathbb{N}$. We define

$$j := \sum_{n=1}^{\infty} \|\chi_{A_n}\|_X^{\frac{-1}{r}} \cdot \chi_{A_n}.$$

Then, similarly the proof of Lemma 1, one can see that $j \in X^p \cap X^q$. In contrast, if $j^2 \in X^r$, then we would have

$$\begin{aligned}\|j^2\|_{X^r} &= \||j^2|^r\|_X \\ &= \left\| \left(\sum_{n=1}^{\infty} \|\chi_{A_n}\|_X^{\frac{-2}{r}} \cdot \chi_{A_n} \right)^r \right\|_X \\ &\geqslant \left\| \|\chi_{A_k}\|_X^{-2} \cdot \chi_{A_k} \right\|_X \geqslant 2^k\end{aligned}$$

for all $k \in \mathbb{N}$, a contradiction. This implies that setting

$$K := \{(f,g) \in X^p \times X^q : fg \in X^r\},$$

we have $(j,j) \in (X^p \times X^q) \setminus K$. Put $h := j^2$. By the above relations, it would be standard to find a sequence (F_n) such that for each distinct $m,n \in \mathbb{N}$, $F_n \cap F_m = \varnothing$, and $h\chi_{F_n} \notin X^r$. This implies that $(j\chi_{F_n}, j\chi_{F_n}) \in (X^p \times X^q) \setminus K$ for all $n \in \mathbb{N}$. Finally, note that the relation $\sim$ defined by

$$(f_1,g_1) \sim (f_2,g_2)$$

if and only if

$$E_{f_1} \cap E_{f_2} = E_{g_1} \cap E_{g_2} = \varnothing$$

for all $f_i \in X^p$ and $g_i \in X^q$ $(i = 1,2)$, satisfies the condition (D). Applying Theorem 4, the proof is complete. $\square$

Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space and $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for $\mathscr{H}$. For each non-empty subset $J \subseteq \mathbb{N}$ we let P_J denote the orthogonal projection onto $\overline{\text{span}}\{e_j\}_{j\in J}$. Also, we set $P_J = 0$ for $J = \varnothing$. For each $T,S \in B(\mathscr{H})$, the space of all bounded linear operators on $\mathscr{H}$, we say that $T \sim S$ if there exist two disjoint subsets $J_1, J_2 \subseteq \mathbb{N}$ such that $P_{J_1}TP_{J_2} = T$ and $P_{J_2}SP_{J_2} = S$. With these notations, we give the next lemma.

LEMMA 2. *The relation $\sim$ on $B(\mathscr{H})$ defined above has the property (D).*

Proof. Suppose that $\{T_n\}_{n\in\mathbb{N}}$ is a sequence in $B(\mathscr{H})$ such that for each distinct m,n we have $T_n \sim T_m$. Let $A := \{n_1,\ldots,n_k\}$ and $B := \{m_1,\ldots,m_l\}$ be two disjoint finite subsets of $\mathbb{N}$. Then, for each $n \in A$ and $m \in B$, there exist some disjoint subsets $J_{(n,m)}, J'_{(n,m)} \subseteq \mathbb{N}$ such that

$$P_{J_{(n,m)}} T_n P_{J_{(n,m)}} = T_n \qquad \text{and} \qquad P_{J'_{(n,m)}} T_m P_{J'_{(n,m)}} = T_m. \tag{5}$$

Without loss of the generality we assume that $T_n \neq 0$ for all index n. By [6, Chapter 2, Section 8, Theorem 4], we have

$$P_{\bigcap_{r=1}^{l} J_{(n,m_r)}} = P_{J_{(n,m_1)}} P_{J_{(n,m_2)}} \ldots P_{J_{(n,m_l)}} \tag{6}$$

for all $n \in A$. Then, (5) implies that

$$P_{\bigcap_{r=1}^{l} J_{(n,m_r)}} T_n P_{\bigcap_{r=1}^{l} J_{(n,m_r)}} = T_n \tag{7}$$

for all $n \in A$. Indeed, by (6) we have

$$\begin{aligned} &P_{\bigcap_{r=1}^{l} J_{(n,m_r)}} T_n P_{\bigcap_{r=1}^{l} J_{(n,m_r)}} \\ &\quad = P_{J_{(n,m_l)}} P_{J_{(n,m_{l-1})}} \ldots P_{J_{(n,m_1)}} T_n P_{J_{(n,m_1)}} P_{J_{(n,m_2)}} \ldots P_{J_{(n,m_l)}} \\ &\quad = T_n \end{aligned}$$

for all $n \in A$. Hence, since $T_n \neq 0$, we have $\bigcap_{r=1}^{l} J_{(n,m_r)} \neq \varnothing$. Put

$$E := \bigcup_{i=1}^{k} \bigcap_{r=1}^{l} J_{(n_i,m_r)}.$$

Then, by [6, Chapter 2, Section 8, Corollary 5], we have

$$P_E P_{\bigcap_{r=1}^{l} J_{(n_i,m_r)}} = P_{\bigcap_{r=1}^{l} J_{(n_i,m_r)}} P_E = P_{\bigcap_{r=1}^{l} J_{(n_i,m_r)}}$$

for every $i \in \{1,\ldots,k\}$. Because of (7),

$$P_E T_{n_i} P_E = T_{n_i}$$

for all $i \in \{1,\ldots,k\}$. This implies that $P_E(\sum_{i=1}^{k} \alpha_i T_{n_i}) P_E = \sum_{i=1}^{k} \alpha_i T_{n_i}$ for all $\alpha_1,\ldots,\alpha_k \in \mathbb{C}$. Similarly, setting $F := \bigcup_{r=1}^{l} \bigcap_{i=1}^{k} J'_{(n_i,m_r)}$ we have $P_F(\sum_{r=1}^{l} \beta_i T_{m_r}) P_F = \sum_{r=1}^{l} \beta_i T_{m_r}$ for all $\beta_1,\ldots,\beta_l \in \mathbb{C}$. Now, since $J_{(n_i,m_r)} \cap J'_{(n_i,m_r)} = \varnothing$ for each $i \in \{1,\ldots,k\}$ and $r \in \{1,\ldots,l\}$, easily we have $E \cap F = \varnothing$. Therefore, $\sim$ satisfies the condition (1) in Definition 2. Next, suppose that $S \in B(\mathscr{H})$ and $\{T_n\}_{n\in\mathbb{N}}$ is a sequence in $B(\mathscr{H})$ such that $T_n \to T$, in operator norm, for some $T \in B(\mathscr{H})$, and $T_n \sim S$ for all n. We can assume that $S \neq 0$ because the proof for the case $S = 0$ is trivial. Then, for each $n \in \mathbb{N}$, there are disjoint subsets $J_n, J_n' \subseteq \mathbb{N}$ such that

$$P_{J_n} T_n P_{J_n} = T_n \qquad \text{and} \qquad P_{J_n'} S P_{J_n'} = S. \tag{8}$$

Again, by [6, Chapter 2, Section 8, Theorem 4] for each $n \in \mathbb{N}$ we have

$$P_{\bigcap_{m=1}^n J_m'} S P_{\bigcap_{m=1}^n J_m'} = P_{J_n'} \dots P_{J_1'} S P_{J_1'} \dots P_{J_n'} = S. \tag{9}$$

Now, the sequence $\{P_{\cap_{m=1}^n J_m'}\}_{n\in\mathbb{N}}$ is a non-increasing sequence of orthogonal projections, hence by [6, Chapter 2, Section 8, Theorem 6],

$$s - \lim_{n\to\infty} P_{\cap_{m=1}^n J_m'} = P_{\cap_{m=1}^\infty J_m'},$$

where $s - \lim$ means the limit in the strong operator topology. From (9) and thanks to [6, Chapter 2, Section 5, Theorem 2] we have

$$\begin{aligned} S &= s - \lim_{n\to\infty} (P_{\cap_{m=1}^n J_m'} S P_{\cap_{m=1}^n J_m'}) \\ &= (s - \lim_{n\to\infty} P_{\cap_{m=1}^n J_m'}) S (s - \lim_{n\to\infty} P_{\cap_{m=1}^n J_m'}) \\ &= P_{\cap_{m=1}^\infty J_m'} S P_{\cap_{m=1}^\infty J_m'}. \end{aligned}$$

By (8) and [6, Chapter 2, Section 8, Corollary 5] we have

$$P_{\bigcup_{m=1}^\infty J_m} T_n P_{\bigcup_{m=1}^\infty J_m} = T_n$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ we get

$$P_{\bigcup_{m=1}^\infty J_m} T P_{\bigcup_{m=1}^\infty J_m} = T,$$

and this completes the proof because $(\bigcup_{m=1}^\infty J_m) \cap (\bigcap_{m=1}^\infty J_m') = \varnothing$. $\square$

DEFINITION 3. Let K be a cone in $B(\mathscr{H})$. We denote

$$\tilde{K} := \bigcup_{J\subseteq\mathbb{N}} P_J K P_J, \tag{10}$$

where $P_J K P_J := \{P_J T P_J : T \in K\}$.

Note that if K is a cone, then $\tilde{K}$ is a cone as well. Moreover, $P_J \tilde{K} P_J \subseteq \tilde{K}$ for all $J \subseteq \mathbb{N}$, and in particular, $K \subseteq \tilde{K}$.

In the sequel, $B_0(\mathscr{H})$ is the space of all compact operators on $\mathscr{H}$.

THEOREM 6. *Let K be a cone in $B(\mathscr{H})$. If there exists a sequence of mutually disjoint subsets $\{J_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ satisfying that $P_{J_n} \tilde{K} P_{J_n} \neq P_{J_n} B(\mathscr{H}) P_{J_n}$ for all $n \in \mathbb{N}$, then $B(\mathscr{H}) \setminus \tilde{K}$ (and consequently $B(\mathscr{H}) \setminus K$) is spaceable in $B(\mathscr{H})$. The statement holds if we consider $B_0(\mathscr{H})$ instead of $B(\mathscr{H})$.*

Proof. We show that the relation $\sim$ defined before Lemma 2 satisfies the conditions in Theorem 4 regarding the cone $\tilde{K}$. Suppose that $T,S \in B(\mathscr{H})$ with $T \sim S$. Then, there exist disjoint subsets $J,J' \subseteq \mathbb{N}$ such that $P_J T P_J = T$ and $P_{J'} S P_{J'} = S$. By disjointness of J and J', from [6, Chapter 2, Section 8, Theorem 2] we have $P_J S P_J = P_J P_{J'} S P_{J'} P_J = 0$. We get

$$\begin{aligned}\|T+S\| &\geqslant \|P_J\|\,\|T+S\|\,\|P_J\| \\ &\geqslant \|P_J(T+S)P_J\| \\ &= \|P_J T P_J\| \\ &= \|T\|.\end{aligned}$$

This shows that the relation $\sim$ satisfies the condition (1) of Theorem 4. Now, if in addition $T+S \in \tilde{K}$, we have

$$\begin{aligned}T &= P_J T P_J \\ &= P_J(T+S)P_J \in \tilde{K}.\end{aligned}$$

Similarly, $S \in \tilde{K}$. So, the condition (2) in Theorem 4 holds with respect to the cone $\tilde{K}$. Finally, consider the sequence $\{J_n\}$ of mutually disjoint subsets of $\mathbb{N}$ which was described in the assumptions. So, for each n we can choose an operator $T_n \in P_{J_n} B(\mathscr{H}) P_{J_n} \setminus P_{J_n} \tilde{K} P_{J_n}$. Then, easily one can see that $\{T_n\}_{n\in\mathbb{N}} \subset B(\mathscr{H}) \setminus \tilde{K}$ and for each distinct $m,n \in \mathbb{N}$ we have

$$T_n = P_{J_n} T_n P_{J_n} \sim P_{J_m} T_m P_{J_m} = T_m,$$

and this completes the proof. $\square$

COROLLARY 2. *The set of all bounded linear operators on $\mathscr{H}$ which are not positive-semidefinite, is spaceable in $B(\mathscr{H})$.*

Proof. Let K be the set of all scalar multiples of positive semidefinite operators on $\mathscr{H}$. Then K is a cone and $P_J K P_J \subseteq K$ for all $J \subseteq \mathbb{N}$ and so $\tilde{K} = K$. By some calculations one can see that the assumptions of Theorem 6 hold in this situation, and therefore $B(\mathscr{H}) \setminus K$ is spaceable. This implies that $B(\mathscr{H}) \setminus B_+(\mathscr{H})$ is spaceable as well, where $B_+(\mathscr{H})$ is the set of all positive semidefinite operators on $\mathscr{H}$. $\square$

The following result is directly concluded from Theorem 6.

COROLLARY 3. *If K is a two sided ideal cone in $B(\mathscr{H})$ and there exists a sequence of mutually disjoint subsets $\{J_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ satisfying the condition $P_{J_n} K P_{J_n} \neq P_{J_n} B(\mathscr{H}) P_{J_n}$ for all $n \in \mathbb{N}$, then $B(\mathscr{H}) \setminus K$ is spaceable in $B(\mathscr{H})$.*

EXAMPLE 1. $B_0(\mathscr{H})$ is a two sided ideal cone in $B(\mathscr{H})$ which satisfies the requirements of Corollary 3. So, the set of all non-compact operators is a spaceable subset of $B(\mathscr{H})$.

REMARK 3. By the same argument, $B_0(\mathscr{H}) \setminus (B_+(\mathscr{H}) \bigcap B_0(\mathscr{H}))$ is spaceable in $B_0(\mathscr{H})$, where $B_+(\mathscr{H})$ is the set of all positive semidefinite operators on $\mathscr{H}$. One can also replace $B_0(\mathscr{H})$ by the real Banach space of all Hermitian operators on $\mathscr{H}$.

REMARK 4. Let $(B_1(\mathscr{H}), \|\cdot\|_1)$ and $(B_2(\mathscr{H}), \|\cdot\|_2)$ denote the Banach space of all trace-class operators equipped with the trace norm and the Banach space of all Hilbert-Schmidt operators equipped with the Hilbert-Schmidt norm, respectively. Since for every $T \in B(\mathscr{H})$, $S \in B_1(\mathscr{H})$ and $G \in B_2(\mathscr{H})$ we have $\|ST\|_1, \|TS\|_1 \leqslant \|T\| \, \|S\|_1$ and $\|TG\|_2, \|GT\|_2 \leqslant \|T\| \, \|G\|_2$, it is not hard to see that a similar argument as in the proof of Lemma 2 and Theorem 6 can be applied to deduce that $B_1(\mathscr{H}) \setminus (B_+(\mathscr{H}) \bigcap B_1(\mathscr{H}))$ and $B_2(\mathscr{H}) \setminus (B_+(\mathscr{H}) \bigcap B_2(\mathscr{H}))$ are spaceable in $B_1(\mathscr{H})$ and $B_2(\mathscr{H})$, respectively. Also, $B_0(\mathscr{H}) \setminus B_1(\mathscr{H})$ and $B_0(\mathscr{H}) \setminus B_2(\mathscr{H})$ are both spaceable in $B_0(\mathscr{H})$.

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