Mathematical
Inequalities
& Applications
Volume 25, Number 3 (2022), 673–678 doi:10.7153/mia-2022-25-42

REFINEMENTS OF TWO DETERMINANTAL INEQUALITIES FOR POSITIVE SEMIDEFINITE MATRICES

YAN HONG AND FENG QI*

Dedicated to retired Professor Junesang Choi at Dongguk University in South Korea

(*Communicated by I. Perić*)

Abstract. Let $A,B,C \in \mathbb{C}^{n\times n}$ be positive semidefinite matrices and let $|A|,|B|,|C|$ be determinants of $A,B,C \in \mathbb{C}^{n\times n}$ respectively. In this paper, the authors prove two determinantal inequalities

$$|A+B+C|+|C| \geqslant |A+C|+|B+C|+\left(2^n-2\right)|AB|^{1/2}+3\left(3^{n-1}-2^n+1\right)|ABC|^{1/3}$$

and

$$|A+B+C|+|A|+|B|+|C| \geqslant |A+B|+|A+C|+|B+C|+3\left(3^{n-1}-2^n+1\right)|ABC|^{1/3}.$$

These two inequalities refine known ones.

1. Introduction

Let $A \in \mathbb{C}^{n\times n}$ be a square matrix of order n. In this paper, we will denote the determinant of the matrix A by $|A|$ and denote eigenvalues of the matrix A by

$$\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A).$$

If eigenvalues of the matrix A are real numbers, we specify

$$\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A).$$

Let $A,B \in \mathbb{C}^{n\times n}$ be positive semidefinite matrices. In [1, p. 465, Corollary], Hartfiel obtained the determinantal inequality

$$|A+B| \geqslant |A|+|B|+(2^n-2)(|A||B|)^{1/2}. \tag{1}$$

In [5, p. 215], Zhang established a determinantal inequality below.

Mathematics subject classification (2020): Primary 15A45; Secondary 15A39, 15A42.
Keywords and phrases: Determinantal inequality, eigenvalue, positive semidefinite matrix, determinant.

* Corresponding author.

THEOREM 1. ([5, p. 215]) *Let $A,B,C \in \mathbb{C}^{n\times n}$ be positive semidefinite matrices. Then*

$$|A+B+C|+|C| \geqslant |A+C|+|B+C|. \tag{2}$$

In [3], Lin established the following determinantal inequality.

THEOREM 2. ([3, Theorem 1.1]) *Let $A,B,C \in \mathbb{C}^{n\times n}$ be positive semidefinite matrices. Then*

$$|A+B+C|+|A|+|B|+|C| \geqslant |A+B|+|A+C|+|B+C|. \tag{3}$$

In this paper, we will refine determinantal inequalities (2) and (3) above.

2. Lemmas

To refine determinantal inequalities (2) and (3) in Theorems 1 and 2, we need the following lemmas.

LEMMA 1. ([4, p. 333]) *Let $A,B \in \mathbb{C}^{n\times n}$ be positive semidefinite matrices. Then*

$$\prod_{i=1}^{n}[\lambda_i(A)+\lambda_{n-i+1}(B)] \geqslant |A+B| \geqslant \prod_{i=1}^{n}[\lambda_i(A)+\lambda_i(B)].$$

By the proof of Theorem 1.1 in [3], we have the following inequality.

LEMMA 2. ([3, Theorem 1.1]) *Let $A,B \in \mathbb{C}^{n\times n}$ be positive semidefinite matrices. Then*

$$\prod_{i=1}^{n}\lambda_i(A+B+I_n)-\prod_{i=1}^{n}\lambda_i(A+B) \geqslant \prod_{i=1}^{n}[\lambda_i(A)+\lambda_i(B)+1]-\prod_{i=1}^{n}[\lambda_i(A)+\lambda_i(B)].$$

LEMMA 3. *Let $a_i,b_i,c_i \geqslant 0$ for $i\in\mathbb{N}$. Then*

$$\prod_{i=1}^{n}(a_i+b_i+c_i)+\prod_{i=1}^{n}a_i+\prod_{i=1}^{n}b_i+\prod_{i=1}^{n}c_i-\prod_{i=1}^{n}(a_i+b_i)-\prod_{i=1}^{n}(a_i+c_i)-\prod_{i=1}^{n}(b_i+c_i)$$
$$\geqslant 3\left(3^{n-1}-2^n+1\right)\prod_{i=1}^{n}(a_ib_ic_i)^{1/3}. \tag{4}$$

Proof. If $n=1,2$, the inequality (4) obviously holds. If $n\geqslant 3$, let $i_1,\dots,i_k$, $j_1,\dots,j_t$, $\ell_1,\dots,\ell_s$ be non-negative positive numbers such that

$$\{i_1,\cdots,i_k,j_1,\cdots,j_t,\ell_1,\cdots,\ell_s\}\setminus\{0\}=\{1,2,\cdots,n\}.$$

Since

$$\prod_{i=1}^{n}(a_i+b_i+c_i)=\sum_{\substack{0\leqslant i_1<\cdots<i_k,0\leqslant j_1<\cdots<j_t,\\ 0\leqslant \ell_1<\cdots<\ell_s,i_k+j_t+\ell_s=n}}\prod_{m=1}^{k}a_{i_m}\prod_{p=1}^{t}b_{j_p}\prod_{q=1}^{s}c_{\ell_q},$$

by virtue of the inequality between the geometric and arithmetic means, we acquire

$$\begin{aligned}\prod_{i=1}^{n}(a_i+b_i+c_i)+\prod_{i=1}^{n}a_i+\prod_{i=1}^{n}b_i+\prod_{i=1}^{n}c_i-\prod_{i=1}^{n}(a_i+b_i)-\prod_{i=1}^{n}(a_i+c_i)-\prod_{i=1}^{n}(b_i+c_i)\\ =\sum_{\substack{1\leqslant i_1<\cdots<i_k,1\leqslant j_1<\cdots<j_t,\\ 1\leqslant \ell_1<\cdots<\ell_s, i_k+j_t+\ell_s=n}}\prod_{m=1}^{k}a_{i_m}\prod_{p=1}^{t}b_{j_p}\prod_{q=1}^{s}c_{\ell_q}\\ \geqslant 3\left(3^{n-1}-2^n+1\right)\prod_{i=1}^{n}(a_ib_ic_i)^{\frac{3^{n-1}-2^n+1}{3(3^{n-1}-2^n+1)}}\\ =3\left(3^{n-1}-2^n+1\right)\prod_{i=1}^{n}(a_ib_ic_i)^{1/3}.\end{aligned}$$

The proof of Lemma 3 is complete. □

REMARK 1. Taking $c_i=1$ for $i=1,2,\dots,n$ in Lemma 3 results in

$$\prod_{i=1}^{n}(a_i+b_i+1)+\prod_{i=1}^{n}a_i+\prod_{i=1}^{n}b_i+1-\prod_{i=1}^{n}(a_i+b_i)-\prod_{i=1}^{n}(a_i+1)-\prod_{i=1}^{n}(b_i+1)\geqslant 3\left(3^{n-1}-2^n+1\right)\prod_{i=1}^{n}(a_ib_i)^{1/3}, \quad (5)$$

where $a_i,b_i\geqslant 0$ for $i\in\mathbb{N}$.

LEMMA 4. *Let $a_i,b_i,c_i\geqslant 0$ for $i\in\mathbb{N}$. Then*

$$\prod_{i=1}^{n}(a_i+b_i+c_i)+\prod_{i=1}^{n}c_i-\prod_{i=1}^{n}(a_i+c_i)-\prod_{i=1}^{n}(b_i+c_i)\geqslant\left(2^n-2\right)\prod_{i=1}^{n}(a_ib_i)^{1/2}+3\left(3^{n-1}-2^n+1\right)\prod_{i=1}^{n}(a_ib_ic_i)^{1/3}. \quad (6)$$

Proof. Using the inequality (1), we obtain

$$\prod_{i=1}^{n}(a_i+b_i)-\prod_{i=1}^{n}a_i-\prod_{i=1}^{n}b_i\geqslant\left(2^n-2\right)\prod_{i=1}^{n}(a_ib_i)^{1/2}.$$

Therefore, by the inequality (4), we arrive at

$$\begin{aligned}\prod_{i=1}^{n}(a_i+b_i+c_i)+\prod_{i=1}^{n}c_i-\prod_{i=1}^{n}(a_i+c_i)-\prod_{i=1}^{n}(b_i+c_i)\\ \geqslant\prod_{i=1}^{n}(a_i+b_i)-\prod_{i=1}^{n}a_i-\prod_{i=1}^{n}b_i+3\left(3^{n-1}-2^n+1\right)\prod_{i=1}^{n}(a_ib_ic_i)^{1/3}\\ \geqslant\left(2^n-2\right)\prod_{i=1}^{n}(a_ib_i)^{1/2}+3\left(3^{n-1}-2^n+1\right)\prod_{i=1}^{n}(a_ib_ic_i)^{1/3}.\end{aligned}$$

The proof of Lemma 4 is complete. □

REMARK 2. Setting $c_i = 1$ for $i = 1, 2, \ldots, n$ in the inequality (6) in Lemma 4 leads to

$$\prod_{i=1}^{n}(a_i+b_i+1)+1-\prod_{i=1}^{n}(a_i+1)-\prod_{i=1}^{n}(b_i+1) \geqslant \left(2^n-2\right)\prod_{i=1}^{n}(a_ib_i)^{1/2}+3\left(3^{n-1}-2^n+1\right)\prod_{i=1}^{n}(a_ib_i)^{1/3}, \quad (7)$$

where $a_i, b_i \geqslant 0$ for $i \in \mathbb{N}$.

REMARK 3. From the inequality (7), it is easy to see that,

1. when $\prod_{i=1}^{n} a_ib_i \geqslant 1$, we have

$$\prod_{i=1}^{n}(a_i+b_i+1)+1-\prod_{i=1}^{n}(a_i+1)-\prod_{i=1}^{n}(b_i+1) \geqslant \left(3^n-2^{n+1}+1\right)\prod_{i=1}^{n}(a_ib_i)^{1/3}.$$

2. when $0 \leqslant \prod_{i=1}^{n} a_ib_i \leqslant 1$, we have

$$\prod_{i=1}^{n}(a_i+b_i+1)+1-\prod_{i=1}^{n}(a_i+1)-\prod_{i=1}^{n}(b_i+1) \geqslant \left(3^n-2^{n+1}+1\right)\prod_{i=1}^{n}(a_ib_i)^{1/2}.$$

3. Refinements of two determinantal inequalities

In this section, we refine determinantal inequalities (2) and (3).

THEOREM 3. *Let $A, B, C \in \mathbb{C}^{n\times n}$ be positive semidefinite matrices. Then*

$$|A+B+C|+|A|+|B|+|C| \geqslant |A+B|+|A+C|+|B+C|+3\left(3^{n-1}-2^n+1\right)|ABC|^{1/3}. \quad (8)$$

Proof. If $|C| = 0$, the inequality (8) evidently holds.

If $|C| \neq 0$, putting

$$A_1 = C^{-1/2}AC^{-1/2} \quad \text{and} \quad B_1 = C^{-1/2}BC^{-1/2}.$$

Using Lemma 2 and the inequality (5), we deduce

$$
\begin{aligned}
&(|A+B+C|+|A|+|B|+|C|-|A+B|-|A+C|-|B+C|)|C|^{-1}\\
=&|A_1+B_1+I_n|+|A_1|+|B_1|+1-|A_1+B_1|-|A_1+I_n|-|B_1+I_n|\\
=&\prod_{i=1}^{n}\lambda_i(A_1+B_1+I_n)+\prod_{i=1}^{n}\lambda_i(A_1)+\prod_{i=1}^{n}\lambda_i(B_1)+1\\
&-\prod_{i=1}^{n}\lambda_i(A_1+B_1)-\prod_{i=1}^{n}\lambda_i(A_1+I_n)-\prod_{i=1}^{n}\lambda_i(B_1+I_n)\\
=&\prod_{i=1}^{n}[\lambda_i(A_1+B_1)+1]+\prod_{i=1}^{n}\lambda_i(A_1)+\prod_{i=1}^{n}\lambda_i(B_1)+1\\
&-\prod_{i=1}^{n}[\lambda_i(A_1+B_1)]-\prod_{i=1}^{n}[\lambda_i(A_1)+1]-\prod_{i=1}^{n}[\lambda_i(B_1)+1]\\
\geqslant&\prod_{i=1}^{n}[\lambda_i(A_1)+\lambda_i(B_1)+1]+\prod_{i=1}^{n}\lambda_i(A_1)+\prod_{i=1}^{n}\lambda_i(B_1)+1\\
&-\prod_{i=1}^{n}[\lambda_i(A_1)+\lambda_i(B_1)]-\prod_{i=1}^{n}[\lambda_i(A_1)+1]-\prod_{i=1}^{n}[\lambda_i(B_1)+1]\\
\geqslant&3\big(3^{n-1}-2^n+1\big)|ABC|^{1/3}|C|^{-1}.
\end{aligned}
$$

The proof of Theorem 3 is complete. □

THEOREM 4. *Let $A,B,C\in\mathbb{C}^{n\times n}$ be positive semidefinite matrices. Then*

$$
|A+B+C|+|C|\geqslant|A+C|+|B+C|+\big(2^n-2\big)|AB|^{1/2}+3\big(3^{n-1}-2^n+1\big)|ABC|^{1/3}. \tag{9}
$$

Proof. By the inequality (8) and the inequality (1), we obtain

$$
\begin{aligned}
&|A+B+C|+|C|-|A+C|-|B+C|\\
\geqslant&|A+B|-|A|-|B|+3\big(3^{n-1}-2^n+1\big)|ABC|^{1/3}\\
\geqslant&\big(2^n-2\big)|AB|^{1/2}+3\big(3^{n-1}-2^n+1\big)|ABC|^{1/3}.
\end{aligned}
$$

The proof of Theorem 4 is complete. □

REMARK 4. Theorem 4 can be proved by Lemma 1 and the inequality (7), as done in the proof of Theorem 3.

REMARK 5. This paper is a corrected and revised version of the preprint [2].

Acknowledgements. The authors are thankful to anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

Funding. This work was partially supported by the National Natural Science Foundation of China (Grant No. 11901322) and by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region (Grant No. NJZY20119), China.

REFERENCES

[1] D. J. HARTFIEL, *An extension of Haynsworth's determinant inequality*, Proc. Amer. Math. Soc. **41** (1973), no. 2, 463–465; available online at https://doi.org/10.2307/2039114.

[2] Y. HONG AND F. QI, *Two determinantal inequalities for positive semidefinite matrices*, Research Square (2021), available online at https://doi.org/10.21203/rs.3.rs-360043/v1.

[3] M. LIN, *A determinantal inequality for positive semidefinite matrices*, Electron. J. Linear Algebra **27** (2014), 821–826; available online at https://doi.org/10.13001/1081-3810.2830.

[4] A. W. MARSHALL, I. OLKIN, AND B. C. ARNOLD, *Inequalities: Theory of Majorization and its Applications*, 2nd ed., Springer Verlag, New York/Dordrecht/Heidelberg/London, 2011; available online at http://dx.doi.org/10.1007/978-0-387-68276-1.

[5] F. ZHANG, *Matrix Theory: Basic Results and Techniques*, 2nd ed., Universitext, Springer, New York, 2011; available online at https://doi.org/10.1007/978-1-4614-1099-7.

(Received December 31, 2021)

Yan Hong
College of Mathematics and Physics
Inner Mongolia Minzu University
Tongliao 028043, Inner Mongolia, China
e-mail: nmhongyan@163.com
https://orcid.org/0000-0002-9916-9097

Feng Qi
Institute of Mathematics
Henan Polytechnic University
Jiaozuo 454003, Henan, China
and
School of Mathematical Sciences
Tiangong University
Tianjin 300387, China
e-mail: qifeng618@gmail.com
qifeng618@hotmail.com
qifeng618@qq.com
https://qifeng618.wordpress.com
https://orcid.org/0000-0001-6239-2968