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GRADIENT ESTIMATES FOR THE p–LAPLACIAN PARABOLIC EQUATIONS WITH A LOW–ORDER TERM

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Abstract. This paper mainly deals with regularity estimates in Orlicz spaces for the following divergence parabolic equations of p-Laplacian type with a low-order term

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) + |u|^{p-2}u = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega_T := \Omega \times (0,T)$$

under some proper assumptions on $\mathbf{f}(x,t)$. Remarkably, the equations we have discussed here contain the low-order term $|u|^{p-2}u$.

1. Introduction

The present paper is devoted to the study of regularity estimates in Orlicz spaces for the following divergence parabolic equations of p-Laplacian type with a low-order term

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) + |u|^{p-2}u = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \qquad \text{in } \Omega_T := \Omega \times (0,T), \tag{1}$$

where $n \geqslant 2$, $p \geqslant 2$, Ω is an open bounded domain in $\mathbb{R}^n$, $\partial_p\Omega_T = \partial\Omega \times [0,T] \cup \Omega \times \{t=0\}$ is the parabolic boundary of Ω_T and $\mathbf{f} = (f^1,\dots,f^n)$ is a given vector field with $\mathbf{f} \in L^p_{loc}(\Omega_T)$. Moreover, we say that $u \in L^p_{loc}(0,T;W^{1,p}_{loc}(\Omega)) \cap L^\infty_{loc}(0,T;L^2_{loc}(\Omega))$ is the local weak solution of (1) if for each $\varphi \in C_0^\infty(\Omega_T)$, we have

$$\int_{\Omega_T} -u\varphi_t + |Du|^{p-2}Du \cdot D\varphi + |u|^{p-2}u\varphi \, dxdt = -\int_{\Omega_T} |\mathbf{f}|^{p-2}\mathbf{f} \cdot D\varphi \, dxdt.$$

Since the Calderón-Zygmund theory (L^p-type regularity) was proved for the simplest elliptic Poisson equation, the related theory has been widely investigated by using various methods for the linear elliptic/parabolic equations with different continuous/discontinuous coefficients and smooth/nonsmooth domains. And then its nonlinear versions have been extensively studied especially for the elliptic problems of p-Laplacian type

$$\operatorname{div}\left((ADu \cdot Du)^{\frac{p-2}{2}} ADu\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \text{ in } \Omega$$

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and the more general case. We remark that in the elliptic case we often use the maximal function approach to get the L^p-type regularity estimates.

Due to the scaling deficit caused by the parabolic nonlinearity, many common approaches used in the elliptic case can no longer be used in the parabolic case of p-Laplacian type

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \operatorname{div}\left(|\mathbf{f}|^{p-2}\mathbf{f}\right) \quad \text{in } \Omega_T. \tag{2}$$

It is worth mentioning that the classical maximal function approach will also be not applicable to the parabolic problems of the p-Laplacian type (2). In order to overcome the difficulty in the scaling-invariant problem, Acerbi and Mingione [1] found a new covering/iteration approach (see [24] for its origin) involving the large-M-inequality principle. Remarkably enough, the above method is a harmonic analysis-free technique since it does not need to use the Calderón-Zygmund decomposition and maximal functions. As a matter of fact, this approach has been widely used in L^p-type regularity theory for various kinds of nonlinear elliptic and parabolic equations. We can also refer to [6, 7, 8, 9, 11, 16, 19, 20, 22, 25] for L^p-type regularity estimates for weak solutions of (2) and the general case with different coefficient and domain assumptions. Furthermore, some authors [3, 4, 27] extended the Calderón-Zygmund theory for the quasilinear parabolic equations of p-Laplacian type to the setting of variable exponents $p(x,t)$-Laplacian case. Meanwhile, we can refer to the book [15] for Hölder estimates of weak solutions of the quasilinear parabolic equations of p-Laplacian type. In addition, several articles [5, 10, 13, 14, 17, 18, 26] have been devoted to the study of the boundedness and Hölder estimates for weak solutions and their gradients of (2) and the more general case.

For convenience of the readers, we first recall some definitions and lemmas (see [2, 23]) about the general Orlicz spaces.

DEFINITION 1. A convex function $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ is said to be a Young function if

$$\lim_{t\to 0+} \frac{\phi(t)}{t} = \lim_{t\to +\infty} \frac{t}{\phi(t)} = 0 \quad \text{and} \quad \phi(0) = 0.$$

Accordingly, the Orlicz class $K^\phi(\Omega)$ is the set of all measurable functions $g : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega \phi(|g|)\, dx < \infty$$

and the Orlicz space $L^\phi(\Omega)$ is the linear hull of $K^\phi(\Omega)$. Moreover, a Young function ϕ is said to satisfy the global Δ_2 condition, denoted by $\phi \in \Delta_2$, if

$$\phi(2t) \leqslant K\phi(t) \quad \text{for every } t > 0 \text{ and some constant } K > 0.$$

Meanwhile, a Young function ϕ satisfies the global ∇_2 condition, denoted by $\phi \in \nabla_2$, if

$$\phi(t) \leqslant \frac{\phi(\theta t)}{2\theta} \quad \text{for every } t > 0 \text{ and some constant } \theta > 1.$$

Let us remark here that $K^{\phi}(\Omega)=L^{\phi}(\Omega)$ and $C_0^{\infty}(\Omega)$ is dense in $L^{\phi}(\Omega)$ for $\phi\in\triangle_2\cap\nabla_2$. Actually, there are two typical examples

$$\phi_1(t)=t^p\in\Delta_2\cap\nabla_2$$

and

$$\phi_2(t)=t^p\ln(1+t)\in\Delta_2\cap\nabla_2\quad\text{for } p>1.$$

Moreover, we would like to point out that $\phi\in\Delta_2\cap\nabla_2$ if and only if there exist constants $A_2\geqslant A_1>0$ and $\alpha_1\geqslant\alpha_2>1$ such that

$$A_1\left(\frac{s}{t}\right)^{\alpha_2}\leqslant\frac{\phi(s)}{\phi(t)}\leqslant A_2\left(\frac{s}{t}\right)^{\alpha_1}\quad\text{for any } 0<t\leqslant s. \tag{3}$$

Our main goal here is to obtain regularity estimates in Orlicz spaces for weak solutions of (1). We would like to remark that regularity theory for the classical parabolic p-Laplacian equation was usually studied for $p>\frac{2n}{n+2}$. Actually, the lower bound $\frac{2n}{n+2}$, which is slightly smaller than 2, on the exponent p is standard and unavoidable (see [5]). In this work we mainly consider the case $p\geqslant 2$. Due to the existence of the lower term and the nonlinearity/inhomogeneity of the equation itself, we shall use a class of parabolic cylinders $\{Q_{z_i}(\lambda^{2-p}\rho_i^p,\rho_i)\}$ (see (7)) whose lengths of the time depend on the solution u and its gradient Du. In the case $p<2$ which is called the singular case, the modulus of ellipticity tends to infinity when $|Du|\to 0$. In this case it is same to the lower term $|u|^{p-2}$ when $|u|\to 0$. In this situation estimates become harder to get.

Now let us state the main result of this work.

THEOREM 1. *Assume that* $\phi\in\Delta_2\cap\nabla_2$. *If* u *is the local weak solution of* (1) *in* $\Omega_T\supset Q_2$, *where the parabolic cylinder* $Q_r:=B_r\times(-r^p,r^p)$, *then we have*

$$|u|^p,|Du|^p\in L^{\phi}_{loc}(\Omega_T)$$

with the estimate

$$\begin{aligned}&\int_{Q_1}\phi\left(|u|^p\right)+\phi\left(|Du|^p\right)dxdt\\ &\leqslant C\int_{Q_2}\phi\left(|\mathbf{f}|^p\right)dxdt+C\phi\left[\left(\int_{Q_2}|u|^p+|Du|^p+|\mathbf{f}|^p+1dxdt\right)^{\frac{p}{2}}\right].\end{aligned} \tag{4}$$

2. Proof of the main result

This section is devoted to the proof of the main result stated in Theorem 1. We first give the following elementary measure theory.

LEMMA 1. (see [12]) *Let* ϕ *be a Young function with* $\phi\in\triangle_2\cap\nabla_2$ *and* $g\in L^{\phi}(\Omega_T)$. *Then we have*

$$\int_{\Omega_T}\phi(|g|)\,dxdt=\int_0^{\infty}|\{z\in\Omega_T:|g|>\lambda\}|\,d[\phi(\lambda)]$$

and

$$\int_0^\infty \frac{1}{\lambda}\left\{\int_{\{z\in\Omega_T:|g|>b_1\lambda\}}|g|dxdt\right\}d\left[\phi(b_2\lambda)\right]\leqslant C(b_1,b_2,\phi)\int_{\Omega_T}\phi(|g|)\,dxdt$$

for any $b_1,b_2>0$.

Next, we will give the iteration-covering procedure which was first introduced by [1, 24]. Instead, it takes advantage of a stopping time argument and Vitali's covering lemma. To begin with, we define

$$\lambda_0^2 := \fint_{Q_2}|u|^p+|Du|^p\,dxdt+\frac{1}{\delta}\fint_{Q_2}|\mathbf{f}|^p\,dxdt+1 \quad \text{for some } \delta\in(0,1), \tag{5}$$

$$J\left[u,\mathbf{f},Q\right] := \fint_{Q}|u|^p+|Du|^p\,dxdt+\frac{1}{\delta}\fint_{Q}|\mathbf{f}|^p\,dxdt \tag{6}$$

for any domain $Q\subset\mathbb{R}^{n+1}$ and the level set

$$E(u,\lambda) := \left\{z=(x,t)\in Q_1 : |u|^p+|Du|^p>\lambda^p\right\}.$$

And then we will decompose the level set $E(u,\lambda)$ into a family of small disjoint cylinders since $|u|^p+|Du|^p$ is bounded in $Q_1\setminus E(u,\lambda)$ for a fixed $\lambda>0$. Meanwhile, we shall obtain the corresponding estimates of these small cylinders.

LEMMA 2. *If u is the weak solution of (1) in $\Omega_T\supset Q_2$, for $\lambda\geqslant\lambda_* =: 10^{n+p}\lambda_0$ there exists a family of disjoint cylinders*

$$\left\{Q_i^0\right\}_{i\geqslant1} := \left\{Q_{z_i}(\lambda^{2-p}\rho_i^p,\rho_i)\right\}_{i\geqslant1} \tag{7}$$

with $z_i=(x_i,t_i)\in E(u,\lambda)$ *and* $\rho_i=\rho(x_i,t_i)\in\left(0,\frac{1}{10}\right]$, *where*

$$Q_{z_i}(\theta,\rho) := B_\rho(x_i)\times(t_i-\theta,t_i+\theta),$$

such that

$$J\left[u,\mathbf{f},Q_i^0\right]=\lambda^p, \tag{8}$$

$$J\left[u,\mathbf{f},Q_{(x_i,t_i)}(\theta,\rho)\right]<\lambda^p \quad \textit{for any } Q_{(x_i,t_i)}(\theta,\rho)\supseteq Q_i^0, \tag{9}$$

$$E(u,\lambda)\subset\bigcup_{i\geqslant1}5Q_i^0\cup \textit{negligible set}, \tag{10}$$

and

$$\begin{aligned}\sum_{i=1}^\infty\left|Q_i^0\right| \leqslant{}& \frac{3}{\lambda^p}\int_{\left\{z\in Q_2:\ |u|^p+|Du|^p>\frac{\lambda^p}{3}\right\}}|u|^p+|Du|^p\,dxdt\\ &+\frac{3}{\delta\lambda^p}\int_{\left\{z\in Q_2:\ |\mathbf{f}|^p>\frac{\delta\lambda^p}{3}\right\}}|\mathbf{f}|^p\,dxdt.\end{aligned} \tag{11}$$

Proof. Fix any $z=(x,t)\in Q_1$ and $\frac{1}{10}\leqslant\rho\leqslant 1$. Then it follows from (5) and (6) that

$$\begin{aligned}
&J\left[u,\mathbf{f},Q_z\left(\lambda^{2-p}\rho^p,\rho\right)\right]\\
&=\fint_{Q_z\left(\lambda^{2-p}\rho^p,\rho\right)}|u|^p+|Du|^p\,dxdt+\frac{1}{\delta}\fint_{Q_z\left(\lambda^{2-p}\rho^p,\rho\right)}|\mathbf{f}|^p\,dxdt\\
&\leqslant\frac{|Q_2|}{\left|Q_z\left(\lambda^{2-p}\rho^p,\rho\right)\right|}\left[\fint_{Q_2}|u|^p+|Du|^p\,dxdt+\frac{1}{\delta}\fint_{Q_2}|\mathbf{f}|^p\,dxdt\right]\\
&\leqslant 20^{n+p}\lambda_0^2\lambda^{p-2}\\
&\leqslant\left(10^{n+p}\lambda_0\right)^2\lambda^{p-2}\\
&\leqslant\lambda^p
\end{aligned}\tag{12}$$

for $\lambda\geqslant\lambda_*=:10^{n+p}\lambda_0$, which implies that

$$\sup_{z=(x,t)\in Q_1}\ \sup_{\rho\in\left[\frac{1}{10},1\right]}J\left[u,\mathbf{f},Q_z\left(\lambda^{2-p}\rho^p,\rho\right)\right]\leqslant\lambda^p.\tag{13}$$

Applying Lebesgue's differentiation theorem, for a.e. $z=(x,t)\in E(u,\lambda)$ we know that

$$\lim_{\rho\to 0}J\left[u,\mathbf{f},Q_z\left(\lambda^{2-p}\rho^p,\rho\right)\right]>\lambda^p,$$

which implies that there exists some $\tilde{\rho}>0$ such that

$$J\left[u,\mathbf{f},Q_z\left(\lambda^{2-p}\tilde{\rho}^p,\tilde{\rho}\right)\right]>\lambda^p.$$

Furthermore, from (13) one can select a radius $\rho_z\in\left(0,\frac{1}{10}\right]$ such that

$$\rho_z:=\max\left\{\rho\in\left(0,\frac{1}{10}\right]:J\left[u,\mathbf{f},Q_z\left(\lambda^{2-p}\rho^p,\rho\right)\right]=\lambda^p\right\},$$

$$J\left[u,\mathbf{f},Q_z\left(\lambda^{2-p}\rho_z^p,\rho_z\right)\right]=\lambda^p$$

and

$$J\left[u,\mathbf{f},Q_z\left(\lambda^{2-p}\rho^p,\rho\right)\right]<\lambda^p\quad\text{for any }\rho>\rho_z.$$

Therefore, we use Vitali's covering lemma to find a family of disjoint cylinders

$$\left\{Q_i^0\right\}_{i\geqslant 1}:=\left\{Q_{z_i}(\lambda^{2-p}\rho_i^p,\rho_i)\right\}_{i\geqslant 1}$$

satisfying (8) and (10). Actually, the first equality of (8) implies that

$$J\left[u,\mathbf{f},Q_i^0\right]=\lambda^p=\fint_{Q_i^0}|u|^p+|Du|^p\,dxdt+\frac{1}{\delta}\fint_{Q_i^0}|\mathbf{f}|^p\,dxdt.$$

Subsequently, by splitting the two integrals above as follows we have

$$\lambda^p|Q_i^0| \leqslant \int_{\left\{z\in Q_i^0:\,|u|^p+|Du|^p>\frac{\lambda^p}{3}\right\}} |u|^p+|Du|^p\,dxdt+\frac{\lambda^p}{3}|Q_i^0|$$
$$+\frac{1}{\delta}\int_{\left\{z\in Q_i^0:\ |\mathbf{f}|^p>\frac{\delta\lambda^p}{3}\right\}} |\mathbf{f}|^p\,dxdt+\frac{\lambda^p}{3}|Q_i^0|,$$

which implies that (11) is true for $\lambda \geqslant \lambda_*$ due to the fact that the cylinders $\{Q_i^0\}$ are disjoint. So we complete the proof. □

Here we are going to derive comparison results between the weak solutions u of (1) and h of the good homogeneous reference equation.

LEMMA 3. *For any $\varepsilon > 0$, there exists a small positive constant $\delta = \delta(\varepsilon,n,p)$ such that if u is the weak solution of (1) with $Q_2 \subset \Omega_T$,*

$$⨍_{Q_2} |u|^p+|Du|^p\,dxdt<1 \quad \textit{and} \quad ⨍_{Q_2}|\mathbf{f}|^p\,dxdt<\delta, \tag{14}$$

then we have

$$⨍_{Q_2} |u-h|^p+|Du-Dh|^p\,dxdt<\varepsilon \tag{15}$$

and

$$\sup_{Q_1}|h|^p+|Dh|^p \leqslant N_1^p \quad \textit{for some}\ N_1>1, \tag{16}$$

where h is the weak solution of

$$h_t - div\left(|Dh|^{p-2}Dh\right)+|h|^{p-2}h=0 \quad in\ Q_2 \tag{17}$$

with $h=u$ on $\partial_p Q_2$.

Proof. If u and h are the weak solutions of (1) and (17) respectively, then by selecting the test function $\varphi = u-h$ which is possible modulo Steklov averages we can show the resulting expressions as

$$I_1+I_2+I_3=I_4,$$

where

$$\begin{aligned}
I_1 &= \frac{d}{dt}\left\{\int_{Q_2}\frac{|u-h|^2}{2}dxdt\right\}=\int_{B_2}\frac{|u(x,2^p)-h(x,2^p)|^2}{2}dx\geqslant 0,\\
I_2 &= \int_{Q_2}\left(|Du|^{p-2}Du-|Dh|^{p-2}Dh\right)\cdot D(u-h)\,dxdt,\\
I_3 &= \int_{Q_2}\left(|u|^{p-2}u-|h|^{p-2}h\right)\cdot(u-h)\,dxdt,\\
I_4 &= -\int_{Q_2}|\mathbf{f}|^{p-2}\mathbf{f}\cdot D(u-h)\,dxdt.
\end{aligned}$$

From the estimates of I_1-I_4, we see that

$$\begin{aligned}&\int_{Q_2} |Dh|^p + |h|^p \, dxdt \\ &\leqslant C \int_{Q_2} |Du|^p + |u|^p + |u|^{p-1}|h| + |h|^{p-1}|u| \\ &\quad + |Du|^{p-1}|Dh| + |Dh|^{p-1}|Du| + |\mathbf{f}|^{p-1}|Dh| + |\mathbf{f}|^{p-1}|Du| \, dxdt,\end{aligned}$$

which implies that

$$\int_{Q_2} |Dh|^p + |h|^p \, dxdt \leqslant C \int_{Q_2} |u|^p + |Du|^p + |\mathbf{f}|^p \, dxdt \leqslant C \tag{18}$$

by Young's inequality with $\varepsilon > 0$ and (14).

Estimates of I_2–I_3. From the elementary inequality

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \geqslant C(p)|\xi - \eta|^p$$

for $p \geqslant 2$ and every $\xi, \eta \in \mathbb{R}^n$, we have

$$I_2 \geqslant C \int_{Q_2} |D(u-h)|^p \, dxdt.$$

Similarly, we have

$$I_3 \geqslant C \int_{Q_2} |u-h|^p \, dxdt.$$

Estimate of I_4. Now we apply Young's inequality to conclude that

$$I_4 \leqslant \tau \int_{Q_2} |D(u-h)|^p \, dxdt + C(\tau) \int_{Q_2} |\mathbf{f}|^p \, dxdt.$$

Finally, we combine all the estimates of I_i $(1 \leqslant i \leqslant 4)$ and choose $\tau > 0$ small enough to see that

$$⨍_{Q_2} |D(u-h)|^p + |u-h|^p \, dxdt \leqslant C ⨍_{Q_2} |\mathbf{f}|^p \, dxdt \leqslant C\delta \leqslant \varepsilon,$$

where we have used (14) and selected δ small enough satisfying the last inequality. Furthermore, we use the above inequality (18) and Theorem 2.1 in [26] to obtain that

$$\sup_{Q_1} |h|^p \leqslant C. \tag{19}$$

Moreover, in view of (19), from the local boundedness of $|Dh|$ (see §5 and §6 in Chapter 8 of [15]) we conclude that

$$\sup_{Q_1} |Dh|^p \leqslant C.$$

Therefore, we finish the proof of this lemma. □

Now we shall finish the proof of the main result: Theorem 1. Here we use an approximation argument (see [1, 12]) to show that the proof of Theorem 1 can be reduced to proving an a priori estimate (4) with the assumption that $|u|^p + |Du|^p \in L^{\phi}_{loc}(\Omega_T)$.

Proof. Fix any $i \in \mathbb{N}$. In view of Lemma 2, for some $\lambda \geqslant \lambda_*$ we can construct the disjoint family of parabolic cylinders $\{Q_i^0\}_{i\in\mathbb{N}}$ satisfying

$$\fint_{Q_i^j} |u|^p + |Du|^p \, dxdt \leqslant \lambda^p \quad \text{and} \quad \fint_{Q_i^j} |\mathbf{f}|^p \, dxdt \leqslant \delta\lambda^p, \tag{20}$$

where $Q_i^j = Q_{z_i}(\lambda^{2-p}(5j\rho_i)^p, 5j\rho_i)$ for $j = 1,2$. Now we rescale by defining

$$\begin{cases} u_\lambda(x,t) = \dfrac{u\left(x, \lambda^{2-p}t\right)}{\lambda}, \\ \mathbf{f}_\lambda(x,t) = \dfrac{\mathbf{f}\left(x, \lambda^{2-p}t\right)}{\lambda}. \end{cases} \tag{21}$$

Then u_λ is still a local weak solution of

$$(u_\lambda)_t - \operatorname{div}\left(|Du_\lambda|^{p-2} Du_\lambda\right) + |u_\lambda|^{p-2} u_\lambda = \operatorname{div}\left(|\mathbf{f}_\lambda|^{p-2}\mathbf{f}_\lambda\right) \quad \text{in } \Omega_T.$$

Furthermore, (20) implies that

$$\fint_{Q_{z_i}((10\rho_i)^p, 10\rho_i)} |u_\lambda|^p + |Du_\lambda|^p dxdt \leqslant 1$$

and

$$\fint_{Q_{z_i}((10\rho_i)^p, 10\rho_i)} |\mathbf{f}_\lambda|^p dxdt \leqslant \delta.$$

Then according to Lemma 3, we find that

$$\sup_{Q_{z_i}((5\rho_i)^p, 5\rho_i)} |Dh|^p + |h|^p \leqslant N_1^p$$

and

$$\fint_{Q_{z_i}((10\rho_i)^p, 10\rho_i)} |u_\lambda - h|^p + |D(u_\lambda - h)|^p dxdt \leqslant \varepsilon,$$

where h is the weak solution of

$$\begin{cases} h_t - \operatorname{div}\left(|Dh|^{p-2} Dh\right) + |h|^{p-2} h = 0 & \text{in } Q_{z_i}((10\rho_i)^p, 10\rho_i), \\ h = u_\lambda & \text{on } \partial_p Q_{z_i}((10\rho_i)^p, 10\rho_i). \end{cases}$$

Now we define h by

$$h(x,t) = h_\lambda\left(x, \lambda^{2-p}t\right) \quad z = (x,t) \in Q_{z_i}((10\rho_i)^p, 10\rho_i).$$

Then by changing variables, we find that

$$\sup_{Q_i^1} |Dh_\lambda|^p + |h_\lambda|^p \leqslant N_1^p \tag{22}$$

and

$$⨍_{Q_i^2} |u/\lambda - h_\lambda|^p + |D(u/\lambda - h_\lambda)|^p dxdt \leqslant \varepsilon. \tag{23}$$

For $\lambda \geqslant \lambda_*$, we deduce from (23) and Lemma 2 that

$$\begin{aligned}
&\left|\left\{z \in Q_i^1 : \ |u|^p + |Du|^p > 2^p N_1^p \lambda^p\right\}\right| \\
&= \left|\left\{z \in Q_i^1 : \ |u/\lambda|^p + |Du/\lambda|^p > 2^p N_1^p\right\}\right| \\
&\leqslant \left|\left\{z \in Q_i^1 : |u/\lambda - h_\lambda|^p + |D(u/\lambda - h_\lambda)|^p > N_1^p\right\}\right| \\
&\quad + \left|\left\{z \in Q_i^1 : |Dh_\lambda|^p + |h_\lambda|^p > N_1^p\right\}\right| \\
&= \left|\left\{z \in Q_i^1 : |u/\lambda - h_\lambda|^p + |D(u/\lambda - h_\lambda)|^p > N_1^p\right\}\right| \\
&\leqslant \frac{1}{N_1^p} \int_{Q_i^1} |u/\lambda - h_\lambda|^p + |D(u/\lambda - h_\lambda)|^p dxdt \\
&\leqslant C\varepsilon \left|Q_i^0\right|,
\end{aligned} \tag{24}$$

where we have used the elementary inequality

$$(a+b)^p \leqslant 2^{p-1}(a^p + b^p) \quad \text{for any} \quad a, b > 0.$$

Furthermore, by using (10)–(11) and the above inequality (24), for $\mu := \lambda^p \geqslant \lambda_*^p$ we have

$$\begin{aligned}
&\left|\left\{z \in Q_1 : \ |u|^p + |Du|^p > 2^p N_1^p \mu\right\}\right| \\
&\leqslant \sum_{i=1}^{\infty} \left|\left\{z \in Q_i^1 : \ |u|^p + |Du|^p > 2^p N_1^p \mu\right\}\right| \\
&\leqslant \frac{C\varepsilon}{\mu} \left\{ \int_{\left\{z \in Q_2 : \ |u|^p + |Du|^p > \frac{\mu}{3}\right\}} |u|^p + |Du|^p \, dxdt + \frac{1}{\delta} \int_{\left\{z \in Q_2 : \ |\mathbf{f}|^p > \frac{\delta\mu}{3}\right\}} |\mathbf{f}|^p \, dxdt \right\}.
\end{aligned} \tag{25}$$

Since ϕ is a convex function with $\phi \in \Delta_2 \cap \nabla_2$, we use Lemma 1, (3) and (25) to compute

$$\begin{aligned}
&\int_{Q_1} \phi\left(|u|^p\right) + \phi\left(|Du|^p\right) dxdt \\
&\leqslant C \int_{Q_1} \phi\left(|u|^p + |Du|^p\right) dxdt \\
&\leqslant C \int_0^\infty \left|\left\{z \in Q_1 : |u|^p + |Du|^p > 2^p N_1^p \mu\right\}\right| d\left[\phi\left(2^p N_1 \mu\right)\right] \\
&= C \left\{ \int_0^{\lambda_*^p} + \int_{\lambda_*^p}^\infty \right\} \left|\left\{z \in Q_1 : |u|^p + |Du|^p > 2^p N_1^p \mu\right\}\right| d\left[\phi\left(2^p N_1 \mu\right)\right]
\end{aligned}$$

$$\leqslant C\phi\left(\lambda_*^p\right)+C\varepsilon\int_0^\infty\frac{1}{\mu}\left\{\int_{\left\{z\in Q_2:\,|u|^p+|Du|^p>\frac{\mu}{3}\right\}}|u|^p+|Du|^p\,dxdt\right\}d\left[\phi\left(2^pN_1\mu\right)\right]$$
$$+C\varepsilon\int_0^\infty\frac{1}{\mu}\left\{\int_{\left\{z\in Q_2:\ |\mathbf{f}|^p>\frac{\delta\mu}{3}\right\}}|\mathbf{f}|^p\,dxdt\right\}d\left[\phi\left(2^pN_1\mu\right)\right],$$

which implies that

$$\begin{aligned}&\int_{Q_1}\phi\left(|u|^p\right)+\phi\left(|Du|^p\right)dxdt\\&\leqslant C\phi\left[\left(\int_{Q_2}|u|^p+|Du|^p+|\mathbf{f}|^p+1\,dxdt\right)^{\frac{p}{2}}\right]\\&\qquad+C\int_{Q_2}\phi\left(|\mathbf{f}|^p\right)dxdt+C\varepsilon\int_{Q_2}\phi\left(|u|^p\right)+\phi\left(|Du|^p\right)dxdt.\end{aligned}$$

Finally, by using a covering and iteration argument (see Lemma 2.1, Chapter 3 in [21]) and choosing $\varepsilon>0$ small enough we obtain

$$\begin{aligned}&\int_{Q_1}\phi\left(|u|^p\right)+\phi\left(|Du|^p\right)dxdt\\&\leqslant C\phi\left[\left(\int_{Q_2}|u|^p+|Du|^p+|\mathbf{f}|^p+1\,dxdt\right)^{\frac{p}{2}}\right]+C\int_{Q_2}\phi\left(|\mathbf{f}|^p\right)dxdt,\end{aligned}$$

which completes the proof. □

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