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FRACTIONAL INTEGRAL OPERATOR WITH ROUGH KERNEL ON VARIOUS CLOSED SUBSPACES OF MORREY SPACES

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Abstract. In this paper we investigate fractional integral operators with rough kernel on certain closed subspaces of Morrey spaces. We prove that the operator maps vanishing Morrey spaces into themselves. In addition, we discuss the behavior of this operator on the closure in Morrey spaces of essentially bounded functions, compactly supported functions, and essentially bounded and compactly supported functions.

1. Introduction

Let $1 \leqslant p < \infty$ and $0 \leqslant \lambda < n$. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy$$

is finite. This space is a Banach space equipped with the norm

$$\|f\|_{L^{p,\lambda}} := \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{1/p}.$$

Morrey spaces were introduced by C. B. Morrey in the study of elliptic partial differential equation (see [18]). Recently, there are many results around the boundedness of integral operators on Morrey spaces. These results are related to the Hardy-Littlewood maximal operator, fractional integral operators, and fractional maximal operators. For instance, Liu et al. [16] studied fractional integrals on Morrey-type spaces over Gauss measure spaces. In addition, Liu et al. [17] study fractional Laplace equations via characterizing the Morrey spaces as well as their preduals by quadratic functions related to the Taylor remainder of the kernel of the Riesz potential. In addition, some related boundedness of the fractional integrals on Riesz-Morrey spaces can be found in [14] and these spaces were introduced in [15, 21, 25].

In this paper, we investigate fractional integral operators with rough kernel. Let us recall some definitions and notation. Let $0 < \alpha < n$ and $\Omega : \mathbb{R}^n \to \mathbb{R}$ with the property

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$\Omega(kx) = \Omega(x)$ for every $k > 0$ and $x \in \mathbb{R}^n$. The fractional integral operator with rough kernel $T_{\Omega,\alpha}$ is defined by

$$T_{\Omega,\alpha} f(x) := \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y)\, dy$$

for every suitable function f. It is known in [20, 12] that the operator is bounded from $L^{p,\lambda}$ to $L^{q,\mu}$ whenever $0 \leqslant \lambda \leqslant \mu < n$, $1 < p < \frac{n-\lambda}{\alpha}$, $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$, and $\Omega \in L^s(\mathbb{S}^{n-1})$, $s \geqslant p'$. Here, $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ and $s' := \frac{s}{s-1}$. We will refine this boundedness result by investigating the operator acting on certain closed subspaces of Morrey spaces. First, let us recall the definiton of vanishing Morrey spaces following the notation in [1, 4].

DEFINITION 1. Let $x \in \mathbb{R}^n$, $r > 0$, $1 \leqslant p < \infty$, and $0 \leqslant \lambda \leqslant n$. For every function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, let us set

$$\mathfrak{M}_{p,\lambda}(f;x,r) := \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, dy \right)^{1/p}.$$

We define the vanishing Morrey spaces $V_0 L^{p,\lambda}$, $V_\infty L^{p,\lambda}$, and $V^{(*)} L^{p,\lambda}$ by

$$V_0 L^{p,\lambda} := \left\{ f \in L^{p,\lambda} : \lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\lambda}(f;x,r) = 0 \right\},$$

$$V_\infty L^{p,\lambda} := \left\{ f \in L^{p,\lambda} : \lim_{r \to \infty} \sup_{x \in \mathbb{R}^n} \mathfrak{M}_{p,\lambda}(f;x,r) = 0 \right\},$$

and

$$V^{(*)} L^{p,\lambda} := \left\{ f \in L^{p,\lambda} : \lim_{N \to \infty} \sup_{x \in \mathbb{R}^n} \int_{B(x,1)} |f(y)|^p \chi_{\mathbb{R}^n \setminus B(0,N)}(y) \, dy = 0 \right\}.$$

In addition to vanishing Morrey spaces, we discuss the following closed subspaces of Morrey spaces.

DEFINITION 2. [5, 19, 24] Let $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant n$.

1. The space $\overline{L^{p,\lambda}}$ is defined to be the closure with respect to Morrey norm of the set of essentially bounded functions in $L^{p,\lambda}$.

2. We denote by $\overset{*}{L}{}^{p,\lambda}$ the closure with respect to Morrey norm of the set of compactly supported functions in $L^{p,\lambda}$.

3. $\widetilde{L^{p,\lambda}} := \overline{L^{p,\lambda}} \cap \overset{*}{L}{}^{p,\lambda}$.

These subspaces were investigated in the research related to the predual of Morrey spaces (see [13, 19, 26]) and also complex interpolation of Morrey spaces (see [7, 8, 9, 10, 11, 23, 24]).

Our main result is that the operator $T_{\Omega,\alpha}$ maps each subspace in Definitions 1 and 2 into the same type of subspaces of Morrey spaces. More precisely, we have the following theorems as our main results.

THEOREM 1. *Let* $0 \leqslant \lambda \leqslant \mu < n$, $1 < p < \frac{n-\lambda}{\alpha}$, *and* $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$. *Assume that* $\Omega \in L^s(\mathbb{S}^{n-1})$, $s \geqslant p'$. *Then*

(i) $T_{\Omega,\alpha}(V_0L^{p,\lambda}) \subseteq V_0L^{q,\mu}$;

(ii) $T_{\Omega,\alpha}(V_\infty L^{p,\lambda}) \subseteq V_\infty L^{q,\mu}$;

(iii) $T_{\Omega,\alpha}(V^{(*)}L^{p,\lambda}) \subseteq V^{(*)}L^{q,\mu}$.

REMARK 1.

1. We note that for the case of $\mu = \frac{\lambda q}{p}$, Theorem 1 point (i) is already obtained in [6, Corollary 5].
2. Related results on the (classical) fractional integral operators on vanishing Morrey spaces can be found in [1]. An extension of this results into generalized Morrey spaces can be seen in [2]. In addition, the preservation of certain vanishing properties of commutators of fractional operators is investigated in [3]. Although the idea of the proof of Theorem 1 is adapted from [1, 3], our proof includes the estimates related to the non-constant rough kernel Ω.

Our result about the operator $T_{\Omega,\alpha}$ on closed subspaces of Morrey spaces in Definition 2 is given in the following theorem.

THEOREM 2. *Let* $0 \leqslant \lambda \leqslant \mu < n$, $1 < p < \frac{n-\lambda}{\alpha}$, *and* $\frac{n-\mu}{q} = \frac{n-\lambda}{p} - \alpha$. *Assume that* $\Omega \in L^s(\mathbb{S}^{n-1})$, $s \geqslant p'$. *Then*

(i) $T_{\Omega,\alpha}(\overline{L^{p,\lambda}}) \subseteq \overline{L^{q,\mu}}$.

(ii) $T_{\Omega,\alpha}(\overset{*}{L}{}^{p,\lambda}) \subseteq \overset{*}{L}{}^{q,\mu}$.

(iii) $T_{\Omega,\alpha}(\widetilde{L^{p,\lambda}}) \subseteq \widetilde{L^{q,\mu}}$.

REMARK 2. Note that Theorem 2 point (iii) is an immediate result from Theorem 2 point (i) and (ii).

We also remark that there are recent research on the boundedness of fractional integral operators on the congruent Riesz-Morrey spaces [22] and related results on special John-Nirenberg-Campanato spaces in [15].

2. Preliminaries

2.1. On vanishing Morrey spaces and characterization of subspaces in Definition 2

In this section, we take a closer view to the relation among subspaces (of the Morrey spaces). For instance, intersection of all those subspaces is non-empty, $\chi_{B(0,1)}$ being inside that intersection.

Let ε, φ be non-negative, and define

$$f_{\varepsilon,\varphi}(x) = \begin{cases} |x|^{\frac{\lambda-n}{p}+\varepsilon} & ,\text{for}|x| \leqslant 1 \\ |x|^{\frac{\lambda-n}{p}-\varphi} & ,\text{for}|x| \geqslant 1. \end{cases}$$

For $\varepsilon > 0$ and $\varphi = 0$, we have $f_{\varepsilon,0} \in V_0 L^{p,\lambda}$ but $f_{\varepsilon,0} \notin V_\infty L^{p,\lambda}$. For $\varepsilon = 0$ and $\varphi > 0$, we have $f_{0,\varphi} \notin V_0 L^{p,\lambda}$ but $f_{0,\varphi} \in V_\infty L^{p,\lambda}$.

We now recall the following characterization of $\overline{L^{p,\lambda}}$, $\overset{*}{L}{}^{p,\lambda}$, and $\widetilde{L^{p,\lambda}}$. In a way then, the characterization is more explicit.

LEMMA 1. [7, 8, 10] *Let* $1 \leqslant p < \infty$ *and* $0 < \lambda < n$. *Then*

$$\overline{L^{p,\lambda}} = \left\{ f \in L^{p,\lambda} : \lim_{R\to\infty} \|f\chi_{\{|f|>R\}}\|_{L^{p,\lambda}} = 0 \right\},$$

$$\overset{*}{L}{}^{p,\lambda} = \left\{ f \in L^{p,\lambda} : \lim_{R\to\infty} \|f\chi_{\mathbb{R}^n\setminus B(0,R)}\|_{L^{p,\lambda}} = 0 \right\},$$

and

$$\widetilde{L^{p,\lambda}} = \left\{ f \in L^{p,\lambda} : \lim_{R\to\infty} \|f\chi_{\{|f|>R\}\cup(\mathbb{R}^n\setminus B(0,R))}\|_{L^{p,\lambda}} = 0 \right\}.$$

According to Lemma 1, we can verify that $\widetilde{L^{p,\lambda}}$ is a proper subset of $\overset{*}{L}{}^{p,\lambda}$ and of $\overline{L^{p,\lambda}}$. In fact, $f(x) := |x|^{\frac{\lambda-n}{p}} \chi_{\{x:|x|>1\}} \in \overline{L^{p,\lambda}} \setminus \widetilde{L^{p,\lambda}}$ and $g(x) := |x|^{\frac{\lambda-n}{p}} \chi_{\{x:|x|\leqslant 1\}} \in \overset{*}{L}{}^{p,\lambda} \setminus \widetilde{L^{p,\lambda}}$. In addition, we have $f \notin \overset{*}{L}{}^{p,\lambda}$ and $g \notin \overline{L^{p,\lambda}}$. Moreover, $\overset{*}{L}{}^{p,\lambda}$ and $\overline{L^{p,\lambda}}$ are proper subsets of $L^{p,\lambda}$ because $h(x) := |x|^{\frac{\lambda-n}{p}}$ belongs to $L^{p,\lambda}$ but it is not a member of both $\overset{*}{L}{}^{p,\lambda}$ and $\overline{L^{p,\lambda}}$.

By Lemma 1, we have the following inclusion relation.

LEMMA 2. *Let* $1 \leqslant p < \infty$ *and* $0 < \lambda < n$. *Then* $\overset{*}{L}{}^{p,\lambda} \subset V^{(*)} L^{p,\lambda} \bigcap V_\infty L^{p,\lambda}$.

Proof. According to Lemma 1, we immediately have $\overset{*}{L}{}^{p,\lambda} \subset V^{(*)} L^{p,\lambda}$. We now need to verify that $\overset{*}{L}{}^{p,\lambda} \subset V_\infty L^{p,\lambda}$. Let $f \in \overset{*}{L}{}^{p,\lambda}$. Given $\varepsilon > 0$, by Lemma 1 we can find K such that for any $R > K$

$$\|f\chi_{\mathbb{R}^n\setminus B(0,R)}\|_{L^{p,\lambda}} < \frac{\varepsilon}{2}.$$

Fix $R > K$ and choose $R^* > R \left(\frac{2\|f\|_{L^{p,\lambda}}}{\varepsilon} \right)^{\frac{p}{\lambda}}$, then

$$\begin{aligned} \mathfrak{M}_{p,\lambda}(f; z, R^*) &\leqslant \|f\chi_{\mathbb{R}^n\setminus B(0,R)}\|_{L^{p,\lambda}} + \mathfrak{M}_{p,\lambda}(f\chi_{B(0,R)}; z, R^*) \\ &< \frac{\varepsilon}{2} + \left(\frac{R^*}{R}\right)^{-\frac{\lambda}{p}} \|f\|_{L^{p,\lambda}} < \varepsilon. \end{aligned}$$

This shows that $f \in V_\infty L^{p,\lambda}$. $\square$

In fact, $\overset{*}{L}^{p,\lambda}$ is a proper subset of $V^{(*)}L^{p,\lambda}$. For example, in dimension one, the function $f(x) = x^{\frac{\lambda-1}{p}}\chi_{[0,\infty)}$ with $0<\lambda<1$ belongs to $V^{(*)}L^{p,\lambda}$ but it is not in $\overset{*}{L}^{p,\lambda}$. Moreover, the function is neither in $V_\infty L^{p,\lambda}$ nor in $V_0 L^{p,\lambda}$.

2.2. On $T_{\Omega,\alpha}$ and Rough maximal operators

The rough maximal operator M_Ω plays an important role in this article. Let us recall the definition of M_Ω. Let $\Omega : \mathbb{R}^n \to \mathbb{R}$ with the property $\Omega(kx) = \Omega(x)$ for every $k>0$ and $x\in\mathbb{R}^n$. The rough maximal operator M_Ω is defined by

$$M_\Omega f(x) := \sup_{r>0} \frac{1}{r^n}\int_{B(x,r)} |\Omega(x-y)f(y)|dy$$

for any suitable function f. It is well known that M_Ω is bounded on L^p for $1<p\leqslant\infty$ if $\Omega\in L^1(\mathbb{S}^{n-1})$, in the sense

$$\|M_\Omega f\|_{L^p} \lesssim \|\Omega\|_{L^1(\mathbb{S}^{n-1})}\|f\|_{L^p}.$$

By [20, Theorem 3.1], we have a pointwise estimate of $T_{\Omega,\alpha}f$ in term of $M_\Omega f$ as follows.

LEMMA 3. *Let $1<p<\frac{n-\lambda}{\alpha}$ and $f\in L^{p,\lambda}$. Assume that $\Omega\in L^s(\mathbb{S}^{n-1})$ with $s\geqslant p'$. Then, for almost every $x\in\mathbb{R}^n$,*

$$|T_{\Omega,\alpha}f(x)| \lesssim \|\Omega\|^{1-u}_{L^s(\mathbb{S}^{n-1})}\|f\|^{1-u}_{L^{p,\lambda}}(M_\Omega f(x))^u$$

where $u = 1-\frac{\alpha p}{n-\lambda}$.

3. Proof of Theorem 1

3.1. Proof of Theorem 1 (i) and (ii)

Fix $z\in\mathbb{R}^n$ and $r>0$. Let $f=f_1+f_2$, where $f_1 = f\chi_{B(z,2r)}$, and $f_2 = f-f_1$. By the linearity of $T_{\Omega,\alpha}$ and the Minkowski inequality for the Lebesgue norm,

$$\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f;z,r) \leqslant \mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f_1;z,r)+\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f_2;z,r).$$

Let us first work on $T_{\Omega,\alpha}f_1$. By Lemma 3 with $u = 1-\frac{\alpha p}{n-\lambda}$, the Hölder inequality with order p/uq, and the boundedness of M_Ω on Lebesgue spaces, we have

$$\begin{aligned}\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f_1;z,r) &\lesssim r^{-\frac{\mu}{q}}\|\Omega\|^{1-u}_{L^s(\mathbb{S}^{n-1})}\|f\|^{1-u}_{L^{p,\lambda}}\|M_\Omega f_1^u\|_{L^q(B(z,r))}\\ &\leqslant r^{\frac{n-\mu}{q}-\frac{un}{p}}\|\Omega\|^{1-u}_{L^s(\mathbb{S}^{n-1})}\|f\|^{1-u}_{L^{p,\lambda}}\|M_\Omega f_1\|^u_{L^p}\\ &\lesssim r^{\frac{n-\mu}{q}-\frac{un}{p}}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|^{1-u}_{L^{p,\lambda}}\|f\|^u_{L^p(B(z,2r))}.\end{aligned}$$

Since $\|f\|_{L^p(B(z,2r))}$ is increasing in r,

$$\begin{aligned}\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f_1;z,r) &\leqslant r^{\frac{n-\mu}{q}+\frac{\alpha\lambda}{n-\lambda}}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|^{1-u}_{L^{p,\lambda}}\int_{2r}^\infty t^{\alpha-\frac{n}{p}-1}\|f\|^u_{L^p(B(z,t))}dt\\ &\leqslant r^{\frac{n-\mu}{q}}\|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|^{1-u}_{L^{p,\lambda}}\int_r^\infty t^{\frac{\mu-n}{q}-1}(\mathfrak{M}_{p,\lambda}(f;z,t))^u dt. \qquad (1)\end{aligned}$$

Let us now work on $T_{\Omega,\alpha}f_2$. By the Fubini Theorem and the Hölder inequality with order p, for any $x \in B(z,r)$

$$\begin{aligned}|T_{\Omega,\alpha}f_2(x)| &\lesssim \int_{\mathbb{R}^n\setminus B(z,2r)} |\Omega(x-y)||f(y)| \int_{|x-y|}^{\infty} t^{\alpha-n-1}dtdy \\ &\lesssim \int_{2r}^{\infty} t^{\alpha-n-1} \int_{B(x,t)} |\Omega(x-y)||f(y)|dydt \\ &\lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p,\lambda}}^{1-u} \int_r^{\infty} t^{\frac{\mu-n}{q}-1}\mathfrak{M}_{p,\lambda}(f;z,t)^u dt.\end{aligned}$$

Hence,

$$\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f_2;z,r) \lesssim r^{\frac{n-\mu}{q}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p,\lambda}}^{1-u} \int_r^{\infty} t^{\frac{\mu-n}{q}-1}\mathfrak{M}_{p,\lambda}(f;z,t)^u dt. \tag{2}$$

Thus, from (1) and (2), we have

$$\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f;z,r) \lesssim r^{\frac{n-\mu}{q}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p,\lambda}}^{1-u} \int_r^{\infty} t^{\frac{\mu-n}{q}-1}\mathfrak{M}_{p,\lambda}(f;z,t)^u dt. \tag{3}$$

Let $f \in V_0L^{p,\lambda}$. For any $\varepsilon > 0$, we can find $\delta < 1$ such that for any $t < \delta$,

$$\mathfrak{M}_{p,\lambda}(f;z,t)^u \|f\|_{L^{p,\lambda}}^{1-u} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} < \frac{\varepsilon}{2}.$$

Therefore, from (3)

$$\begin{aligned}\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f;z,r) &\lesssim r^{\frac{n-\mu}{q}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p,\lambda}}^{1-u} \int_r^{\delta} t^{\frac{\mu-n}{q}-1}\mathfrak{M}_{p,\lambda}(f;z,t)^u dt \\ &\quad + r^{\frac{n-\mu}{q}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p,\lambda}}^{1-u} \int_{\delta}^{\infty} t^{\frac{\mu-n}{q}-1}\mathfrak{M}_{p,\lambda}(f;z,t)^u dt \\ &\lesssim \frac{\varepsilon}{2} + \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p,\lambda}} (\frac{r}{\delta})^{\frac{n-\mu}{q}}.\end{aligned}$$

Hence, we can choose a small r such that $\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f;z,r) \lesssim \varepsilon$. We conclude that $T_{\Omega,\alpha}f \in V_0L^{p,\lambda}$.

Let $f \in V_{\infty}L^{p,\lambda}$. For any $\varepsilon > 0$, we can find $K > 0$, such that for any $t > K$, we have $\mathfrak{M}_{p,\lambda}(f;z,t)^u \|f\|_{L^{p,\lambda}}^{1-u} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} < \varepsilon$. Therefore, for any $r > K$, we have

$$\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f;z,r) \lesssim r^{\frac{n-\mu}{q}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}\|f\|_{L^{p,\lambda}}^{1-u} \int_r^{\infty} t^{\frac{\mu-n}{q}-1}\mathfrak{M}_{p,\lambda}(f;z,t)^u dt \lesssim \varepsilon.$$

We conclude that $T_{\Omega,\alpha}f \in V_{\infty}L^{q,\mu}$, and this proves part (ii). □

3.2. Proof of Theorem 1 (iii)

For any $f \in L^{p,\lambda}$ and $N \geqslant 0$, let us define

$$\mathscr{A}_{N,p}(f) := \sup_{x\in\mathbb{R}^n} \int_{B(x,1)\cap B^c(0,N)} |f(y)|^p dy$$

where $B^c(0,N) = \mathbb{R}^n \setminus B(0,N)$ for $N \geqslant 0$. It is clear that $f \in V^{(*)}L^{p,\lambda}$ if $\mathscr{A}_{N,p}(f) \to 0$ as $N \to \infty$.

From [1], we note that $M_1(V^{(*)}L^{p,\lambda}) \subset V^{(*)}L^{p,\lambda}$ for $p > 1$ and we extend this result as follows.

THEOREM 3. *For $p > 1$, $s \geqslant p'$, and $\Omega \in L^s(\mathbb{S}^{n-1})$, $M_{\Omega}(V^{(*)}L^{p,\lambda}) \subset V^{(*)}L^{p,\lambda}$.*

Proof. Let $f \in V^{(*)}L^{p,\lambda}$ be not equivalent to zero. For $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$, we decompose f into f_1 and f_2, with $f_1 = f\chi_{B(x,2)\cap B^c(0,N/2)}$ and $f_2 = f - f_1$. Since M_Ω is a sublinear operator,

$$\mathscr{A}_{N,p}(M_\Omega f) \lesssim \mathscr{A}_{N,p}(M_\Omega f_1) + \mathscr{A}_{N,p}(M_\Omega f_2).$$

Let us treat $\mathscr{A}_{N,p}(M_\Omega f_1)$ first. By the boundedness of M_Ω on L^p,

$$\int_{B(x,1)\cap B^c(0,N)} (M_\Omega f_1(y))^p dy \lesssim \|f_1\|_{L^p}^p = \int_{B(x,2)\cap B^c(0,N/2)} |f(y)|^p dy. \tag{4}$$

Note that we can cover $B(x,2)$ with a finite number of open unit balls. Thus,

$$B(x,2) \subset \bigcup_{j=1}^{K_0} B(x_j,1)$$

Hence, from (4), we have

$$\begin{aligned}\int_{B(x,1)\cap B^c(0,N)} (M_\Omega f_1(y))^p dy &\lesssim \sum_{j=1}^{K_0} \int_{B(x_j,1)\cap B^c(0,N/2)} |f(y)|^p dy \\ &\lesssim \mathscr{A}_{N/2,p}(f). \end{aligned}\tag{5}$$

Since the right-hand-side of (5) is independent of x, we conclude that $\mathscr{A}_{N,p}(M_\Omega f_1) \to 0$ as $N \to \infty$.

Let us now handle $\mathscr{A}_{N,p}(M_\Omega f_2)$. For any given $\varepsilon > 0$, we can choose $t_1 > 1$ such that $t^{\lambda-n}\|f\|_{L^{p,\lambda}}^p < \varepsilon$ for all $t \geqslant t_1$. Then,

$$\begin{aligned}&\int_{B(x,1)\cap B^c(0,N)} (M_\Omega f_2(y))^p dy \\ &\quad\lesssim \int_{B(x,1)\cap B^c(0,N)} \left(\sup_{0<t<t_1} t^{-n} \int_{B(y,t)} |\Omega(y-z)||f_2(z)|dz\right)^p dy \\ &\qquad + \int_{B(x,1)\cap B^c(0,N)} \left(\sup_{t\geqslant t_1} t^{-n} \int_{B(y,t)} |\Omega(y-z)||f_2(z)|dz\right)^p dy \\ &\quad= I_1(x,N) + I_2(x,N). \end{aligned}\tag{6}$$

For $I_1(x,N)$, we note that $z \in B(y,t)$ and also $z \in \mathbb{R}^n \setminus (B(x,2)\cap B^c(0,N/2))$. If $z \in B(0,N/2)$, then $t \geqslant |z-y| \geqslant |y| - |z| \geqslant N/2$. Since we can choose $N > 2t_1$, we can ignore the case of $z \in B(0,N/2)$. For $z \in B^c(x,2)$, we have $t \geqslant |z-y| \geqslant |z-x| - |y-x| \geqslant 1$. These results tell us that the supremum part of $I_1(x,N)$ only takes places in $1 < t < t_1$. By the Hölder inequality,

$$\begin{aligned}I_1(x,N) &\lesssim \int_{B(x,1)\cap B^c(0,N)} \left(\sup_{1<t<t_1} t^{-\frac{n}{p}} \left(\int_{B(y,t)} |f(z)|^p dz\right)^{\frac{1}{p}}\right)^p dy \\ &\lesssim \int_{B(x,1)\cap B^c(0,N)} \int_{B(0,t_1)} |f(y-z)|^p dz dy \\ &\lesssim t_1^n \sup_{w\in\mathbb{R}^n} \int_{B(w,1)\cap B^c(0,N-t_1)} |f(y)|^p dy.\end{aligned}$$

Hence, $I_1(x,N) \to 0$ as $N \to \infty$.

For $I_2(x,N)$, we can use the Hölder inequality to obtain

$$I_2(x,N) \lesssim \int_{B(x,1)} \left(\sup_{t \geqslant t_1} t^{-\frac{n}{p}} \left(\int_{B(y,t)} |f(z)|^p dz \right)^{\frac{1}{p}} \right)^p dy$$
$$\lesssim \int_{B(x,1)} \left(\sup_{t \geqslant t_1} t^{\frac{\lambda - n}{p}} \|f\|_{L^{p,\lambda}} \right)^p dy \lesssim \varepsilon.$$

Therefore, by (6), we conclude that $\mathscr{A}_{N,p}(M_\Omega f_2) \to 0$ as $N \to \infty$. This completes the proof of Theorem 3. $\square$

By Lemma 3 with $u = 1 - \frac{\alpha p}{n - \lambda}$, and the Hölder inequality with order p/uq, we have

$$\mathscr{A}_{N,q}(T_\Omega f) \lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^{q-qu} \|f\|_{L^{p,\lambda}}^{q-qu} \mathscr{A}_{N,uq}(M_\Omega f)$$
$$\lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^{q-qu} \|f\|_{L^{p,\lambda}}^{q-qu} (\mathscr{A}_{N,p}(M_\Omega f))^{\frac{uq}{p}}.$$

Followed by Theorem 3, the proof of Theorem 1 (iii) is now complete. $\square$

4. Proof of Theorem 2

4.1. Proof of Theorem 2 point (i)

Let $f \in \overline{L^{p,\lambda}}$. We will prove that $T_{\Omega,\alpha} f \in \overline{L^{q,\mu}}$. According to Lemma 1, it suffices to show that

$$\lim_{R \to \infty} \|\chi_{\{|T_{\Omega,\alpha} f| > R\}} T_{\Omega,\alpha} f\|_{L^{q,\mu}} = 0. \tag{7}$$

Let $0 < R_1 < R < \infty$. Note that,

$$\|\chi_{\{|T_{\Omega,\alpha} f| > R\}} T_{\Omega,\alpha} f\|_{L^{q,\mu}} \leqslant \|\chi_{\{|T_{\Omega,\alpha} f| > R\}} T_{\Omega,\alpha}(f\chi_{|f| \leqslant R_1})\|_{L^{q,\mu}}$$
$$+ \|T_{\Omega,\alpha}(f\chi_{|f| > R_1})\|_{L^{q,\mu}}. \tag{8}$$

According to Lemma 3 and the boundedness of M_Ω on L^∞, we have

$$\|\chi_{\{|T_{\Omega,\alpha} f| > R\}} T_{\Omega,\alpha}(f\chi_{|f| \leqslant R_1})\|_{L^{q,\mu}}$$
$$\lesssim \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^{1-u} \|f\|_{L^{p,\lambda}}^{1-u} \|\chi_{\{|T_{\Omega,\alpha} f| > R\}} M_\Omega(f\chi_{|f| \leqslant R_1})^u\|_{L^{q,\mu}}$$
$$\lesssim R_1^u \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{L^{p,\lambda}}^{1-u} \|\chi_{\{|T_{\Omega,\alpha} f| > R\}}\|_{L^{q,\mu}}. \tag{9}$$

The inequality (9) and the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ and $L^{q,\mu}$ yield

$$\|\chi_{\{|T_{\Omega,\alpha} f| > R\}} T_{\Omega,\alpha}(f\chi_{|f| \leqslant R_1})\|_{L^{q,\mu}} \lesssim \frac{R_1^u}{R} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{L^{p,\lambda}}^{1-u} \|\chi_{\{|T_{\Omega,\alpha} f| > R\}} R\|_{L^{q,\mu}}$$
$$\leqslant \frac{R_1^u}{R} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|f\|_{L^{p,\lambda}}^{1-u} \|T_{\Omega,\alpha} f\|_{L^{q,\mu}}$$
$$\lesssim \frac{R_1^u}{R} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}^2 \|f\|_{L^{p,\lambda}}^{2-u}. \tag{10}$$

Using the boundedness of $T_{\Omega,\alpha}$ from $L^{p,\lambda}$ and $L^{q,\mu}$ once again, we get

$$\|T_{\Omega,\alpha}(f\chi_{\{|f| > R_1\}})\|_{L^{q,\mu}} \lesssim \|f\chi_{\{|f| > R_1\}}\|_{L^{p,\lambda}} \tag{11}$$

Combining the inequalities (8), (10), and (11), we obtain

$$\|\chi_{\{|T_{\Omega,\alpha}f|>R\}}T_{\Omega,\alpha}f\|_{L^{q,\mu}} \lesssim \frac{R_1^u}{R}\|\Omega\|^2_{L^s(\mathbb{S}^{n-1})}\|f\|^{2-u}_{L^{p,\lambda}} + \|f\chi_{\{|f|>R_1\}}\|_{L^{p,\lambda}}.$$

Taking $R\to\infty$, we get

$$\limsup_{R\to\infty}\|\chi_{\{|T_{\Omega,\alpha}f|>R\}}T_{\Omega,\alpha}f\|_{L^{q,\mu}} \lesssim \|f\chi_{\{|f|>R_1\}}\|_{L^{p,\lambda}}. \tag{12}$$

By $f\in\overline{L^{p,\lambda}}$ and Lemma 1, we have $\lim\limits_{R_1\to\infty}\|f\chi_{\{|f|>R_1\}}\|_{L^{p,\lambda}}=0$. Thus, (7) follows from this limit and the inequality (12). □

4.2. Proof of Theorem 2 point (ii)

Let $f\in\overset{*}{L}{}^{p,\lambda}$, then by Lemma 2 $f\in V_\infty L^{p,\lambda}$. According to Theorem 1 point (ii), $T_{\Omega,\alpha}f\in V_\infty L^{q,\mu}$. Given $\varepsilon>0$, we can find K such that for any $r>K$ and $x\in\mathbb{R}^n$

$$\mathfrak{M}_{q,\mu}(T_{\Omega,\alpha}f;x,r)<\varepsilon.$$

Hence,

$$\|\chi_{B^c(0,R)}T_{\Omega,\alpha}f\|_{L^{q,\mu}} < \sup_{x\in\mathbb{R}^n,r\leqslant K}\mathfrak{M}_{q,\mu}(\chi_{B^c(0,R)}T_{\Omega,\alpha}f;x,r)+\varepsilon.$$

Let $f_1=f\chi_{B(0,\frac{R}{2})}$ and $f_2=f-f_1$. By the linearity of $T_{\Omega,\alpha}$, we now only need to show that

$$\sup_{x\in\mathbb{R}^n,r\leqslant K}\mathfrak{M}_{q,\mu}(\chi_{B^c(0,R)}T_{\Omega,\alpha}f_i;x,r)<\varepsilon$$

for both $i=1$ and $i=2$.

Let us now work on $i=1$. For $z\in B^c(0,R)$ and $y\in B(0,\frac{R}{2})$, we have $|z-y|>\frac{R}{2}$. Therefore, for $z\in B^c(0,R)$, by the Hölder inequality, we have

$$|T_{\Omega,\alpha}f_1(z)| \lesssim R^{\alpha-n}\int_{B(0,\frac{R}{2})}|\Omega(z-y)||f(y)|dy \lesssim R^{\alpha-\frac{n-\lambda}{p}}\|f\|_{L^{p,\lambda}}.$$

Hence, for sufficiently large enough R, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^n,r\leqslant K}\mathfrak{M}_{q,\mu}(\chi_{B^c(0,R)}T_{\Omega,\alpha}f_1;x,r) &\lesssim R^{\alpha-\frac{n-\lambda}{p}}\|f\|_{L^{p,\lambda}}\sup_{r\leqslant K}r^{\frac{n-\mu}{q}}\\ &\leqslant K^{\frac{n-\mu}{q}}R^{\alpha-\frac{n-\lambda}{p}}\|f\|_{L^{p,\lambda}}<\varepsilon.\end{aligned}$$

For $i=2$, by boundedness of $T_{\Omega,\alpha}:L^{p,\lambda}\to L^{q,\mu}$, and the fact $f\in\overset{*}{L}{}^{p,\lambda}$, we can find large enough R, such that

$$\sup_{x\in\mathbb{R}^n,r\leqslant K}\mathfrak{M}_{q,\mu}(\chi_{B^c(0,R)}T_{\Omega,\alpha}f_2;x,r)\leqslant\|T_{\Omega,\alpha}f_2\|_{L^{q,\mu}}\lesssim\|f\chi_{B^c(0,\frac{R}{2})}\|_{L^{p,\lambda}}<\varepsilon.$$

This proves Theorem 2 (ii). □

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