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AN APPLICATION OF THE WEIGHTED DISCRETE HARDY INEQUALITY

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Abstract. In this work, we use a well-known characterization of a weighted version of the classical discrete Hardy inequality to exhibit a sufficient condition for the existence of solutions of the differential equation $\operatorname{div}\mathbf{u} = f$ in weighted Sobolev spaces over a certain irregular planar domain. The solvability of this equation is fundamental for the analysis of the Stokes equations.

The proof follows from a local-to-global argument based on a certain decomposition of functions which is also of interest for its applications to other inequalities or related results in Sobolev spaces, such as the Korn inequality.

1. Introduction

Given $p > 1$, *the discrete Hardy inequality* states

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p, \tag{1}$$

for any non-negative sequence $\{a_n\}_{n\geqslant 1}$, where the constant in the inequality $(p/(p-1))^p$ is optimal. This inequality has been widely studied and many generalizations have been shown. For details concerning the prehistory and history of the Hardy inequality we refer the reader to [10] and [11] respectively. In this article, we use a weighted version of the classical discrete Hardy inequality which says:

$$\left(\sum_{n=1}^{\infty} u_n \left(\sum_{k=1}^{n} a_k\right)^p\right)^{1/p} \leqslant C \left(\sum_{n=1}^{\infty} v_n a_n^p\right)^{1/p}. \tag{2}$$

The existence of a constant C, that makes inequality (2) valid for any non-negative sequence $\{a_n\}_{n\geqslant 1}$, depends only on p and the sequence weights $\{u_n\}_{n\geqslant 1}$ and $\{v_n\}_{n\geqslant 1}$.

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There are several characterizations of the sequence weights in the previous inequality, but the one used in these notes states that the constant C in (2) exists if and only if

$$A = \sup_{k \geqslant 1} \left(\sum_{i=k}^{\infty} u_i \right)^{1/p} \left(\sum_{i=1}^{k} v_i^{1-q} \right)^{1/q} < \infty,$$

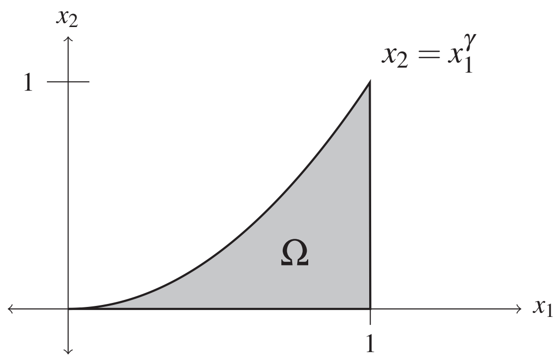
where $q = p/(p-1)$. We refer the reader to [4], [11, Chapter 6], [12, page 56] and [16] for details about this and other characterizations. The existence of a characterization for the sequence weights in (2) is key to prove our main result on the solvability of the divergence equation in weighted Sobolev spaces. We deal with the existence of weighted Sobolev solutions of the equation $\operatorname{div} \mathbf{u} = f$ for weights $v_1(x), v_2(x) : \Omega \to \mathbb{R}_{>0}$, where Ω is the planar domain

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1 \text{ and } 0 < x_2 < x_1^{\gamma}\}, \tag{3}$$

for $\gamma \geqslant 1$. Specifically, we are looking for sufficient conditions on the weights $v_1(x)$ and $v_2(x)$ such that, for any $f \in L^2(\Omega, v_2(x))$ with vanishing mean value, there exists a solution $\mathbf{u}$ of $\operatorname{div} \mathbf{u} = f$ in the Sobolev spaces $H_0^1(\Omega, v_1(x))^2 := \overline{C_0^{\infty}(\Omega)^2}$ with the following estimate

$$\int_{\Omega} |D\mathbf{u}(x)|^2 v_1(x)\, \mathrm{d}x \leqslant C^2 \int_{\Omega} |f(x)|^2 v_2(x)\, \mathrm{d}x, \tag{4}$$

where $D\mathbf{u}(x)$ denotes the differential matrix of $\mathbf{u}$. The weights considered here satisfy that $v_1(x) = x_1^{2(\gamma-1)} v_2(x)$ and v_1, v_2 depend only on the first component of x (i.e. $v_1(x) = v_1(x_1)$ and $v_2(x) = v_2(x_1)$). Notice that if $\gamma > 1$, the domain Ω has a singularity (cusp) at the origin, while the domain is regular (convex) if $\gamma = 1$. The factor $x_1^{2(\gamma-1)}$ in the definition of $v_1(x)$ is there to deal with the singularity at the origin and disappears when Ω is regular ($\gamma = 1$), in which case we have the same weights in both sides of the estimate (4). The exponent in the factor $x_1^{2(\gamma-1)}$ is optimal in the following sense: if $v_2(x) = 1$ and $v_1(x) = x_1^a$, with $a < 2(\gamma - 1)$, the solvability of $\operatorname{div} \mathbf{u} = f$ with estimate (4) fails in general (we refer to [2] for counterexamples).



The solvability of the divergence equation is fundamental for the variational analysis of the Stokes equations and strongly depends on the geometry of the domain, which has been studied on Lipschitz domains, star-shaped domains with respect to a ball, John domains, Hölder-α domains, among others. We refer to [1] and references therein for an extensive description of the solvability of this equation on domains under several geometric conditions. The domain Ω of our interest and defined in (3) was already considered in [6, 13]. The authors in [6] use the Piola transform of an explicit solution on a regular domain whose analysis required the use of the theory of singular integral operators and Muckenhoupt weights. In [13], the author uses a technique similar to the one treated in this article, where the discrete weighted Hardy inequality (2) is replaced by a Hardy-type operator on weighted $L^p(\Omega)$ spaces. The reason to work with (2) instead of the Hardy-type operator defined in [13] relies on the simplicity of the discrete inequality and the characterization of the weights for which the inequality remains valid.

Now, in order to prove our main results, we decompose Ω into a collection of infinitely many regular (star-shaped with respect to a ball) subdomains $\{\Omega_i\}_{i\geqslant 0}$ where the weights can be assumed to be constant. In that case the solvability of the divergence equation has been proved. Then, we extend by zero the solutions in Ω_i to the whole domain and add them up to obtain a solution in Ω. Inequality (2) appears when we estimate the norm of the "global solution" in terms of the estimation of the "local solutions". The decomposition $\{\Omega_i\}_{i\geqslant 0}$ of Ω mentioned above is:

$$\Omega_i := \{(x_1,x_2)\in\Omega \,:\, 2^{-(i+2)} < x_1 < 2^{-i}\}. \tag{5}$$

This is the main result of the paper.

THEOREM 1. *Let $\omega:\Omega\to\mathbb{R}$ be an admissible weight in the sense of Definition 1, for $p=2$, such that the following weighted Hardy inequality is valid for any non-negative sequence $\{d_n\}_{n\geqslant 1}$:*

$$\sum_{j=1}^{\infty} u_i\left(\sum_{i=1}^{j} d_i\right)^2 \leqslant C_H^2 \sum_{j=1}^{\infty} u_j d_j^2,$$

where

$$u_i := |\Omega_i|\,\omega^2(2^{-i}).$$

Then, there exists a constant C such that for any f in $L^2(\Omega,\omega^{-2}(x_1))$, with vanishing mean value, there exists a solution $\mathbf{u}:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ of the equation $\operatorname{div}\mathbf{u}=f$ *in $H_0^1(\Omega,x_1^{2(\gamma-1)}\omega^{-2}(x_1))^2$ such that*

$$\int_\Omega |D\mathbf{u}(x)|^2 x_1^{2(\gamma-1)}\omega^{-2}(x_1)\,\mathrm{d}x \leqslant C^2\int_\Omega |f(x)|^2\omega^{-2}(x_1)\,\mathrm{d}x.$$

Moreover,

$$C^2 \leqslant \gamma^2 2^{12+4\gamma} C_\omega^8 C_H^2,$$

where C_ω appears in Definition 1.

REMARK 1. The strong connection between the solvability of the equation $\operatorname{div}\mathbf{u} = f$ and the validity of the Korn inequality in the second case is well-known (see [7, 9, 1]). Thus, it is worth observing that in [3] the authors use the weighted discrete Hardy inequality (2) to prove the validity of the Korn inequality on domains with a single singularity on the boundary by using a different local-to-global argument.

The following result considers the case where the weights are power functions.

COROLLARY 1. (Power weights) *Let $\Omega \subset \mathbb{R}^2$ be the domain defined in (3) and $\beta > \frac{-\gamma-1}{2}$. Then, there exists a positive constant C such that for any $f \in L^2(\Omega, \omega(x_1)^{-2})$, with $\int_\Omega f = 0$, there exists a solution $\mathbf{u} \in H_0^1\left(\Omega, x_1^{2(\gamma-1)}\omega(x_1)^{-2}\right)^2$ of $\operatorname{div}\mathbf{u} = f$ that satisfies*

$$\int_\Omega |D\mathbf{u}(x)|^2 x_1^{2(\gamma-1)}\omega(x_1)^{-2}\,\mathrm{d}x \leqslant C^2 \int_\Omega |f(x)|^2 \omega(x_1)^{-2}\,\mathrm{d}x, \tag{6}$$

where $\omega(x_1) := x_1^\beta$. Moreover, if $\beta \leqslant 0$, the constant C in (6) satisfies the following estimate:

$$C \leqslant \frac{M}{1-2^{-2\left(\beta+\frac{\gamma+1}{2}\right)}},$$

where the constant M is independent of β.

Notice that the distance from (x_1,x_2) in Ω to the origin is comparable to x_1, thus the weights here can be understood as powers of the distance to the origin or the cusp if $\gamma > 1$. Indeed,

$$x_1 \leqslant \sqrt{x_1^2+x_2^2} \leqslant \sqrt{2}x_1.$$

for all $(x_1,x_2) \in \Omega$.

The existence of a solution of the divergence equation in this planar domain Ω with the estimate (6) was first obtained in [6, Theorem 4.1] for β in $\left(\frac{-\gamma-1}{2}, \frac{3\gamma-1}{2}\right)$, and later in [13, Theorem 5.1] for $\beta \geqslant 0$. In this case, we recover both results as a corollary of our main theorem. In addition, an estimate of the constant that bounds its blow-up as β tends to $\frac{-\gamma-1}{2}$ is exhibited. Finally, notice that if $\beta \leqslant \frac{-\gamma-1}{2}$ then $L^2(\Omega, x_1^{-2\beta}) \not\subset L^1(\Omega)$ and the vanishing mean value condition in the divergence problem is not well-defined. Hence, the condition $\beta > \frac{-\gamma-1}{2}$ is optimal for the current setting. For an example of a non-integrable function in $L^2(\Omega, x_1^{-2\beta})$, when $\beta \leqslant \frac{-\gamma-1}{2}$, one can consider $f(x) = (1-\ln(x_1))^{-1}x_1^{-\gamma-1}$.

The following result considers the case where the weights are powers of a logarithmic function.

COROLLARY 2. (Powers of logarithmic weights) *Let $\Omega \subset \mathbb{R}^2$ be the domain defined in (3) and $\alpha \in \mathbb{R}$. Then, there exists a positive constant C such that for any $f \in$*

$L^2(\Omega,\omega(x_1)^{-2})$, *with* $\int_\Omega f=0$, *there exists a solution* $\mathbf{u}\in H^1_0\left(\Omega,x_1^{2(\gamma-1)}\omega(x_1)^{-2})\right)^2$ *of* $\operatorname{div}\mathbf{u}=f$ *that satisfies*

$$\int_\Omega |D\mathbf{u}(x)|^2 x_1^{2(\gamma-1)}\omega(x_1)^{-2}\,\mathrm{d}x \leqslant C^2\int_\Omega |f(x)|^2\omega(x_1)^{-2}\,\mathrm{d}x,$$

where $\omega(x_1):=(1-\ln(x_1))^\alpha$.

The article is organized as follows: In Section 2, we show that the weighted discrete Hardy inequality, with some appropriate weights, implies the validity of a certain decomposition of functions in which our local-to-global argument is based. The main result in this section might be of interest for applications to other inequalities and related results in Sobolev spaces. In Section 3, we prove the validity of the corollaries stated in the introduction that claim the solvability of the divergence equation in weighted spaces for power weights and powers of logarithmic weights.

The novelty of this work lies in the use of the well-studied weighted discrete Hardy inequality to get new sufficient conditions on the weights that imply the solvability of the divergence equation, recovering the existing results in [6, 13] when the weights are powers of the distance to the cusp/origin. The second corollary using powers of logarithmic weights is also new.

2. A decomposition of functions and applications

During this section, we assume that $1<p,q<\infty$, with $\frac{1}{p}+\frac{1}{q}=1$, except for the proof of Theorem 1 which uses the estimate of the constant in the divergence equation provided by Costabel and Dauge [5] for $p=q=2$. In addition, we refer by a *weight* $v:\Omega\to\mathbb{R}$ to a positive and Lebesgue-measurable function, and by a *sequence weight* $\{v_i\}_{i\geqslant 1}$ to a sequence of positive real numbers. We also denote by $x=(x_1,x_2)$ a general point in $\mathbb{R}^2$.

DEFINITION 1. A weight $\omega:\Omega\to\mathbb{R}$ is called *admissible* if $\omega^p\in L^1(\Omega)$ and there exists a uniform constant C_ω such that

$$\operatorname*{ess\,sup}_{x\in\Omega_i}\omega(x)\leqslant C_\omega \operatorname*{ess\,inf}_{x\in\Omega_i}\omega(x), \tag{7}$$

for all $i\geqslant 0$. Notice that admissible weights are subordinate to a partition $\{\Omega_i\}_{i\geqslant 0}$ of Ω introduced in (5), and $1<p<\infty$.

EXAMPLE 1. The function $\omega(x):=x_1^\beta$, where $\beta>\frac{-\gamma-1}{p}$, is an admissible weight with $C_\omega=2^{2|\beta_0|}$, where $\beta_0:=\frac{-\gamma-1}{p}$.

REMARK 2. Notice that decomposition $\{\Omega_i\}_{i\geqslant 0}$ of Ω defined in (5) and condition (7) for admissible weights are natural assumptions to address the singularity of the domain and to consider the weights of our interest, which involve the distance to the cusp. For other domains or weights, the partition of the domain and the conditions on the weights might be different. Other settings might also require to deal with weighted discrete Hardy-type inequalities on trees.

DEFINITION 2. Given $g : \Omega \to \mathbb{R}$ integrable function with vanishing mean value, i.e. $\int g = 0$, we refer by *a $\mathscr{C}$-orthogonal decomposition of g* subordinate to $\{\Omega_i\}_{i\geqslant 0}$ to a collection of integrable functions $\{g_i\}_{i\geqslant 0}$ with the following properties:

1. $g(x) = \sum_{i\geqslant 0} g_i(x)$.
2. $\operatorname{supp}(g_i) \subset \Omega_i$, for all $i \geqslant 0$.
3. $\int_{\Omega_i} g_i = 0$, for all $i \geqslant 0$.

The letter $\mathscr{C}$ in the previous definition refers to the space of constant functions. Notice that having vanishing mean value could also be understood as being orthogonal to the functions in $\mathscr{C}$. Other applications of this type of decomposition of functions require to have orthogonality to other spaces (see [14, 15]). We also refer the reader to [8] for applications to a fractional Poincaré type inequality.

We show the existence of a $\mathscr{C}$-orthogonal decomposition by using a constructive argument introduced in [13]. Let us describe the idea of this argument when Ω is the union of three subdomains. Thus, let $f \in L^1(\Omega)$ be a function with vanishing mean value. Then, using a partition of the unity $\{\phi_i\}_{0\leqslant i\leqslant 2}$ subordinate to $\{\Omega_i\}_{0\leqslant i\leqslant 2}$ we can write g as:

$$g = f_0 + f_1 + f_2 = g\phi_0 + g\phi_1 + g\phi_2.$$

However, this partition might not be orthogonal to $\mathscr{C}$. In order to get this property we make the following arrangements:

$$g = f_0 + \left(f_1 + \frac{\chi_{B_2}}{|B_2|}\int_{\Omega_2} f_2\right) + \underbrace{\left(f_2 - \frac{\chi_{B_2}}{|B_2|}\int_{\Omega_2} f_2\right)}_{f_2 - h_2},$$

where $B_2 := \Omega_2 \cap \Omega_1$. Note that the function $f_2 - h_2$ has its support in Ω_2 and $\int f_2 - h_2 = 0$. Finally, we repeat the process with the first two functions. Thus, if $B_1 := \Omega_1 \cap \Omega_0$ we have that

$$\begin{aligned} f = & \overbrace{\left(f_0 + \frac{\chi_{B_1}}{|B_1|}\int_{\Omega_1\cup\Omega_2} f_1 + f_2\right)}^{f_0 - h_0} \\ & + \underbrace{\left(f_1 + \frac{\chi_{B_2}}{|B_2|}\int_{\Omega_2} f_2 - \frac{\chi_{B_1}}{|B_1|}\int_{\Omega_1\cup\Omega_2} f_1 + f_2\right)}_{f_1 - h_1} + \underbrace{\left(f_2 - \frac{\chi_{B_2}}{|B_2|}\int_{\Omega_2} f_2\right)}_{f_2 - h_2}, \end{aligned} \tag{8}$$

obtaining the claimed decomposition. Observe that we have used the vanishing mean value of f only to prove that $f_0 - h_0$ integrates zero.

Let us introduce the following weighted discrete Hardy-type inequalities:

$$\sum_{j=1}^{\infty} |\Omega_j| \omega_j^p \left(\sum_{i=1}^{j} d_i \right)^p \leqslant C_H^p \sum_{j=1}^{\infty} |\Omega_j| \omega_j^p d_j^p, \tag{9}$$

and

$$\sum_{i=1}^{\infty} |\Omega_i|^{1-q} \omega_i^{-q} \left(\sum_{j=i}^{\infty} b_j \right)^q \leqslant C_H^q \sum_{i=1}^{\infty} |\Omega_i|^{1-q} \omega_i^{-q} b_i^q. \tag{10}$$

The first one is inequality (2) to the p power where the sequence weight $u_n = v_n = |\Omega_n| \omega_n^p$, and the second one is its dual version. The following lemma follows from this duality.

LEMMA 1. *Given a sequence weight $\{\omega_i\}_{i\geqslant 1}$, inequality (10) is valid for any non-negative sequence $\{b_i\}_{i\geqslant 1}$ if and only if inequality (9) is valid for any non-negative sequence $\{d_j\}_{j\geqslant 1}$, with the same constant C_H.*

Proof. By using the duality between l^p and l^q, and defining $\tilde{d}_j := |\Omega_j|^{1/p} \omega_j d_j$ and $\tilde{b}_i := |\Omega_i|^{-1/p} \omega_i^{-1} b_i$, it follows that inequality (9) and (10) can be written as

$$\sup_{\|\tilde{d}\|_{l^p}=1} \sup_{\|\tilde{b}\|_{l^q}=1} \sum_{j=1}^{\infty} \tilde{b}_j |\Omega_j|^{1/p} \omega_j \sum_{i=1}^{j} |\Omega_i|^{-1/p} \omega_i^{-1} \tilde{d}_i \leqslant C_H \tag{11}$$

and

$$\sup_{\|\tilde{b}\|_{l^q}=1} \sup_{\|\tilde{d}\|_{l^p}=1} \sum_{i=1}^{\infty} \tilde{d}_i |\Omega_i|^{-1/p} \omega_i^{-1} \sum_{j=i}^{\infty} |\Omega_j|^{1/p} \omega_j \tilde{b}_j \leqslant C_H. \tag{12}$$

Finally, one can obtain (12) from (11), and viceversa, by changing the order of the summations. $\square$

THEOREM 2. *Let $\omega : \Omega \to \mathbb{R}$ be an admissible weight that satisfies (9) for the sequence weight $\omega_i := \omega(2^{-i})$. Then, given $g \in L^1(\Omega)$, with $\int_\Omega g = 0$, there exists $\{g_t\}_{t\in\Gamma}$, a $\mathscr{C}$-decomposition of g subordinate to $\{\Omega_i\}_{i\geqslant 0}$ (see Definition 2), such that*

$$\sum_{i=0}^{\infty} \int_{\Omega_i} |g_i(x)|^q \omega^{-q}(x) \,\mathrm{d}x \leqslant C_d^q \int_\Omega |g(x)|^q \omega^{-q}(x) \,\mathrm{d}x. \tag{13}$$

Moreover, we have the following estimate for the optimal constant C_d:

$$C_d \leqslant 2^{2+1/q} C_\omega^2 C_H. \tag{14}$$

Proof. The decomposition treated here follows the example with three subdomains that appears in page 776. Indeed, let $\{\phi_i\}_{i\geqslant 0}$ be a partition of unity subordinate to the collection $\{\Omega_i\}_{i\geqslant 0}$. Namely, a collection of smooth functions such that $\sum_{i\geqslant 0} \phi_i = 1$, $0 \leqslant \phi_i \leqslant 1$ and $supp(\phi_i) \subset \Omega_i$. Thus, g can be cut-off into $g = \sum_{i\geqslant 0} f_i$

by taking $f_i = g\phi_i$. This decomposition satisfies conditions 1. and 2. in Definition 2 but not necessarily 3.. Thus, we make the following modifications to $\{f_i\}_{i\geqslant 0}$ to obtain a collection of functions that also satisfies item 3.. Indeed, for any $i \geqslant 1$,

$$g_i(x) := f_i(x) + h_{i+1}(x) - h_i(x), \tag{15}$$

where

$$\begin{aligned} h_i(x) &:= \frac{\chi_i(x)}{|B_i|} \int_{W_i} \sum_{k\geqslant i} f_k, \\ B_i &:= \Omega_i \cap \Omega_{i-1}, \\ W_i &:= \bigcup_{k\geqslant i} \Omega_k. \end{aligned} \tag{16}$$

We denote by χ_i the characteristic function of B_i. Notice that the auxiliary function h_i is not defined for $i = 0$, thus g_0 follows in this other way

$$g_0(x) = f_0(x) + h_1(x).$$

This decomposition was introduced in [13] in a more general way where the natural numbers in the subindex set is replaced by a set with a partial order given by a structure of tree (i.e. connected graph without cycles). We also use in this article inequality (9) instead of another Hardy type inequality on trees introduced in [13]. Thus, it only remains to show estimate (13). Notice that h_i and h_{i+1} have disjoint supports thus

$$|h_{i+1}(x) - h_i(x)|^q = |h_{i+1}(x)|^q + |h_i(x)|^q.$$

Next, using that $|a+b|^q \leqslant 2^{q-1}(|a|^q + |b|^q)$ for all $a, b \in \mathbb{R}$, we have

$$\begin{aligned} &\sum_{i=0}^{\infty} \int_{\Omega_i} |g_i(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x \\ &\leqslant 2^{q-1} \left(\sum_{i=0}^{\infty} \int_{\Omega_i} |f_i(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x + 2\sum_{i=1}^{\infty} \int_{\Omega_i} |h_i(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x \right) \\ &\leqslant 2^q \left(\int_{\Omega} |g(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x + \sum_{i=1}^{\infty} \int_{\Omega_i} |h_i(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x \right). \end{aligned} \tag{17}$$

Let us work over the sum on the right hand side in the previous inequality by using the weighted discrete Hardy inequality. Notice that from the definition of the auxiliary functions in (16) and inequality (7) in Definition 1 it follows that

$$|h_i(x)| \leqslant \frac{\chi_i(x)}{|B_i|} \sum_{k=i}^{\infty} \int_{\Omega_i} |g|$$

and

$$\int_{\Omega_i} \frac{\chi_i(x)}{|B_i|^q} \omega^{-q}(x_1)\,\mathrm{d}x \leqslant C_\omega^q \omega_i^{-q} |B_i|^{1-q}.$$

Therefore, since $|\Omega_i| < 2|B_i|$ for any $i \geqslant 1$, the sum in inequality (17) is bounded by

$$\sum_{i=1}^{\infty} \int_{\Omega_i} |h_i(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x \leqslant C_\omega^q \sum_{i=1}^{\infty} |B_i|^{1-q} \omega_i^{-q} \left(\sum_{k=i}^{\infty} \int_{\Omega_k} |g| \right)^q$$
$$\leqslant 2^{q-1} C_\omega^q \sum_{i=1}^{\infty} |\Omega_i|^{1-q} \omega_i^{-q} \left(\sum_{k=i}^{\infty} \int_{\Omega_k} |g| \right)^q.$$

Next, by using Lemma 1 with $b_i = \int_{\Omega_i} |g|$, $i \geqslant 1$, and Hölder inequality, we can conclude that

$$\sum_{i=1}^{\infty} \int_{\Omega_i} |h_i(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x \leqslant 2^{q-1} C_\omega^q C_H^q \sum_{i=1}^{\infty} |\Omega_i|^{1-q} \omega_i^{-q} \left(\int_{\Omega_i} |g| \right)^q$$
$$\leqslant 2^{q-1} C_\omega^q C_H^q \sum_{i=1}^{\infty} \omega_i^{-q} \left(\int_{\Omega_i} |g|^q \right)$$
$$\leqslant 2^{q-1} C_\omega^{2q} C_H^q \sum_{i=1}^{\infty} \int_{\Omega_i} |g(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x$$
$$\leqslant 2^q C_\omega^{2q} C_H^q \int_{\Omega} |g(x)|^q \omega^{-q}(x_1)\,\mathrm{d}x.$$

Finally, from inequality (17) it follows (13). $\square$

In order to prove the solvability of the divergence equation on the subdomains Ω_i we use the following result proved by M. Costabel and M. Dauge in [5] for star-shaped domains. Let us recall the definition of this class of domains. A domain U is *star-shaped with respect to a ball B* if and only if any segment with an end-point in U and the other one in B is contained in U.

THEOREM 3. *Let $U \subset \mathbb{R}^2$ be a domain contained in a ball of radius R, star-shaped with respect to a concentric ball of radius r. Then, for any $g \in L^2(U)$ with vanishing mean value there exists a solution $\mathbf{u} \in H_0^1(U)^2$ of the equation $\operatorname{div}\mathbf{u} = g$ satisfying the estimate*

$$\left(\int_U |D\mathbf{u}(x)|^2\,\mathrm{d}x \right)^{1/2} \leqslant \frac{2R}{r} \left(\int_U |g(x)|^2\,\mathrm{d}x \right)^{1/2}.$$

Proof of Theorem 1. Let f be a function in $L^2(\Omega, \omega^{-2}(x_1))$ with vanishing mean value. Notice that, since ω is an admissible weight for $p=2$, $L^2(\Omega, \omega^{-2}(x_1)) \subset L^1(\Omega)$ and the mean value of f is well-defined. Then, from Theorem 2, there exists a $\mathscr{C}$-decomposition $\{f_i\}_{i\geqslant 0}$ of f subordinate to $\{\Omega_i\}_{i\geqslant 0}$ satisfying (13). Now, let us assume, to be shown later in this proof, that Ω_i is included in a ball with radius $R_i = 2^{-i+1}$ and star-shaped with respect to a concentric ball A_i with radius $r_i = 2^{-\gamma(i+2)-1}/\gamma$. Then, from Theorem 3, there exists a solution of $\operatorname{div}\mathbf{v}^i = f_i$ in Ω_i that satisfies

$$\int_{\Omega_i} |D\mathbf{v}^i(x)|^2\,\mathrm{d}x \leqslant \gamma^2 2^{6+4\gamma} 2^{2(\gamma-1)i} \int_{\Omega_i} |f_i(x)|^2\,\mathrm{d}x.$$

Hence, by extending $\mathbf{v}^i$ by zero, the vector field $\mathbf{u}(x) := \sum_{i\geqslant 0}\mathbf{v}^i(x)$ satisfies that

$$\operatorname{div}\mathbf{u}(x) = \operatorname{div}\sum_{i\geqslant 0}\mathbf{v}^i(x) = \sum_{i\geqslant 0} f_i(x) = f(x),$$

and

$$\begin{aligned}
&\int_\Omega |D\mathbf{u}(x)|^2 x_1^{2(\gamma-1)}\omega^{-2}(x_1)\,\mathrm{d}x \\
&\leqslant 2\sum_{i\geqslant 0}\int_{\Omega_i} |D\mathbf{v}^i(x)|^2 x_1^{2(\gamma-1)}\omega^{-2}(x_1)\,\mathrm{d}x \\
&\leqslant 2C_\omega^2\sum_{i\geqslant 0} 2^{-2i(\gamma-1)}\omega^{-2}(2^{-i})\int_{\Omega_i}|D\mathbf{v}^i(x)|^2\,\mathrm{d}x \\
&\leqslant \gamma^2 2^{7+4\gamma}C_\omega^2\sum_{i\geqslant 0}\omega^{-2}(2^{-i})\int_{\Omega_i}|f_i(x)|^2\,\mathrm{d}x \\
&\leqslant \gamma^2 2^{7+4\gamma}C_\omega^4\sum_{i\geqslant 0}\int_{\Omega_i}|f_i(x)|^2\omega(x_1)^{-2}\,\mathrm{d}x \\
&\leqslant \gamma^2 2^{12+4\gamma}C_\omega^8 C_H^2\int_\Omega |f(x)|^2\,\mathrm{d}x.
\end{aligned}$$

Finally, let us show that Ω_i is included in a ball with radius $R_i = 2^{-i+1}$ and star-shaped with respect to a concentric ball A_i. Notice that Ω_i is included in the square $[0,2^{-i}]^2$ with diameter $2^{-i+1/2}$. Thus, any ball with center at a point in Ω_i and radius $R_i = 2^{-i+1}$ contains Ω_i. We define A_i as the ball with radius $r_i := \rho_i/2\gamma$, and center $c_i := (2^{-i} - r_i, r_i)$, where $\rho_i = 2^{-\gamma(i+2)}$, as shown in Picture 1.

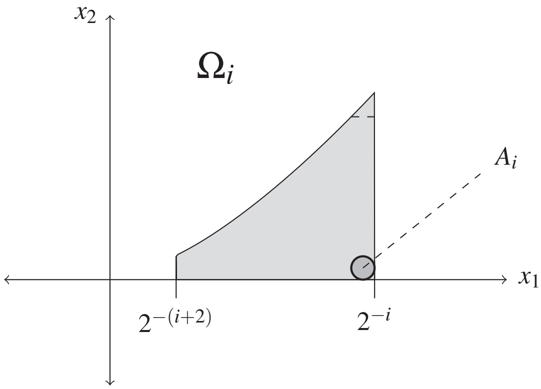


Figure 1: Ω_i *is a star-shaped domain.*

Now, given $y \in \Omega_i$ and $x \in A_i$, we have to show that the segment $\overline{xy}$ with end-points at y and x is included in Ω_i.

The open rectangle D_t with sides parallel to the axis and vertices $(2^{-i},0)$ and (t,t^γ), for $2^{-i-2} \leqslant t \leqslant 2^{-i} - 2r_i$, is convex, contains B_i and is included in Ω_i. Thus, the segment $\overline{xy}$ is included in Ω_i if y belongs to D_t, for any t in the interval $[2^{-i-2}, 2^{-i} - 2r_i]$.

Hence, it is sufficient to prove the case where $y = (y_1, y_2)$ belongs to the region above (or over) the dashed line in Picture 1: $2^{-i} - 2r_i < y_1 < 2^{-i}$ and $(2^{-i} - 2r_i)^\gamma \leqslant y_2 < y_1^\gamma$. Moreover, observe that if the segment $\overline{xy}$ is not included in Ω_i then its slope must be equal to $\gamma t^{\gamma-1}$, for some $2^{-i} - 2r_i < t < 2^{-i}$. Hence, it is sufficient to show that the slope of $\overline{xy}$ is larger than γ i.e.

$$\frac{|y_2 - x_2|}{|y_1 - x_1|} \geqslant \gamma.$$

Now, it follows from some straightforward estimations that

$$|y_2 - x_2| \geqslant 2^{-i\gamma} - |2^{-i\gamma} - y_2| - |x_2|,$$

where, $|x_2| \leqslant \rho_i$ and

$$|2^{-i\gamma} - y_2| \leqslant |(2^{-i})^\gamma - (2^{-i} - 2r_i)^\gamma| < \gamma 2r_i = \rho_i.$$

Then,

$$\frac{|y_2 - x_2|}{|y_1 - x_1|} \geqslant \frac{2^{2\gamma}\rho_i - 2\rho_i}{\rho_i/\gamma} \geqslant 2\gamma. \quad \square$$

3. The weighted discrete Hardy inequality

In this chapter, we prove the two corollaries stated in the Introduction about the solvability of the divergence equation in weighted Sobolev spaces for the weights $\omega(x) = x_1^\beta$ and $\omega(x) = (1 - \ln(x_1))^\alpha$. Notice that Theorem 1 requires $p = 2$, however, we analyze the general case $1 < p < \infty$ since Theorem 2, which does not have the constraint $p = 2$, can be used to obtain other inequalities (such as the weighted fractional Poincaré inequality [8]) in our cuspidal domain Ω.

Let us recall the characterization of the weighted discrete Hardy inequality (2) that we use for our applications (see [4]).

THEOREM 4. *Let $\{u_i\}_{i\geqslant 1}$ and $\{v_i\}_{i\geqslant 1}$ be sequence weights, and $1 < p < \infty$ with its conjugate exponent $q = \frac{p}{p-1}$, then inequality (2) is valid if and only if*

$$A = \sup_{k\geqslant 1} \left(\sum_{i=k}^{\infty} u_i\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} v_i^{1-q}\right)^{\frac{1}{q}} < \infty.$$

In addition, if C_H represents the optimal constant in (2), then

$$A \leqslant C_H \leqslant C_p A,$$

where $C_p = \min\{q p^{1/p}, \ p q^{1/q}\}$.

Thus, we use this characterization to determine the exponents β and α for which the weights $\omega(x) = x_1^{\beta}$ and $\omega(x) = (1-\ln(x_1))^{\alpha}$ satisfy the sufficient condition in Theorem 1:

$$\sum_{j=1}^{\infty} u_i \left(\sum_{i=1}^{j} d_i\right)^2 \leqslant C_H^2 \sum_{j=1}^{\infty} u_i d_j^2,$$

where

$$u_i := |\Omega_i|\, \omega^2(2^{-i}).$$

Let us start by calculating the measure of the subdomains Ω_i:

$$\begin{aligned}
|\Omega_i| &= \int_{2^{-(i+2)}}^{2^{-i}} \int_0^{x_1^{\gamma}} \mathrm{d}x_2\, \mathrm{d}x_1 = \int_{2^{-(i+2)}}^{2^{-i}} x_1^{\gamma}\, \mathrm{d}x_1 \\
&= \frac{x_1^{\gamma+1}}{\gamma+1}\Bigg|_{2^{-(i+2)}}^{2^{-i}} \\
&= \frac{1}{\gamma+1}\left(2^{-i(\gamma+1)} - 2^{-(i+2)(\gamma+1)}\right) \\
&= \frac{1-2^{-2(\gamma+1)}}{\gamma+1} 2^{-i(\gamma+1)} \\
&= C_{\gamma} 2^{-(\gamma+1)i},
\end{aligned}$$

where

$$C_{\gamma} = \frac{1-2^{-2(\gamma+1)}}{\gamma+1}. \tag{18}$$

For simplicity, we include some basic calculations on geometric sums which will be used in the following proofs:

$$\sum_{i=k}^{\infty} r^i = \frac{r^k}{1-r}, \text{ for } 0<r<1,$$

$$\sum_{i=1}^{k} r^{i(1-q)} = \frac{(r^{1-q})^{k+1} - r^{1-q}}{r^{1-q}-1}, \text{ for } r>0, \text{ and } q>1.$$

The following lemma considers the power weights $\omega(x_1) = x_1^{\beta}$.

LEMMA 2. *Let $\Omega \subset \mathbb{R}^2$ be the domain defined in (3) and $1<p<\infty$ with its conjugate exponent $q = \frac{p}{p-1}$. Then, the weight $\omega : \Omega \to \mathbb{R}$ defined by $\omega(x) = x_1^{\beta}$, with $\beta > \frac{-\gamma-1}{p}$, is an admissible weight in the sense of Definition 1, and satisfies weighted discrete Hardy inequality (9) where $\omega_i := \omega(2^{-i})$.*

Moreover,

$$C_H < C_p \left(\frac{1}{r(1-r)}\right)^{1/p} \left(\frac{1}{r^{1-q}-1}\right)^{1/q},$$

where

$$r := 2^{-p\beta-\gamma-1} \quad \textit{and} \quad C_p = \min\{q\,p^{1/p},\ p\,q^{1/q}\}.$$

Proof. First, let us show that $x_1^{p\beta} \in L^1(\Omega)$:

$$\int_0^1 \int_0^{x_1^\gamma} x_1^{\beta p}\, \mathrm{d}x_2\, \mathrm{d}x_1 = \int_0^1 x_1^{\beta p+\gamma} dx_1,$$

which is finite if and only if $\beta p + \gamma > -1$, equivalently, $\beta > \frac{-\gamma-1}{p}$. Moreover, it is easy to prove that inequality (7) is valid with $C_\omega = 2^{2|\beta_0|}$, where $\beta_0 := \frac{-\gamma-1}{p}$.

Now, we have to show that the weighted discrete Hardy inequality (9) is satisfied for the sequence weight $\omega_i := 2^{-i\beta}$, with $\beta > \frac{-\gamma-1}{p}$. Thus, by Theorem 4, it is necessary and sufficient to show that

$$A = \sup_{k\geqslant 1} \left(\sum_{i=k}^{\infty} |\Omega_i| 2^{-i\beta p} \right)^{1/p} \left(\sum_{i=1}^{k} (|\Omega_i| 2^{-i\beta p})^{1-q} \right)^{1/q} < \infty.$$

Hence, let us denote

$$|\Omega_i| 2^{-i\beta p} = C_\gamma \left(2^{-(\gamma+1)-p\beta} \right)^i =: C_\gamma r^i,$$

where C_γ was introduced in (18). Notice that $r \in (0,1)$. Thus,

$$\begin{aligned}
A =& \sup_{k\geqslant 1} \left(\sum_{i=k}^{\infty} C_\gamma r^i \right)^{1/p} \left(\sum_{i=1}^{k} (C_\gamma r^i)^{(1-q)} \right)^{1/q} \\
=& C_\gamma^{1/p+(1-q)/q} \sup_{k\geqslant 1} \left(\sum_{i=k}^{\infty} r^i \right)^{1/p} \left(\sum_{i=1}^{k} r^{i(1-q)} \right)^{1/q} \\
=& \sup_{k\geqslant 1} \left(\frac{r^k}{1-r} \right)^{1/p} \left(\frac{(r^{1-q})^{k+1} - r^{1-q}}{r^{1-q}-1} \right)^{1/q} \\
=& \left(\frac{1}{1-r} \right)^{1/p} \left(\frac{r^{1-q}}{r^{1-q}-1} \right)^{1/q} \sup_{k\geqslant 1} r^{k/p} \left(r^{k(1-q))} - 1 \right)^{1/q} \\
<& \left(\frac{1}{1-r} \right)^{1/p} \left(\frac{r^{1-q}}{r^{1-q}-1} \right)^{1/q} \sup_{k\geqslant 1} r^{k/p} r^{k(1-q)/q} \\
=& \left(\frac{1}{1-r} \right)^{1/p} \left(\frac{r^{1-q}}{r^{1-q}-1} \right)^{1/q} < \infty.
\end{aligned}$$

Moreover, using again Theorem 4, it follows that

$$C_H \leqslant C_p A < C_p \left(\frac{1}{r(1-r)} \right)^{1/p} \left(\frac{1}{r^{1-q}-1} \right)^{1/q},$$

where

$$r := 2^{-p\beta-\gamma-1}. \quad \square$$

Proof of Corollary 1. It follows from Theorem 1 and Lemma 2. $\square$

LEMMA 3. *Let* $\Omega \subset \mathbb{R}^2$ *be the domain defined in* (3) *and* $1 < p < \infty$ *with its conjugate exponent* $q = \frac{p}{p-1}$. *Then, the weight* $\omega : \Omega \to \mathbb{R}$ *defined by* $\omega(x) = (1 - \ln(x_1))^\alpha$, *with* $\alpha \in \mathbb{R}$, *is an admissible weight in the sense of Definition* 1, *and satisfies the weighted discrete Hardy inequality* (9) *for* $\omega_i := \omega(2^{-i})$.

Proof. If α is zero, then $\omega(x) = 1$. This weight was studied in Lemma 2, for $\beta = 0$, which is admissible and satisfies the discrete Hardy inequality (9) with

$$C_H < C_p \left(\frac{1}{r(1-r)}\right)^{1/p} \left(\frac{1}{r^{1-q}-1}\right)^{1/q},$$

for

$$r := 2^{-(\gamma+1)} \quad \text{and} \quad C_p = \min\{q p^{1/p},\ p q^{1/q}\}.$$

Thus, we have to consider the case when α is different from 0.

First, let us show that $\omega^p(x) = (1 - \ln(x_1))^{p\alpha} \in L^1(\Omega)$:

$$\int_0^1 \int_0^{x_1^\gamma} (1 - \ln(x_1))^{p\alpha}\, \mathrm{d}x_2\, \mathrm{d}x_1 = \int_0^1 x_1^\gamma (1 - \ln(x_1))^{p\alpha}\, \mathrm{d}x_1.$$

If α is positive, then the function $f(x_1) = x_1^\gamma((1 - \ln(x_1))^{p\alpha}$ tends to 0 as x_1 tends to 0 from the right, then the integral of this continuous function is finite:

$$\lim_{x_1 \to 0^+} x_1^\gamma (1 - \ln(x_1))^{p\alpha} = \lim_{x_1 \to 0^+} \left(\frac{1 - \ln(x_1)}{x_1^{-\gamma/p\alpha}}\right)^{p\alpha} = 0,$$

since

$$\lim_{x_1 \to 0^+} \frac{1 - \ln(x_1)}{x_1^{-\gamma/p\alpha}} = \lim_{x_1 \to 0^+} \frac{-x_1^{-1}}{(-\gamma/p\alpha) x_1^{-\gamma/p\alpha - 1}} = \lim_{x_1 \to 0^+} \frac{p\alpha}{\gamma} x_1^{\gamma/p\alpha} = 0.$$

If α is negative then $0 < x_1^\gamma (1 - \ln(x_1))^{p\alpha} < 1$, thus $\omega^p(x) \in L^1(\Omega)$.

Now, let us estimate the constant C_ω in inequality (7):

$$\sup_{x \in \Omega_i} \omega(x) \leqslant C_\omega \inf_{x \in \Omega_i} \omega(x).$$

If α is positive, then $\omega(x)$ is decreasing with respect to x_1, then

$$\sup_{x \in \Omega_i} \omega(x) = \omega(2^{-i-2}) = (1 + (i+2)\ln(2))^\alpha$$
$$\inf_{x \in \Omega_i} \omega(x) = \omega(2^{-i}) = (1 + i \ln(2))^\alpha,$$

hence,

$$\frac{\omega(2^{-i-2})}{\omega(2^{-i})} = \left(1 + \frac{2\ln(2)}{1 + i\ln(2)}\right)^\alpha \leqslant (1 + 2\ln(2))^\alpha.$$

If α is negative, then $\omega(x)$ is increasing with respect to x_1, then

$$\begin{aligned}\sup_{x\in\Omega_i}\omega(x) &= \omega(2^{-i}) = (1+i\ln(2))^{\alpha}\\ \inf_{x\in\Omega_i}\omega(x) &= \omega(2^{-i-2}) = (1+(i+2)\ln(2))^{\alpha},\end{aligned}$$

hence,

$$\frac{\omega(2^{-i})}{\omega(2^{-i-2})} = \left(1+\frac{2\ln(2)}{1+i\ln(2)}\right)^{-\alpha} \leqslant (1+2\ln(2))^{-\alpha}.$$

Thus, $C_\omega := (1+2\ln(2))^{|\alpha|}$ satisfies estimate (7).

Third, let us study the weighted discrete Hardy inequality for this weight. We use the characterization stated in Theorem 4 thus we have to estimate the following supremum

$$\begin{aligned}A = \sup_{k\geqslant 1}\left(\sum_{i=k}^{\infty}u_i\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}{u_i}^{1-q}\right)^{\frac{1}{q}}\\ = \sup_{k\geqslant 1}\left(\sum_{i=k}^{\infty}2^{-(\gamma+1)i}(1+i\ln(2))^{p\alpha}\right)^{\frac{1}{p}}\\ \left(\sum_{i=1}^{k}\left(2^{-(\gamma+1)i}(1+i\ln(2))^{p\alpha}\right)^{1-q}\right)^{\frac{1}{q}}.\end{aligned} \tag{19}$$

If α is negative, then $p\alpha<0$ and $(1+i\ln(2))^{p\alpha}\leqslant(1+k\ln(2))^{p\alpha}$ for all $i\geqslant k$. Similarly, $p\alpha(1-q)>0$ and $(1+i\ln(2))^{p\alpha(1-q)}\leqslant(1+k\ln(2))^{p\alpha(1-q)}$ for all $i\leqslant k$. Thus,

$$\begin{aligned}A &\leqslant \sup_{k\geqslant 1}(1+k\ln(2))^{\alpha+p\alpha(1-q)/q}\left(\sum_{i=k}^{\infty}r^i\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}r^{i(1-q)}\right)^{\frac{1}{q}}\\ &= \sup_{k\geqslant 1}\left(\sum_{i=k}^{\infty}r^i\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}r^{i(1-q)}\right)^{\frac{1}{q}}\\ &\leqslant \left(\frac{1}{1-r}\right)^{1/p}\left(\frac{r^{1-q}}{r^{1-q}-1}\right)^{1/q} < \infty,\end{aligned}$$

where $r = 2^{-(\gamma+1)}$.

For α positive, we define $a = p\alpha > 0$ and $f(t) = r^t(1+t\ln(2))^a$. By a straightforward calculation, it can be seen that f is positive and decreasing for t sufficiently large. Thus, there exists $k_0\in\mathbb{N}$ such

$$\sum_{i=k}^{\infty}2^{-(\gamma+1)i}(1+i\ln(2))^{p\alpha} \leqslant \int_{k-1}^{\infty} r^t(1+t\ln(2))^a\,\mathrm{d}t.$$

for $k \geqslant k_0$. Next, by using integration by parts, we obtain

$$\begin{aligned} I &:= \int_{k-1}^{\infty} r^t (1+t\ln(2))^a \,\mathrm{d}t \\ &\leqslant \frac{-r^k}{r\ln(r)}(1+k\ln(2))^a + \int_{k-1}^{\infty} r^t (1+t\ln(2))^a \left[\frac{a\ln(2)}{-\ln(r)(1+t\ln(2))}\right] \mathrm{d}t. \end{aligned}$$

Now, we assume that k_0 is sufficiently large such that the function between brackets in the previous line is less than $1/2$. Thus,

$$I \leqslant \frac{-r^k}{r\ln(r)}(1+k\ln(2))^a + \frac{1}{2}I,$$

and

$$\frac{1}{2}I \leqslant \frac{-r^k}{r\ln(r)}(1+k\ln(2))^a.$$

Thus, it follows that

$$\begin{aligned} \sum_{i=k}^{\infty} r^i (1+i\ln(2))^a &\leqslant \int_{k-1}^{\infty} r^t (1+t\ln(2))^a \,\mathrm{d}t = I \\ &\leqslant \frac{-2r^k}{r\ln(r)}(1+k\ln(2))^a, \end{aligned} \tag{20}$$

for $k \geqslant k_0$.

Let us study the second sum in the estimation of A in (19). Thus, we define

$$\tilde{r} := 2^{-(\gamma+1)(1-q)} = r^{1-q} = r^{\frac{-q}{p}} > 1, \tag{21}$$

and

$$\tilde{a} := a(1-q) = -aq/p < 0. \tag{22}$$

Notice that the function $g(t) := \tilde{r}^t (1+t\ln(2))^{\tilde{a}}$ is positive and increasing for t sufficiently large. Thus, there exists a constant $C_2 > 1$ such that

$$\sum_{i=1}^{k} \tilde{r}^i (1+i\ln(2))^{\tilde{a}} \leqslant C_2 \int_1^{k+1} \tilde{r}^t (1+t\ln(2))^{\tilde{a}} \,\mathrm{d}t, \tag{23}$$

for all $k \geqslant 1$. Finally, notice that to show that A in (19) is finite it is sufficient to consider the case where the supremum runs over $k \geqslant k_0$ and estimate its power q. Thus, from (20) and (23), we have

$$\sup_{k \geqslant k_0} \left(\sum_{i=k}^{\infty} r^i (1+i\ln(2))^a\right)^{\frac{q}{p}} \left(\sum_{i=1}^{k} \tilde{r}^i (1+i\ln(2))^{\tilde{a}}\right) \leqslant C_2 \frac{\int_1^{k+1} \tilde{r}^t (1+t\ln(2))^{\tilde{a}} \,\mathrm{d}t}{r^{\frac{-kq}{p}} (1+k\ln(2))^{\frac{-aq}{p}}},$$

for another constant C_2, which is independent of k, denoted with the same letter for simplicity.

Finally, we calculate the limit of the above quotient as k goes to infinity, understanding k as a continuous variable. We use for this analysis definitions (21) and (22), and L'Hospital rule:

$$
\begin{aligned}
&\lim_{k\to\infty} \frac{\int_1^{k+1} \tilde{r}^t (1+t\ln(2))^{\tilde{a}}\,\mathrm{d}t}{r^{\frac{-kq}{p}}(1+k\ln(2))^{\frac{-aq}{p}}} = \lim_{k\to\infty} \frac{\int_1^{k+1} \tilde{r}^t (1+t\ln(2))^{\tilde{a}}\,\mathrm{d}t}{\tilde{r}^k(1+k\ln(2))^{\tilde{a}}} \\
&= \lim_{k\to\infty} \frac{\tilde{r}^{k+1}(1+(k+1)\ln(2))^{\tilde{a}}}{\ln(\tilde{r})\tilde{r}^k(1+k\ln(2))^{\tilde{a}} + \tilde{a}\ln(2)\tilde{r}^k(1+k\ln(2))^{\tilde{a}-1}} \\
&= \lim_{k\to\infty} \tilde{r}\left(\frac{1+(k+1)\ln(2)}{1+k\ln(2)}\right)^{\tilde{a}} \frac{1}{\ln(\tilde{r}) + \frac{\tilde{a}\ln(2)}{1+k\ln(2)}} \\
&= \frac{\tilde{r}}{\ln(\tilde{r})}.
\end{aligned}
$$

Therefore, the sequence is convergent and bounded, which implies that A is finite. □

Proof of Corollary 2. It follows immediately from Theorem 1 and Lemma 3. □

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