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NECESSARY AND SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF COMMUTATORS OF BILINEAR HARDY–LITTLEWOOD MAXIMAL FUNCTION

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Abstract. Let $\mathscr{M}$ be the bilinear Hardy-Littlewood maximal function and $\vec{b}=(b,b)$ be a collection of locally integrable functions. In this paper, the authors establish characterizations of the weighted BMO space in terms of several different commutators of bilinear Hardy-Littlewood maximal function, respectively; these commutators include the maximal iterated commutator $\mathscr{M}_{\Pi\vec{b}}$, the maximal linear commutator $\mathscr{M}_{\Sigma\vec{b}}$, the iterated commutator $[\Pi\vec{b},\mathscr{M}]$ and the linear commutator $[\Sigma\vec{b},\mathscr{M}]$.

1. Introduction

A locally integrable function f is said to belong to BMO space if there exists a constant $C>0$ such that for any cube $Q\subset\mathbb{R}^n$,

$$\frac{1}{|Q|}\int_Q |f(x)-f_Q|dx\leqslant C,$$

where $f_Q=\frac{1}{|Q|}\int_Q f(x)dx$ and the minimal constant C is defined by $\|f\|_*$.

There are a number of classical results that demonstrate BMO functions are the right collections to do harmonic analysis on the boundedness of commutators. A well known result of Coifman, Rochberg and Weiss [8] states that the commutator

$$[b,T](f)=bT(f)-T(bf)$$

is bounded on some L^p, $1<p<\infty$, if and only if $b\in$ BMO, where T be the classical Calderón-Zygmund operator. Chanillo [6] proved that if $b\in$ BMO, the commutator

$$[b,I_\alpha]f(x)=b(x)I_\alpha f(x)-I_\alpha(bf)(x)$$

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is bounded from L^p to L^q with $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$, where I_α be a fractional integral operator. He also showed that $b\in BMO$ is necessary for the boundedness of $[b,I_\alpha]$, when $n-\alpha$ is an even number. A complete characterization of BMO via the commutator $[b,I_\alpha]$ was shown by Ding [9]. During the past thirty years, the theory was then extended and generalized to several directions. For instance, Bloom [3] investigated the characterization of BMO spaces in the weighted setting. Due to its interest, the singular integral operators were replaced by maximal operators as an object of study. The first result in this direction was done in [14]. In 1991, García-Cuerva, Harboure, Segovia and Torrea [12] showed that the maximal commutator

$$M_b(f)(x)=\sup_{Q\ni x}\frac{1}{|Q|}\int_Q|b(x)-b(y)||f(y)|dy$$

is bounded on L^p, $1<p<\infty$, if and only if $b\in \mathrm{BMO}$. In 2000, Bastero, Milman and Ruiz [1] studied the necessary and sufficient conditions for the boundedness of $[b,M]$ on L^p spaces when $1<p<\infty$. They showed that the commutator of Hardy-Littlewood maximal operator

$$[b,M](f)(x)=b(x)M(f)(x)-M(bf)(x)$$

is bounded on L^p, $1<p<\infty$, if and only if $b\in \mathrm{BMO}$ with $b^-\in L^\infty$, where $b^-(x)=-\min\{b(x),0\}$. In 2014, Zhang [29] considered the characterization of BMO via the commutator of the fractional maximal function on variable exponent Lebesgue spaces.

In the multilinear setting, the boundedness of commutators has been extensively studied already, as in Pérez and Torres' [19], Tang's [22], Lerner, Ombrosi, Pérez, Torres, and Trujillo-González's [13] and Chen and Xue's [7], and Pérez, Pradolini, Torres, and Trujillo-González's [18]. Specially, Chaffee and Torres [5], Wang, Pan and Jiang [25] and Zhang [28] contributed the theory of characterization of BMO spaces by considering the linear commutator of Multilinear operators, respectively. In [23, 24], we replace the linear commutators by iterated commutators. In this paper, we establish characterizations of the weighted BMO space in terms of several different commutators of bilinear Hardy-Littlewood maximal function, respectively; these commutators include the maximal iterated commutator $\mathscr{M}_{\Pi\vec{b}}$, the maximal linear commutator $\mathscr{M}_{\Sigma\vec{b}}$, the iterated commutator $[\Pi\vec{b},\mathscr{M}]$ and the linear commutator $[\Sigma\vec{b},\mathscr{M}]$. Our main results as follows.

THEOREM 1. *Let $1<p_1,p_2<\infty$, $\vec{b}=(b,b)$, $1/p=1/p_1+1/p_2$ and $\omega\in A_1$. Then the following are equivalent,*

(A1) $b\in \mathrm{BMO}(\omega)$*;*

(A2) $\mathscr{M}_{\Sigma\vec{b}}$ *is bounded from $L^{p_1}(\omega)\times L^{p_2}(\omega)$ to $L^p(\omega^{1-p})$;*

(A3) $\mathscr{M}_{\Pi\vec{b}}$ *is bounded from $L^{p_1}(\omega)\times L^{p_2}(\omega)$ to $L^p(\omega^{1-2p})$.*

THEOREM 2. *Let $1<p_1,p_2<\infty$, $\vec{b}=(b,b)$, $1/p=1/p_1+1/p_2$ and $\omega\in A_1$. Then the following are equivalent,* (B1) $b \in \mathrm{BMO}(\omega)$ *and* $b^-/\omega \in L^\infty$*;*

(B2) $[\Sigma\vec{b},\mathscr{M}]$ *is bounded from* $L^{p_1}(\omega)\times L^{p_2}(\omega)$ *to* $L^p(\omega^{1-p})$*;*

(B3) $[\Pi\vec{b},\mathscr{M}]$ *is bounded from* $L^{p_1}(\omega)\times L^{p_2}(\omega)$ *to* $L^p(\omega^{1-2p})$.

2. Some preliminaries and notations

In 2009, Lerner, Ombrosi, Pérez, Torres and Trujillo-González [12] introduced the following multilinear maximal function that adapts to the multilinear Calderón-Zygmund theory. In this paper, we only consider the bilinear case. A similar argument also works for the multilinear cases.

DEFINITION 1. For a collection of locally integrable functions $\vec{f}=(f_1,f_2)$, the bilinear maximal function $\mathscr{M}$ is defined by

$$\mathscr{M}(\vec{f})(x)=\sup_{Q\ni x}\prod_{i=1}^{2}\frac{1}{|Q|}\int_Q |f_i(y_i)|dy_i.$$

We now give the definitions of the maximal commutators and the commutators related to the bilinear maximal function $\mathscr{M}$. The definition is motivated by [18].

DEFINITION 2. For two collections of locally integrable functions $\vec{f}=(f_1,f_2)$ and $\vec{b}=(b_1,b_2)$, the maximal linear commutator $\mathscr{M}_{\Sigma\vec{b}}$ is defined by

$$\mathscr{M}_{\Sigma\vec{b}}(\vec{f})(x)=\sum_{i=1}^{2}\mathscr{M}_{b_i}^{(i)}(\vec{f})(x),$$

where

$$\mathscr{M}_{b_i}^{(i)}(\vec{f})(x)=\sup_{Q\ni x}\frac{1}{|Q|^2}\int_Q\int_Q |b_i(x)-b_i(y_i)|\prod_{j=1}^{2}|f_j(y_j)|dy_1dy_2.$$

The maximal iterated commutator $\mathscr{M}_{\Pi\vec{b}}$ is defined by

$$\mathscr{M}_{\Pi\vec{b}}(\vec{f})(x)=\sup_{Q\ni x}\frac{1}{|Q|^2}\int_Q\int_Q\prod_{i=1}^{2}|b_i(x)-b_i(y_i)||f_i(y_i)|dy_1dy_2.$$

The linear commutator of $\mathscr{M}$ is defined by

$$[\Sigma\vec{b},\mathscr{M}](\vec{f})(x)=[b_1,\mathscr{M}]^{(1)}(\vec{f})(x)+[b_2,\mathscr{M}]^{(2)}(\vec{f})(x),$$

where

$$[b_1,\mathscr{M}]^{(1)}(\vec{f})(x)=b_1(x)\mathscr{M}(\vec{f})(x)-\mathscr{M}(b_1f_1,f_2)(x)$$

and

$$[b_2,\mathscr{M}]^{(2)}(\vec{f})(x)=b_2(x)\mathscr{M}(\vec{f})(x)-\mathscr{M}(f_1,b_2f_2)(x).$$

The iterated commutator of $\mathscr{M}$ is defined by

$$[\Pi\vec{b},\mathscr{M}](\vec{f})(x) = b_1(x)b_2(x)\mathscr{M}(\vec{f})(x) - b_1(x)\mathscr{M}(f_1,b_2f_2)(x) \\ -b_2(x)\mathscr{M}(b_1f_1,f_2)(x)+\mathscr{M}(b_1f_1,b_2f_2)(x).$$

We now recall the definition of A_p weight introduced by Muckenhoupt [15].

DEFINITION 3. For $1<p<\infty$ and a nonnegative locally integrable function ω on $\mathbb{R}^n$, ω is in the Muckenhoupt A_p class if it satisfies the condition

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-\frac{1}{p-1}}dx\right)^{p-1} < \infty.$$

And a weight function ω belongs to the class A_1 if there exists $C>0$ such that for every cube Q,

$$\frac{1}{|Q|}\int_Q \omega(x)dx \leqslant C \operatorname{ess\,inf}_{x\in Q} \omega(x).$$

We write $A_\infty = \bigcup_{1\leqslant p<\infty} A_p$.

Now, we recall the definition of weighted BMO function, which is was first defined in the relevant work [16]. This work was improved recently in [4].

DEFINITION 4. Let $1\leqslant p<\infty$. Given a a nonnegative locally integrable function ω, the weighted BMO space $\mathrm{BMO}^p(\omega)$ is defined be the set of all functions $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\mathrm{BMO}^p(w)} := \sup_Q \left(\frac{1}{w(Q)}\int_Q |f(y)-f_Q|^p\omega(y)^{1-p}dy\right)^{1/p} < \infty,$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ and $\omega(Q)=\int_Q \omega(x)dx$. We write $\mathrm{BMO}^1(\omega)=\mathrm{BMO}(\omega)$ simple.

REMARK. For $1\leqslant p<\infty$ and $\omega\in A_1$, García-Cuerva [11] proved that $\mathrm{BMO}(\omega) = \mathrm{BMO}^p(\omega)$ with equivalence of the corresponding norms.

Standard real analysis tools as the weighted maximal function $M_\omega(f)$, the sharp maximal function $M^\sharp(f)$ carries over to this context, namely,

$$M_\omega(f)(x) = \sup_{Q\ni x}\frac{1}{\omega(Q)}\int_Q |f(y)|\omega(y)dy;$$

$$M^\sharp(f)(x) = \sup_{Q\ni x}\inf_c \frac{1}{|Q|}\int_Q |f(y)-c|dy \approx \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)-f_Q|dy.$$

A variant of weighted maximal function and sharp maximal operator $M_{\omega,s}(f)(x) = \big(M_\omega(f^s)\big)^{1/s}$ and $M^\sharp_\delta(f)(x) = \big(M^\sharp(f^\delta)(x)\big)^{1/\delta}$, which will become the main tool in our scheme.

3. Main lemmas

To prove Theorem 1 and Theorem 2, we need the following results. The key idea of using $M_\delta^\sharp$ for commutators (with the main point that $\delta < 1$) begun in reference [17], and further in references [13] and [18].

LEMMA 1. *Let* $0 < \delta_1 < \delta_2 < \varepsilon < 1/2$, $\omega \in A_1$, $\vec{b} = (b,b)$ *and* $b \in \mathrm{BMO}(\omega)$. *Then*

$$\begin{aligned} M_{\delta_1}^\sharp\big(\mathscr{M}_{\Pi\vec{b}}(\vec{f})\big)(x) \lesssim{}& \|b\|_{\mathrm{BMO}(\omega)}^2\omega(x)^2 M_\varepsilon(\mathscr{M}(\vec{f})(x)) \\ &+\|b\|_{\mathrm{BMO}(\omega)}^2\omega(x)^2\prod_{i=1}^2 M_{\omega,s}(f_i)(x) \\ &+\sum_{i=1}^2\|b\|_{\mathrm{BMO}(\omega)}\omega(x)M_{\delta_2}(\mathscr{M}_b^{(i)}(\vec{f}))(x), \end{aligned}$$

for any $1 < s < \infty$ *and bounded compact supported functions* f_1, f_2.

Proof. First of all, we give the definition of the following auxiliary maximal function, which has been studied in [20] and [21] for the linear case. Let $\varphi(x) \geqslant 0$ be a smooth function such that $\varphi_\varepsilon(t) = \varepsilon^{-2n}\varphi(\frac{t}{\varepsilon})$, $|\varphi'(t)| \lesssim t^{-1}$ and $\chi_{[0,1]}(t) \leqslant \varphi(t) \leqslant \chi_{[0,2]}(t)$.

Let

$$\Phi(f_1,f_2)(x) = \sup_{\varepsilon>0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varphi_\varepsilon(|x-y_1|+|x-y_2|)\prod_{i=1}^2|f_i(y_i)|dy_1dy_2,$$

and

$$\Phi_{\Pi\vec{b}}(f_1,f_2)(x) = \sup_{\varepsilon>0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varphi_\varepsilon(|x-y_1|+|x-y_2|)\prod_{i=1}^2|b(x)-b(y_i)||f_i(y_i)|dy_1dy_2.$$

We first show that

$$\Phi(f_1,f_2)(x) \approx \mathscr{M}(f_1,f_2)(x).$$

In fact, let $B_\varepsilon = \{y \in \mathbb{R}^n : |x-y| \leqslant \varepsilon\}$. It is easy to see that

$$B_{\frac{\varepsilon}{2}} \times B_{\frac{\varepsilon}{2}} \subset \big\{(y_1,y_2) : |x-y_1|+|x-y_2| \leqslant \varepsilon\big\} \subset B_\varepsilon \times B_\varepsilon.$$

The bounded compact supported condition of φ gives

$$\begin{aligned} \Phi(f_1,f_2)(x) &= \sup_{\varepsilon>0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\varphi_\varepsilon(|x-y_1|+|x-y_2|)|f_1(y_1)||f_2(y_2)|dy_1dy_2 \\ &\leqslant \sup_{\varepsilon>0}\frac{1}{\varepsilon^{2n}}\int_{B_\varepsilon}\int_{B_\varepsilon}\varphi\Big(\frac{|x-y_1|+|x-y_2|}{\varepsilon}\Big)|f_1(y_1)||f_2(y_2)|dy_1dy_2 \\ &\lesssim \mathscr{M}(f_1,f_2)(x) \end{aligned}$$

and

$$\begin{aligned}\Phi(f_1,f_2)(x) &\geqslant \sup_{\varepsilon>0}\frac{1}{\varepsilon^{2n}}\int_{B_{\frac{\varepsilon}{2}}}\int_{B_{\frac{\varepsilon}{2}}}\varphi\Big(\frac{|x-y_1|+|x-y_2|}{\varepsilon}\Big)|f_1(y_1)||f_2(y_2)|dy_1dy_2\\ &\gtrsim \mathscr{M}(f_1,f_2)(x).\end{aligned}$$

We can also obtain that $\Phi_{\Pi\vec{b}}(f_1,f_2)(x)\approx\mathscr{M}_{\Pi\vec{b}}(f_1,f_2)(x)$.

Now, we shall estimate the sharp maximal function of the auxiliary maximal function. Let Q be a cube and $x\in Q$. Then, for any $z\in Q$ we have

$$\begin{aligned}\big|\Phi_{\Pi\vec{b}}(f_1,f_2)(z)-c_Q\big| &\leqslant \big|b(z)-b_Q\big|^2\Phi(f_1,f_2)(z)\\ &\quad+\big||b(z)-b_Q|\Phi(f_1,(b-b_Q)f_2)(z)\big|\\ &\quad+\big||b(z)-b_Q|\Phi((b-b_Q)f_1,f_2)(z)\big|\\ &\quad+\big|\Phi((b-b_Q)f_1,(b-b_Q)f_2)(z)-c_Q\big|\\ &=: A_1^Q(z)+A_2^Q(z)+A_3^Q(z)+A_4^Q(z),\end{aligned}$$

where $c_Q=\big(\Phi((b-b_Q)f_1^\infty,(b-b_Q)f_2^\infty)\big)_Q$ and f_i^∞ will be defined later.

Therefore,

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q\Big|\big|\Phi_{\Pi\vec{b}}(f_1,f_2)(z)\big|^{\delta_1}-|c_Q|^{\delta_1}\Big|dz\right)^{1/\delta_1} &\lesssim \left(\frac{1}{|Q|}\int_Q\Big|\Phi_{\Pi\vec{b}}(f_1,f_2)(z)-c_Q|^{\delta_1}\Big|\right)^{1/\delta_1}\\ &\lesssim \sum_{j=1}^4 A_j,\end{aligned}$$

where $A_j=\big(\frac{1}{|Q|}\int_Q\big(A_j^Q(z)\big)^\delta dz\big)^{1/\delta}$, $j=1,2,3,4$.

Let us consider first the term A_1. By averaging A_1^Q over Q, we get

$$\begin{aligned}A_1 &= \left(\frac{1}{|Q|}\int_Q\Big(|b(z)-b_Q|^2\Phi(f_1,f_2)(z)\Big)^{\delta_1}dz\right)^{1/\delta_1}\\ &\lesssim \|b\|^2_{\mathrm{BMO}(\omega)}\omega(x)^2M_\varepsilon(\mathscr{M}(f_1,f_2))(x).\end{aligned}$$

Let us consider next the term A_2. We write

$$\begin{aligned}A_2^Q(z) &= \big|b(z)-b_Q|\Phi(f_1,|b-b_Q|f_2)(z)\\ &\leqslant \big|b(z)-b_Q|\Phi(f_1,\big(|b(z)-b_Q|+|b(z)-b|\big)f_2)(z)\\ &\leqslant \big|b(z)-b_Q\big|^2\mathscr{M}(f_1,f_2)(z)+\big|b(z)-b_Q|\mathscr{M}_b^{(2)}(f_1,f_2)(z)\\ &=: A_{21}^Q(z)+A_{22}^Q(z).\end{aligned}$$

For $A_{21}^Q(z)$, the fact that $\Phi(f_1,f_2)(z)\lesssim\mathscr{M}(f_1,f_2)(z)$ gives

$$A_{21}:=\left(\frac{1}{|Q|}\int_Q\big(A_{21}^Q(z)\big)^\delta dz\right)^{1/\delta}\lesssim\|b\|^2_{\mathrm{BMO}(\omega)}\omega(x)^2M(\mathscr{M}(f_1,f_2))(x).$$

For $A_{22}^Q(z)$,

$$\begin{aligned} A_{22} &:= \left(\frac{1}{|Q|}\int_Q \left(A_{22}^Q(z)\right)^{\delta_1} dz\right)^{1/\delta_1} \\ &\lesssim \omega(x)\|b\|_{\mathrm{BMO}(\omega)} \frac{1}{|Q|^2}\left(\int_Q \left|\mathscr{M}_b^{(2)}(f_1,f_2)(z)\right|^{\delta_2}\right)^{1/\delta_2} \\ &\lesssim \omega(x)\|b\|_{\mathrm{BMO}(\omega)} M_{\delta_2}(\mathscr{M}_b^{(2)}(f_1,f_2))(x). \end{aligned}$$

The same process also follows that

$$A_3 \lesssim \|b\|_{\mathrm{BMO}(\omega)}^2 \omega(x)^2 M_\varepsilon(\mathscr{M}(f_1,f_2))(x) + \omega(x)\|b\|_{\mathrm{BMO}(\omega)} M_{\delta_2}(\mathscr{M}_b^{(1)}(f_1,f_2))(x).$$

To estimate A_4, we split f_j to $f_j = f_j^0 + f_j^\infty$ with $f_j^0 = f_j\chi_{2Q}$. We write

$$\begin{aligned} A_4^Q \leqslant\ & \left|\Phi((b-b_Q)f_1^0,(b-b_Q)f_2^0)(z)\right| \\ & + \left|\Phi(((b-b_Q)f_1^0,(b-b_Q)f_2^\infty)(z)\right| \\ & + \left|\Phi((b-b_Q)f_1^\infty,(b-b_Q)f_2^0)(z)\right| \\ & + \left|\Phi((b-b_Q)f_1^\infty,(b-b_Q)f_2^\infty)(z) - c_Q\right| \\ =:\ & A_{41}^Q(z) + A_{42}^Q(z) + A_{43}^Q(z) + A_{44}^Q(z). \end{aligned}$$

Then

$$\begin{aligned} A_4 &\leqslant \left(\frac{1}{|Q|}\int_Q \Big(\sum_{j=1}^4 A_{4j}^Q(z)\Big)^{\delta_1} dz\right)^{1/\delta_1} \\ &\lesssim \sum_{j=1}^4 \left(\frac{1}{|Q|}\int_Q \left(A_{4j}^Q(z)\right)^{\delta_1} dz\right)^{1/\delta_1} \\ &\lesssim \sum_{j=1}^4 A_{4j}. \end{aligned}$$

By Kolmogorov inequality and the fact that $\mathscr{M}$ is bounded from $L^1 \times L^1$ to $L^{1/2,\infty}$, we have

$$\begin{aligned} A_{41} &\leqslant \frac{C}{|Q|^2}\|\Phi((b-b_Q)f_1^0,(b-b_Q)f_2^0)\|_{L^{1/2,\infty}} \\ &\leqslant \frac{C}{|Q|^2}\|\mathscr{M}((b-b_Q)f_1^0,(b-b_Q)f_2^0)\|_{L^{1/2,\infty}} \\ &\leqslant \frac{C}{|Q|^2}\prod_{i=1}^2 \int_{2Q} |b(y_i)-b_Q||f_i(y_i)|dy_i \\ &\leqslant \frac{C}{|Q|^2}\prod_{i=1}^2 \left(\int_{2Q} |b(y_i)-b_Q|^{s'}\omega^{1-s'}(y_i)dy_i\right)^{1/s'} \left(\int_{2Q} |f_i(y_i)|^s \omega(y_i)dy_i\right)^{1/s} \\ &\lesssim \prod_{i=1}^2 \|b\|_{\mathrm{BMO}^{s'}(\omega)} \omega(x) M_{\omega,s}(f_i)(x). \end{aligned}$$

For A_{42}, it is easy to see that

$$\varphi_\varepsilon(|z-y_1|+|z-y_2|) \lesssim \frac{1}{\big(|z-y_1|+|z-y_2|\big)^{2n}},$$

then

$$\begin{aligned}
A_{42} &\lesssim \frac{1}{|Q|}\int_Q\int_{2Q}\int_{\mathbb{R}^n\setminus 2Q} \frac{|b(y_1)-b_Q||f_1(y_1)||b(y_2)-b_Q||f_2(y_2)|}{\big(|z-y_1|+|z-y_2|\big)^{2n}}dy_1dy_2dz\\
&\lesssim \int_Q\int_{2Q}\int_{\mathbb{R}^n\setminus 2Q} \frac{|b(y_1)-b_Q||f_1(y_1)||b(y_2)-b_Q||f_2(y_2)|}{\big(|z-y_1|+|z-y_2|\big)^{2n}}dy_1dy_2dz\\
&\lesssim \frac{1}{|Q|}\int_{2Q}|b(y_1)-b_Q||f_1(y_1)|dy_1\int_Q\int_{\mathbb{R}^n\setminus 2Q}\frac{|b(y_2)-b_Q||f_2(y_2)|}{|z-y_2|^{2n}}dy_2dz\\
&\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)\sum_{k=1}^{\infty}\frac{2^{-kn}}{|2^kQ|}\int_{2^kQ}|b(y_2)-b_Q||f_2(y_2)|dy_2\\
&\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)\sum_{k=1}^{\infty}\frac{2^{-kn}}{|2^kQ|}\\
&\quad\times\left[\int_{2^kQ}|b(y_1)-m_{2^kQ}(b)||f_2(y_2)|dy_2+\int_{2^kQ}|m_{2^kQ}(b)-b_Q||f_2(y_2)|dy_2\right]\\
&\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)\sum_{k=1}^{\infty}\frac{2^{-kn}}{|2^kQ|}\\
&\quad\times\left[\|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_2)(x)+k\|b\|_{\mathrm{BMO}(\omega)}\omega(x)M(f_2)(x)\right]\\
&\lesssim \prod_{i=1}^{2}\|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_i)(x).
\end{aligned}$$

Similarly, for A_{43}, we have

$$A_{43} \lesssim \prod_{i=1}^{2}\|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_i)(x).$$

For $|z-z'| \leqslant \frac{1}{2}\max\{|z-y_1|,|z-y_2|\}$,

$$\big|\varphi_\varepsilon(|z-y_1|+|z-y_2|)-\varphi_\varepsilon(|z'-y_1|+|z'-y_2|)\big| \lesssim \frac{|z-z'|}{\big(|z-y_1|+|z-y_2|\big)^{2n+1}}.$$

Therefore,

$$\begin{aligned}
&\big|\Phi((b(z)-b)f_1^\infty,(b(z)-b)f_2^\infty)(z)-\Phi((b(z)-b)f_1^\infty,(b(z)-b)f_2^\infty)(z')\big|\\
&\lesssim \sup_{\varepsilon>0}\int_{\mathbb{R}^n\setminus 2Q}\int_{\mathbb{R}^n\setminus 2Q}\Big|\varphi_\varepsilon(|z-y_1|+|z-y_2|)-\varphi_\varepsilon(|z'-y_1|+|z'-y_2|)\Big|\\
&\quad\times\prod_{i=1}^{2}|b(y_i)-b_Q||f_i(y_i)|dy_1dy_2
\end{aligned}$$

$$\lesssim \prod_{i=1}^{2} \int_{\mathbb{R}^n \setminus 2Q} \frac{|z-z'|^{\varepsilon_i}}{|z-y_i|^{n+\varepsilon_i}} |b(y_i)-b_Q||f_i(y_i)|dy_i$$
$$\lesssim \prod_{i=1}^{2} \sum_{k=1}^{\infty} \frac{-2^{kn\varepsilon_i}}{|2^kQ|} \int_{2^kQ} |b(y_i)-b_Q||f_i(y_i)|dy_i$$
$$\lesssim \prod_{i=1}^{2} \|b\|_{\mathrm{BMO}^{s'}(\omega)} \omega(x) M_{\omega,s}(f_i)(x),$$

where $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 = 1$.

Collecting our estimates, we have shown that

$$\begin{aligned} M^{\sharp}_{\delta_1}\left(\mathscr{M}_{\Pi\vec{b}}(\vec{f})\right)(x) \lesssim \ & \|b\|^2_{\mathrm{BMO}(\omega)} \omega(x)^2 M_{\varepsilon}(\mathscr{M}(\vec{f})(x)) \\ & + \|b\|^2_{\mathrm{BMO}(\omega)} \omega(x)^2 \prod_{i=1}^{2} M_{\omega,s}(f_i)(x) \\ & + \sum_{i=1}^{2} \|b\|_{\mathrm{BMO}(\omega)} \omega(x) M_{\delta_2}(\mathscr{M}_b^{(i)}(\vec{f}))(x), \end{aligned}$$

for any $1 < s < \infty$ and bounded compact supported functions f_1, f_2. $\square$

LEMMA 2. *Let* $0 < \delta_2 < \varepsilon < 1/2$, $\omega \in A_1$, $\vec{b} = (b,b)$ *and* $b \in \mathrm{BMO}(\omega)$. *Then there exist a constant* C *such that*

$$\begin{aligned} M^{\sharp}_{\delta_2}\left(\mathscr{M}^{(1)}_b(\vec{f})\right)(x) \lesssim \ & \|b\|_{\mathrm{BMO}(\omega)} \omega(x) M_{\varepsilon}(\mathscr{M}(\vec{f})(x)) \\ & + \|b\|_{\mathrm{BMO}(\omega)} \omega(x) M_{\omega,s}(f_1)(x) M(f_2)(x), \end{aligned}$$

for any $1 < s < \infty$ *and bounded compact supported functions* f_1, f_2.

Proof. Let Q be a cube and $x \in Q$. Then, for $z \in Q$ we have

$$\begin{aligned} \left|\Phi^{(1)}_b(f_1,f_2)(z) - c_Q\right| \leqslant \ & \left|b(z)-b_Q\right|\Phi(f_1,f_2)(z) \\ & + \left|\Phi((b-b_Q)f_1,f_2)(z) - c_Q\right| \\ =: \ & B^Q_1(z) + B^Q_2(z). \end{aligned}$$

Therefore,

$$\begin{aligned} & \left(\frac{1}{|Q|}\int_Q \left|\left|\Phi^{(1)}_b(f_1,f_2)(z)\right|^{\delta_2} - |c_Q|^{\delta_2}\right| dz\right)^{1/\delta_2} \\ & \lesssim \left(\frac{1}{|Q|}\int_Q \left|\Phi^{(1)}_b(f_1,f_2)(z) - c_Q\right|^{\delta_2} dz\right)^{1/\delta_2} \\ & \lesssim \sum_{j=1}^{2} B_j, \end{aligned}$$

where $B_j = \left(\frac{1}{|Q|}\int_Q \left(B_j^Q(z)\right)^{\delta_2} dz\right)^{1/\delta_2}$, $j=1,2$.

Let us consider first the term B_1. By averaging B_1^Q over Q, we get

$$\begin{aligned} B_1 &= \left(\frac{1}{|Q|}\int_Q \left(|b(z)-b_Q|\Phi(f_1,f_2)(z)\right)^{\delta_2} dz\right)^{1/\delta_2} \\ &\lesssim \|b\|_{\mathrm{BMO}(\omega)}\omega(x)M_\varepsilon(\mathscr{M}(f_1,f_2))(x). \end{aligned}$$

Let us consider next the term B_2. We split f_j to $f_j = f_j^0 + f_j^\infty$ with $f_j^0 = f_j\chi_{2Q}$. We write

$$\begin{aligned} B_2^Q \leqslant\ & \left|\Phi((b-b_Q)f_1^0, f_2^0)(z)\right| + \left|\Phi(((b-b_Q)f_1^0, f_2^\infty)(z)\right| \\ &+ \left|\Phi((b-b_Q)f_1^\infty, f_2^0)(z)\right| + \left|\Phi((b-b_Q)f_1^\infty, f_2^\infty)(z) - c_Q\right| \\ =:\ & B_{21}^Q(z) + B_{22}^Q(z) + B_{23}^Q(z) + B_{24}^Q(z). \end{aligned}$$

By Kolmogorov inequality and the fact that $\mathscr{M}$ is bounded from $L^1 \times L^1$ to $L^{1/2,\infty}$, we have

$$\begin{aligned} B_{21} &\leqslant \frac{C}{|Q|^2}\|\Phi((b-b_Q)f_1^0, f_2^0)\|_{L^{1/2,\infty}} \\ &\lesssim \frac{1}{|Q|^2}\|\mathscr{M}((b-b_Q)f_1^0, f_2^0)\|_{L^{1/2,\infty}} \\ &\lesssim \frac{1}{|Q|^2}\int_{2Q}|b(y_1)-b_Q||f_1(y_1)|dy_1\int_{2Q}|f_2(y_2)|dy_2 \\ &\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)M(f_2)(x). \end{aligned}$$

For B_{22},

$$\begin{aligned} B_{22} &\lesssim \frac{1}{|Q|}\int_Q\int_{2Q}\int_{\mathbb{R}^n\setminus 2Q}\frac{|b(y_1)-b_Q||f_1(y_1)||f_2(y_2)|}{\left(|z-y_1|+|z-y_2|\right)^{2n}}dy_1dy_2dz \\ &\lesssim \frac{1}{|Q|}\int_Q\int_{2Q}\int_{\mathbb{R}^n\setminus 2Q}\frac{|b(y_1)-b_Q||f_1(y_1)||f_2(y_2)|}{\left(|z-y_1|+|z-y_2|\right)^{2n}}dy_1dy_2dz \\ &\lesssim \frac{1}{|Q|}\int_{2Q}|b(y_1)-b_Q||f_1(y_1)|dy_1\int_Q\int_{\mathbb{R}^n\setminus 2Q}\frac{|f_2(y_2)|}{|z-y_2|^{2n}}dy_2dz \\ &\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)\sum_{k=1}^\infty\frac{2^{-kn}}{|2^kQ|}\int_{2^kQ}|f_2(y_2)|dy_2 \\ &\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)M(f_2)(x). \end{aligned}$$

For B_{23}, we have

$$\begin{aligned} B_{23} &\lesssim \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus 2Q}\int_{2Q}\frac{|b(y_1)-b_Q||f_1(y_1)||f_2(y_2)|}{\left(|z-y_1|+|z-y_2|\right)^{2n}}dy_1dy_2dz \\ &\lesssim \frac{1}{|Q|}\int_Q\int_{2Q}\int_{\mathbb{R}^n\setminus 2Q}\frac{|b(y_1)-b_Q||f_1(y_1)||f_2(y_2)|}{\left(|z-y_1|+|z-y_2|\right)^{2n}}dy_1dy_2dz \end{aligned}$$

$$\begin{aligned}&\lesssim \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\backslash 2Q}\frac{|b(y_1)-b_Q||f_1(y_1)|}{|z-y_1|^{2n}}dy_1dz\int_{2Q}|f_2(y_2)|dy_2\\&\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)M(f_2)(x).\end{aligned}$$

Concerning the last estimate for B_{24}. For any $z'\in Q$ and $y_1,y_2\in\mathbb{R}^n\backslash 2Q$, we have

$$\begin{aligned}&\left|\Phi((b-b_Q)f_1^\infty,f_2^\infty)(z)-\Phi((b-b_Q)f_1^\infty,f_2^\infty)(z')\right|\\&\lesssim \sup_{\varepsilon>0}\int_{\mathbb{R}^n\backslash 2Q}\int_{\mathbb{R}^n\backslash 2Q}\left|\varphi_\varepsilon(|z-y_1|+|z-y_2|)-\varphi_\varepsilon(|z'-y_1|+|z'-y_2|)\right|\\&\quad\times|b(y_1)-b_Q||f_1(y_1)||f_2(y_2)|dy_1dy_2\\&\lesssim \int_{\mathbb{R}^n\backslash 2Q}\frac{|b(y_1)-b_Q||f_1(y_1)|}{|z-y_1|^{2n+\varepsilon_1}}dy_1\int_{\mathbb{R}^n\backslash 2Q}\frac{|f_2(y_2)|}{|z-y_2|^{\varepsilon_2}}dy_2\\&\lesssim \sum_{k=2}^\infty\frac{2^{-kn\varepsilon_1}}{|2^kQ|}\int_{2^kQ}|b(y_1)-b_Q||f_1(y_1)|dy_1\sum_{i=2}^\infty\frac{2^{-kn\varepsilon_2}}{|2^kQ|}\int_{2^iQ}|f_2(y_2)|dy_2\\&\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)M(f_2)(x).\end{aligned}$$

where $\varepsilon_1,\varepsilon_2>0$ with $\varepsilon_1+\varepsilon_2=1$. Taking the mean over Q for z and z' respectively, we obtain

$$\begin{aligned}B_{24}&\lesssim \frac{1}{|Q|}\int_Q\left|\Phi((b-b_Q)f_1^\infty,f_2^\infty)(z)-c_Q\right|dz\\&\lesssim \frac{1}{|Q|}\int_Q\frac{1}{|Q|}\int_Q\left|\Phi((b-b_Q)f_1^\infty,f_2^\infty)(z)-\Phi((b-b_Q)f_1^\infty,f_2^\infty)(z')\right|dzdz'\\&\lesssim \|b\|_{\mathrm{BMO}^{s'}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)M(f_2)(x).\end{aligned}$$

Collecting our estimates, we have shown that

$$\begin{aligned}M^\sharp_{\frac12}\left(\mathscr{M}_b^{(1)}(\vec f)\right)(x)&\lesssim \|b\|_{\mathrm{BMO}(\omega)}\omega(x)M(\mathscr{M}(\vec f)(x))\\&\quad+\|b\|_{\mathrm{BMO}(\omega)}\omega(x)M_{\omega,s}(f_1)(x)M(f_2)(x),\end{aligned}$$

for any $1<s<\infty$ and bounded compact supported functions f_1,f_2. □

Similarly, we have

LEMMA 3. *Let $0<\delta_2<\varepsilon<1/2$, $\omega\in A_1$, $\vec b=(b,b)$ and $b\in\mathrm{BMO}(\omega)$. Then there exist a constant C such that*

$$\begin{aligned}M^\sharp_{\delta_2}\left(\mathscr{M}_b^{(2)}(\vec f)\right)(x)&\lesssim \|b\|_{\mathrm{BMO}(\omega)}\omega(x)M_\varepsilon(\mathscr{M}(\vec f)(x))\\&\quad+\|b\|_{\mathrm{BMO}(\omega)}\omega(x)M_{\omega,s}(f_2)(x)M(f_1)(x),\end{aligned}$$

for any $1<s<\infty$ and bounded compact supported functions f_1,f_2.

LEMMA 4. *Let* $\omega \in A_1$ *and* $0 < p < \infty$. *Then* $\omega^{1-p} \in A_\infty$.

Proof. If $0 < p \leqslant 1$, then $1-p \in [0,1)$. It is easy to see that $\omega^{1-p} \in A_1 \subset A_\infty$. If $1 < p < \infty$, if follows from $\omega \in A_1 \subset A_p$ that $\omega^{1-p} \in A_{p'} \subset A_\infty$. □

LEMMA 5. *Let* $\omega \in A_1$, $1 < s < p_1, p_2 < \infty$ *and* $1/p = 1/p_1 + 1/p_2$. *Then both* $\mathscr{M}(\vec{f})$ *and* $\prod_{i=1}^{2} M_{\omega,s}(f_i)$ *are bounded from* $L^{p_1}(\omega) \times L^{p_2}(\omega)$ *to* $L^p(\omega)$.

Proof. From the fact that $M(f)(x) \lesssim M_{\omega,s}(f)(x)$ and $M_{\omega,s}(f)(x)$ is bounded on $L^p(\omega)$ for $1 < s < p_1, p_2 < \infty$, it is easy to obtain that both $\mathscr{M}(\vec{f})$ and $\prod_{i=1}^{2} M_{\omega,s}(f_i)$ are bounded from $L^{p_1}(\omega) \times L^{p_2}(\omega)$ to $L^p(\omega)$. □

The following relationships between M_δ and $M^\sharp$ to be used is a version of the classical ones due to Fefferman and Stein [10].

LEMMA 6. *Let* $0 < p, \delta < \infty$ *and* $\omega \in A_\infty$. *There exist a positive* C *such that*

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \omega(x) dx \leqslant C \int_{\mathbb{R}^n} (M_\delta^\sharp f(x))^p \omega(x) dx,$$

for any smooth function f *for which the left-hand side is finite.*

LEMMA 7. *Let* Q_0 *be any fixed cube and* b *be a locally integral function. Then, for any* $x \in Q_0$, *we get*

$$\mathscr{M}(\chi_{Q_0}, \chi_{Q_0})(x) \equiv 1; \tag{1}$$

$$\mathscr{M}(b\chi_{Q_0}, \chi_{Q_0})(x) = \mathscr{M}(\chi_{Q_0}, b\chi_{Q_0})(x) = \mathscr{M}_{Q_0}(b)(x); \tag{2}$$

$$\mathscr{M}(b\chi_{Q_0}, b\chi_{Q_0})(x) = \mathscr{M}_{Q_0}^2(b)(x), \tag{3}$$

where $M_{Q_0}(b)(x) = \sup_{Q_0 \supset Q \ni x} \frac{1}{|Q|} \int_Q |b(y)| dy$.

Proof. We only give the proof of (3) and the proof of (1),(2) are similar. For any $x \in Q_0$, we have

$$\begin{aligned} M_{Q_0}^2(b)(x) &= \Big(\sup_{Q_0 \supset Q \ni x} \frac{1}{|Q|} \int_Q |b(y)| dy \Big)^2 \\ &= \sup_{Q_0 \supset Q \ni x} \frac{1}{|Q|} \int_Q |b(y_1)| \chi_{Q_0}(y_1) dy_1 \cdot \frac{1}{|Q|} \int_Q |b(y_2)| \chi_{Q_0}(y_2) dy_2 \\ &\leqslant \mathscr{M}(b\chi_{Q_0}, b\chi_{Q_0})(x). \end{aligned}$$

On the other hand, for any cube $Q \subset \mathbb{R}^n$, we can construct a cube Q_1 such that

$$Q_0 \supset Q_1 \supset Q_0 \cap Q \ni x$$

and $|Q_1| \leqslant |Q|$. Therefore,

$$\frac{1}{|Q|} \int_{Q \cap Q_0} |b(y)| dy \leqslant \frac{1}{|Q_1|} \int_{Q_1} |b(y)| dy \leqslant M_{Q_0}(b)(x).$$

Thus,

$$\mathscr{M}(b\chi_{Q_0}, b\chi_{Q_0})(x) = \sup_{Q \ni x}\Big(\frac{1}{|Q|}\int_Q |b(y)|\chi_{Q_0}(y)dy\Big)^2 \leqslant M^2_{Q_0}(b)(x),$$

then (3) is proved. □

4. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. $(A1) \Rightarrow (A2)$: It is enough to prove Theorem 1 for f_1, f_2 being bounded functions with compact support. We observe that to use the Fefferman-Stein inequality, one needs to verify that certain terms in the left-hand side of the inequalities are finite. Applying a similar argument as in [13, pp. 32–33], the boundedness properties of $\mathscr{M}$ and Fatou's lemma, one gets the desired result.

Since Lemma 4 and $\omega \in A_1$, then $\omega^{1-p} \in A_\infty$. By Lemma 2 and Lemma 3 with $1 < s < \min\{p_1, p_2\}$, from a standard argument that we can obtain for $0 < \delta_2 < \varepsilon < 1/2$,

$$\begin{aligned}
\|\mathscr{M}_{\Sigma\vec{b}}(\vec{f})\|_{L^p(\omega^{1-p})} &\lesssim \|M_{\delta_2}(\mathscr{M}_{\Sigma\vec{b}}(\vec{f}))\|_{L^p(\omega^{1-p})} \lesssim \|M^\sharp_{\delta_2}(\mathscr{M}_{\Sigma\vec{b}}(\vec{f}))\|_{L^p(\omega^{1-p})}\\
&\lesssim \|b\|_{\mathrm{BMO}(\omega)}\Big(\big\|M_\varepsilon(\mathscr{M}(\vec{f}))\big\|_{L^p(\omega)} + \Big\|\prod_{i=1}^2 M_{\omega,s}(f_i)\Big\|_{L^p(\omega)}\Big)\\
&\lesssim \|b\|_{\mathrm{BMO}(\omega)}\prod_{i=1}^2\|f_i\|_{L^{p_i}(\omega)}.
\end{aligned}$$

$(A2) \Rightarrow (A1)$: Let Q be any fixed cube. Suppose that $\mathscr{M}_{\Sigma\vec{b}}$ is bounded from $L^{p_1}(\omega) \times L^{p_2}(\omega)$ into $L^p(\omega^{1-p})$, then

$$\|\mathscr{M}_{\Sigma\vec{b}}(\chi_Q, \chi_Q)\|_{L^p(\omega^{1-p})} \lesssim \|\chi_Q\|_{L^{p_1}(\omega)}\|\chi_Q\|_{L^{p_2}(\omega)} \lesssim \omega(Q)^{\frac{1}{p}},$$

which implies that

$$\begin{aligned}
&\frac{2}{\omega(Q)}\int_Q |b(x) - b_Q|dx\\
&\leqslant \frac{1}{\omega(Q)}\int_Q |Q|^{-2}\int_Q\int_Q |b(x) - b(y_1)|\chi_Q(y_1)\chi_Q(y_2)dy_1dy_2dx\\
&\qquad + \frac{1}{\omega(Q)}\int_Q |Q|^{-2}\int_Q\int_Q |b(x) - b(y_2)|\chi_Q(y_1)\chi_Q(y_2)dy_1dy_2dx\\
&\lesssim \frac{1}{\omega(Q)}\int_Q \mathscr{M}_{\Sigma\vec{b}}(\chi_Q, \chi_Q)(x)dx\\
&\lesssim \frac{1}{\omega(Q)}\Big(\int_Q \big|\mathscr{M}_{\vec{b}}(\chi_Q, \chi_Q)(x)\big|^p \omega(x)^{1-p}dx\Big)^{1/p}\Big(\int_Q \omega(x)dx\Big)^{1/p'}\\
&\lesssim \frac{1}{\omega(Q)^{1/p}}\|\mathscr{M}_{\Sigma\vec{b}}(\chi_Q, \chi_Q)\|_{L^p(\omega^{1-p})}\\
&\lesssim \|\mathscr{M}_{\Sigma\vec{b}}\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-p})}.
\end{aligned}$$

Thus showing that $b \in \mathrm{BMO}(\omega)$.

$(A1) \Rightarrow (A3)$: Since $\omega \in A_1$, Lemma 4 implies that $\omega^{1-2p} \in A_\infty$. From Lemma 1, Lemma 2 and Lemma 3 with $1 < s < \min\{p_1, p_2\}$, we get

$$\begin{aligned}\|\mathscr{M}_{\Pi\vec{b}}(\vec{f})\|_{L^p(\omega^{1-2p})} &\lesssim \|M_{\delta_1}\left(\mathscr{M}_{\Pi\vec{b}}(\vec{f})\right)\|_{L^p(\omega^{1-2p})} \lesssim \|M^\sharp_{\delta_1}\left(\mathscr{M}_{\Pi\vec{b}}(\vec{f})\right)\|_{L^p(\omega^{1-2p})}\\ &\lesssim \|b\|^2_{\mathrm{BMO}(\omega)}\left(\big\|M_\varepsilon\big(\mathscr{M}(\vec{f})\big)\big\|_{L^p(\omega)} + \Big\|\prod_{i=1}^2 M_{\omega,s}(f_i)\Big\|_{L^p(\omega)}\right)\\ &\quad + \sum_{i=1}^2 \|b\|_{\mathrm{BMO}(\omega)}\big\|M_{\delta_2}(\mathscr{M}^{(i)}_b(\vec{f}))(x)\big\|_{L^p(\omega^{1-p})}\\ &\lesssim \|b\|^2_{\mathrm{BMO}(\omega)}\prod_{i=1}^2\|f_i\|_{L^{p_i}(\omega)}.\end{aligned}$$

$(A3) \Rightarrow (A1)$: By $\mathscr{M}_{\Pi\vec{b}}$ is bounded from $L^{p_1}(\omega)\times L^{p_2}(\omega)$ into $L^p(\omega^{1-2p})$, we get

$$\begin{aligned}&\frac{1}{\omega(Q)}\int_Q |b(x)-b_Q|^2\omega(x)^{-1}dx\\ &\lesssim \frac{1}{\omega(Q)}\int_Q \omega(x)^{-1}|Q|^{-2}\int_Q\int_Q |b(x)-b(y_1)||b(x)-b(y_2)|dy_1dy_2dx\\ &\lesssim \frac{1}{\omega(Q)}\int_Q \mathscr{M}_{\Pi\vec{b}}(\chi_Q,\chi_Q)(x)\omega(x)^{-1}dx\\ &\lesssim \frac{1}{\omega(Q)}\Big(\int_Q \big|\mathscr{M}_{\Pi\vec{b}}(\chi_Q,\chi_Q)(x)\big|^p\omega(x)^{1-2p}dx\Big)^{1/p}\Big(\int_Q\omega(x)dx\Big)^{1/p'}\\ &\lesssim \frac{1}{\omega(Q)^{1/p}}\|\mathscr{M}_{\Pi\vec{b}}(\chi_Q,\chi_Q)\|_{L^p(\omega^{1-2p})}\\ &\lesssim \|\mathscr{M}_{\Pi\vec{b}}\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-2p})}.\end{aligned}$$

Thus we complete the proof of Theorem 1. □

Proof of Theorem 2. $(B1) \Rightarrow (B2)$: By the definition of $\mathscr{M}(\vec{f})$, we have

$$M(bf_1,f_2)(x) = M(|b|f_1,f_2)(x),\ \ M(f_1,bf_2)(x) = M(f_1,|b|f_2)(x).$$

Then

$$\begin{aligned}&\big|[b,\mathscr{M}]^{(1)}(\vec{f})(x) - [|b|,\mathscr{M}]^{(1)}(\vec{f})(x)\big|\\ &\lesssim \Big|b(x)\mathscr{M}(\vec{f})(x) - \mathscr{M}(bf_1,f_2)(x) - |b(x)|\mathscr{M}(\vec{f})(x) + \mathscr{M}(|b|f_1,f_2)(x)\Big|\\ &\lesssim b^-(x)\mathscr{M}(\vec{f})(x).\end{aligned}$$

Similarly, we also have $\big|[b,\mathscr{M}]^{(2)}(\vec{f})(x) - [|b|,\mathscr{M}]^{(2)}(\vec{f})(x)\big| \lesssim b^-(x)\mathscr{M}(\vec{f})(x)$. Since $\big||a|-|c|\big| \leqslant |a-c|$ for any real numbers a and c, there holds

$$\big|[|b|,\mathscr{M}]^{(i)}(f_1,f_2)(x)\big| \leqslant \mathscr{M}^{(i)}_b(f_1,f_2)(x),$$

for $i = 1,2$. This shows that

$$\left|\Sigma\vec{b},\mathscr{M}](\vec{f})(x)\right| \lesssim \mathscr{M}_{\Sigma\vec{b}}(\vec{f})(x) + b^{-}(x)\mathscr{M}(\vec{f})(x). \tag{4}$$

Applying (4) and Theorem 1 we have

$$\begin{aligned}\left\|[\Sigma\vec{b},\mathscr{M}](\vec{f})(x)\right\|_{L^p(\omega^{1-p})} &\lesssim \left\|\mathscr{M}_{\Sigma\vec{b}}(\vec{f})\right\|_{L^p(\omega^{1-p})} + \left\|b^{-}\mathscr{M}(\vec{f})\right\|_{L^p(\omega^{1-p})}\\ &\lesssim \left(\|b^{-}/\omega\|_{L^\infty} + \|b\|_{\mathrm{BMO}(\omega)}\right)\|f_1\|_{L^{p_1}(\omega)}\|f_2\|_{L^{p_2}(\omega)}.\end{aligned}$$

Therefore, $b \in \mathrm{BMO}(\omega)$ with $b^{-}/\omega \in L^\infty$ implies that $[\Sigma\vec{b},\mathscr{M}]$ is bounded from $L^{p_1}(\omega)\times L^{p_2}(\omega)$ to $L^p(\omega^{1-p})$.

$(B2) \Rightarrow (B1)$: Let Q_0 be any fixed cube. By Lemma 7, for any $x \in Q_0$,

$$b(x) = b(x)\mathscr{M}(\chi_{Q_0},\chi_{Q_0})(x),$$

$$M_{Q_0}(b)(x) = \mathscr{M}(b\chi_{Q_0},\chi_{Q_0})(x) = \mathscr{M}(\chi_{Q_0},b\chi_{Q_0})(x),$$

Then,

$$\begin{aligned}&\frac{2}{\omega(Q_0)}\int_{Q_0}|b(x)-M_{Q_0}(b)(x)|dx\\ &= \frac{2}{\omega(Q_0)}\int_{Q_0}|b(x)\mathscr{M}(\chi_{Q_0},\chi_{Q_0})(x)-\mathscr{M}(b\chi_{Q_0},\chi_{Q_0})(x)|dx\\ &= \frac{1}{\omega(Q_0)}\int_{Q_0}|b(x)\mathscr{M}(\chi_{Q_0},\chi_{Q_0})(x)-\mathscr{M}(b\chi_{Q_0},\chi_{Q_0})(x)\\ &\qquad\qquad +b(x)\mathscr{M}(\chi_{Q_0},\chi_{Q_0})(x)-\mathscr{M}(\chi_{Q_0},b\chi_{Q_0})(x)|dx\\ &\lesssim \frac{1}{\omega(Q_0)}\int_{Q_0}\left|[\Sigma\vec{b},\mathscr{M}](\chi_{Q_0},\chi_{Q_0})(x)\right|dx\\ &\lesssim \frac{1}{\omega(Q_0)}\left(\int_{Q_0}\left|[\Sigma\vec{b},\mathscr{M}](\chi_{Q_0},\chi_{Q_0})(x)\right|^p\omega(x)^{1-p}dx\right)^{1/p}\cdot\left(\int_{Q_0}\omega(x)dx\right)^{1/p'}\\ &\lesssim \frac{1}{\omega(Q_0)^{1/p}}\left\|[\Sigma\vec{b},\mathscr{M}](\chi_{Q_0},\chi_{Q_0})\right\|_{L^p(\omega^{1-p})}\\ &\lesssim \|[\Sigma\vec{b},\mathscr{M}]\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-p})}.\end{aligned}$$

Now, we have all the ingredients to prove $b \in \mathrm{BMO}(\omega)$ and $b^{-}/\omega \in L^\infty$.

$$\begin{aligned}\frac{1}{\omega(Q_0)}\int_{Q_0}|b(x)-b_{Q_0}|dx &\lesssim \frac{1}{\omega(Q_0)}\int_{Q_0}|b(x)-M_{Q_0}(b)(x)|dx\\ &\lesssim \|[\Sigma\vec{b},\mathscr{M}]\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-p})}.\end{aligned}$$

which implies that $b \in \mathrm{BMO}(\omega)$.

In order to show show that $b^{-}/\omega \in L^\infty$, observe that for any $x \in Q_0$, $M_{Q_0}(b)(x) \geqslant |b(x)|$. Therefore,

$$0 \leqslant b^{-}(x) \lesssim M_{Q_0}(b)(x) - b^{+}(x) + b^{-}(x) = M_{Q_0}(b)(x) - b(x),$$

which gives

$$\begin{aligned}\frac{1}{|Q_0|}\int_{Q_0}\frac{b^-(x)}{\omega(x)}dx &\lesssim \frac{1}{|Q_0|}\int_{Q_0} b^-(x)dx\cdot\frac{1}{\inf_{x\in Q_0}\omega(x)}\\ &\lesssim \frac{1}{|Q_0|}\int_{Q_0}|b(x)-M_{Q_0}(b)(x)|dx\cdot\frac{|Q_0|}{\omega(Q_0)}\\ &\lesssim \|[\Sigma\vec{b},\mathscr{M}]\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-p})},\end{aligned}$$

this yields that

$$(b^-/\omega)_{Q_0}\lesssim \|[\Sigma\vec{b},\mathscr{M}]\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-p})}.$$

Thus, the boundedness of b^-/ω follows from Lebesgue's differentiation theorem.

$(B1)\Rightarrow(B3)$: Let $\vec{B}=(|b|,b)$ and $\vec{\mathbb{B}}=(|b|,|b|)$. Then

$$\begin{aligned}&\Big|[\Pi\vec{b},\mathscr{M}](f_1,f_2)(x)-[\Pi\vec{B},\mathscr{M}](f_1,f_2)(x)\Big|\\ &\lesssim\Big|b(x)b(x)\mathscr{M}(\vec{f})(x)-b(x)\mathscr{M}(f_1,bf_2)(x)\\ &\qquad -|b(x)|b(x)\mathscr{M}(\vec{f})(x)+|b(x)|\mathscr{M}(f_1,bf_2)(x)\Big|\\ &\lesssim b^-(x)\big|[b,\mathscr{M}]^{(2)}(f_1,f_2)(x)\big|.\end{aligned}$$

Similarly, we also have

$$\begin{aligned}&\Big|[\Pi\vec{\mathbb{B}},\mathscr{M}](f_1,f_2)(x)-[\Pi\vec{B},\mathscr{M}](f_1,f_2)(x)\Big|\\ &\lesssim\Big||b(x)||b(x)|\mathscr{M}(\vec{f})(x)-|b(x)|\mathscr{M}(bf_1,f_2)(x)\\ &\qquad -|b(x)|b(x)\mathscr{M}(\vec{f})(x)+|b(x)|\mathscr{M}(bf_1,f_2)(x)\Big|\\ &\lesssim b^-(x)\big|[|b|,\mathscr{M}]^{(1)}(f_1,f_2)(x)\big|.\end{aligned}$$

Noting that

$$\big|[\Pi\vec{\mathbb{B}},\mathscr{M}](\vec{f})(x)\big|\leqslant\mathscr{M}_{\Pi\vec{b}}(\vec{f})(x),$$

which yields that

$$\big|[\Pi\vec{b},\mathscr{M}](\vec{f})(x)\big|\lesssim\mathscr{M}_{\Pi\vec{b}}(\vec{f})(x)+b^-(x)\mathscr{M}_{\Sigma\vec{b}}(\vec{f})(x)+(b^-(x))^2\mathscr{M}(\vec{f})(x).$$

It follows from Theorem 1 and $b^-/\omega\in L^\infty$ that

$$\begin{aligned}&\big\|[\Pi\vec{b},\mathscr{M}](\vec{f})(x)\big\|_{L^p(\omega^{1-2p})}\\ &\lesssim\big\|\mathscr{M}_{\Pi\vec{b}}(\vec{f})\big\|_{L^p(\omega^{1-2p})}+\big\|b^-\mathscr{M}_{\Sigma\vec{b}}(\vec{f})\big\|_{L^p(\omega^{1-2p})}+\big\|(b^-)^2\mathscr{M}(\vec{f})\big\|_{L^p(\omega^{1-2p})}\\ &\lesssim\|b\|^2_{\mathrm{BMO}(\omega)}\|f_1\|_{L^{p_1}(\omega)}\|f_2\|_{L^{p_2}(\omega)}+\|b^-/\omega\|_{L^\infty}\big\|\mathscr{M}_{\Sigma\vec{b}}(\vec{f})\big\|_{L^p(\omega^{1-p})}\\ &\qquad+\|b^-/\omega\|^2_{L^\infty}\|\mathscr{M}(\vec{f})\|_{L^p(\omega)}\\ &\lesssim\big(\|b^-/\omega\|_{L^\infty}+\|b\|_{\mathrm{BMO}(\omega)}\big)^2\|f_1\|_{L^{p_1}(\omega)}\|f_2\|_{L^{p_2}(\omega)},\end{aligned}$$

this leads to our results.

$(B3) \Rightarrow (B1)$: Let Q_0 be any fixed cube. By Lemma 5, for any $x \in Q_0$,

$$b(x)^2 = b(x)^2 \mathscr{M}(\chi_{Q_0}, \chi_{Q_0})(x),$$
$$b(x)M_{Q_0}(b)(x) = b(x)\mathscr{M}(b\chi_{Q_0}, \chi_{Q_0})(x) = b(x)\mathscr{M}(\chi_{Q_0}, b\chi_{Q_0})(x),$$
$$M_{Q_0}^2(b)(x) = \mathscr{M}(b\chi_{Q_0}, b\chi_{Q_0})(x).$$

Then,

$$\begin{aligned}
&\frac{1}{\omega(Q_0)}\int_{Q_0} |b(x) - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx \\
&= \frac{1}{\omega(Q_0)}\int_{Q_0} \Big(b(x)^2 - 2b(x)M_{Q_0}(b)(x) + M_{Q_0}^2(b)(x)\Big)\omega(x)^{-1} dx \\
&= \frac{1}{\omega(Q_0)}\int_{Q_0} [\Pi\vec{b}, \mathscr{M}](\chi_{Q_0}, \chi_{Q_0})(x)\omega(x)^{-1} dx \\
&\lesssim \frac{1}{\omega(Q_0)}\left(\int_{Q_0} \Big|[\Pi\vec{b}, \mathscr{M}](\chi_{Q_0}, \chi_{Q_0})(x)\Big|^p \omega(x)^{1-2p} dx\right)^{1/p} \cdot \left(\int_{Q_0} \omega(x) dx\right)^{1/p'} \\
&= \frac{1}{\omega(Q_0)^{1/p}} \big\|[\Pi\vec{b}, \mathscr{M}](\chi_{Q_0}, \chi_{Q_0})\big\|_{L^p(\omega^{1-2p})} \\
&\lesssim \|[\Pi\vec{b}, \mathscr{M}]\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-2p})}.
\end{aligned}$$

Now, we have all the ingredients to prove $b \in \mathrm{BMO}(\omega)$ and $b^-/\omega \in L^\infty$.

$$\begin{aligned}
&\frac{1}{\omega(Q_0)}\int_{Q_0} |b(x) - b_{Q_0}|^2 \omega(x)^{-1} dx \\
&\lesssim \frac{1}{\omega(Q_0)}\int_{Q_0} |b(x) - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx \\
&\qquad + \frac{1}{\omega(Q_0)}\int_{Q_0} |b_{Q_0} - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx \\
&\lesssim \frac{1}{\omega(Q_0)}\int_{Q_0} |b(x) - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx \\
&\lesssim \|[\Pi\vec{b}, \mathscr{M}]\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-2p})}.
\end{aligned}$$

which implies that $b \in \mathrm{BMO}^2(\omega)$; that is, $b \in \mathrm{BMO}(\omega)$.

For any $x \in Q_0$, we have

$$\begin{aligned}
\frac{1}{|Q_0|}\int_{Q_0} \frac{b^-(x)}{\omega(x)} dx &\lesssim \frac{1}{|Q_0|}\int_{Q_0} b^-(x) dx \cdot \frac{1}{\inf_{x\in Q_0}\omega(x)} \\
&\lesssim \frac{1}{|Q_0|}\int_{Q_0} |b(x) - M_{Q_0}(b)(x)| dx \cdot \frac{|Q_0|}{\omega(Q_0)} \\
&\lesssim \left(\frac{1}{\omega(Q_0)}\int_{Q_0} |b(x) - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx\right)^{1/2} \\
&\lesssim \|[\Pi\vec{b}, \mathscr{M}]\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-2p})},
\end{aligned}$$

which yields

$$(b^-/\omega)_{Q_0} \lesssim \|[\Pi\vec{b}, \mathscr{M}]\|_{L^{p_1}(\omega)\times L^{p_2}(\omega)\to L^p(\omega^{1-2p})}.$$

Thus, the boundedness of b^-/ω follows from Lebesgue's differentiation theorem.

The proof of Theorem 2 is complete. □

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