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TWO CHERNOFF–TYPE INEQUALITIES AND THEIR STABILITY PROPERTIES

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Abstract. Two Chernoff-type inequalities are obtained by Fourier series expansion, and a conjecture on the reverse isoperimetric inequality made by Pan et al. [Math. Inequal. Appl. **13** (2010), 329–338] is proved as an application of these inequalities. Furthermore, stability versions of the inequalities are obtained based on the Hausdorff distance and the L_2 metric, respectively.

1. Introduction

A compact convex set K in the n-dimensional Euclidean space $\mathbb{R}^n$ is called a *convex body* if it has a non-empty interior. When $n=2$, it is called a *planar convex body*. For a planar convex body K with perimeter L_K and area A_K, the classical isoperimetric inequality states that

$$L_K^2 - 4\pi A_K \geqslant 0, \tag{1}$$

where equality holds if and only if K is a disc. There have been many proofs, sharpened forms, generalizations, and applications of the classical isoperimetric inequality (1); see, for instance, [1, 4, 8, 13, 17] and the literature therein.

Let K be a planar convex body with area A_K. In 1969, Chernoff [2] proved the following inequality involving the width function $w(K,\theta)$ of K:

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\left(K,\theta+\frac{\pi}{2}\right)d\theta \geqslant 2A_K, \tag{2}$$

where equality holds if and only if K is a disc. His proof was based on a Fourier series expansion derived from Hurwitz's method (see Courant [3, p. 213] or Groemer [9, pp. 135–136]) for proving the classical isoperimetric inequality (1). Results in higher dimensions and recent works on this topic can be found in Lutwak [11], Mao and Yang [12], Ou and Pan [14], Zhang [18], and Zhang and Yang [19], among others.

Pan and Zhang [16] established the following reverse isoperimetric inequality for a planar convex body K with perimeter L_K and area A_K:

$$L_K^2 \leqslant 4\pi(A_K + |\tilde{A}_K|),$$

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where $\tilde{A}_K$ is the oriented area of K (see Section 2) and equality holds if and only if K is a disc. Pan et al. [15] improved the above result to

$$L_K^2 \leqslant 4\pi A_K + 2\pi|\tilde{A}_K|,$$

and they also conjectured that the inequality

$$L_K^2 \leqslant 4\pi A_K + \varepsilon|\tilde{A}_K| \tag{3}$$

holds for the best constant $\varepsilon = \pi$. Gao [5] showed that the inequality (3) does indeed hold and proved that it is sharp. He also pointed out that equality holds in (3) if and only if the support function of K is of the form

$$h(K,\theta) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta \tag{4}$$

for certain real numbers a_0, a_1, a_2, b_1 and b_2. This means that K is not necessarily a disc. Gao et al. [7] systematically studied the planar convex body with support function (4) and showed that its boundary is a polynomial curve with degree at most 6. For example, if the support function of K is $h(K,\theta) = 10 + 2\cos 2\theta$ (see Figure 1), then the boundary ∂K has the following simplified polynomial representation (see [7, pp. 458–459]):

$$y^6 + (3x^2 - 400)y^4 + (3x^4 - 512x^2 + 53\,248)y^2 + x^6 + 320x^4 + 12\,288x^2 = 2\,359\,296.$$

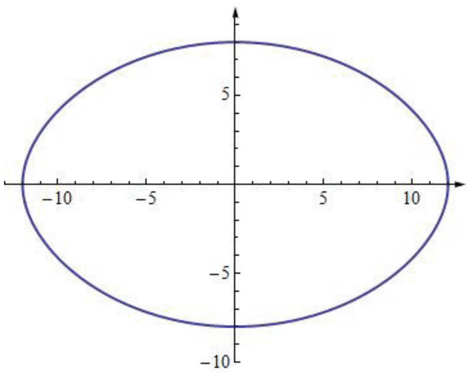

Figure 1: *Planar convex body with support function* $h(K,\theta) = 10 + 2\cos 2\theta$.

To derive a stronger version of (2), Gao and Wang [6] obtained its stability versions based on the Hausdorff distance and the L_2 metric (see Section 2) between two planar convex bodies.

In this short paper, we focus on the inequality (2) and consider the following question:

QUESTION 1. *Does the inequality* (2) *have a sharpened form and an appropriate reverse version?*

Inspired by the work of Chernoff [2] and of Ou and Pan [14], we give the answer to this question and derive the following two Chernoff-type inequalities:

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\left(K,\theta+\frac{\pi}{2}\right)d\theta \geqslant \frac{L_K^2+2\pi A_K}{3\pi} \tag{5}$$

and

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\left(K,\theta+\frac{\pi}{2}\right)d\theta \leqslant 2A_K+\frac{1}{3}|\tilde{A}_K|, \tag{6}$$

where equalities hold in (5) and (6) if and only if the planar convex body has support function (4), that is, the boundary is a polynomial curve with degree at most 6.

Obviously, the inequality (5) is a sharpened form of (2) through the classical isoperimetric inequality (1), and the inequality (6) is a reverse version of (2). As a by-product, we give another proof of the reverse isoperimetric inequality (3) using the inequalities (5) and (6).

The other aim of this paper is to derive stability versions of the two Chernoff-type inequalities based on the Hausdorff distance and the L_2 metric, respectively.

2. Preliminaries

Denote by $\mathbb{R}^n$ the usual n-dimensional Euclidean space with canonical inner product $\langle\cdot,\cdot\rangle$. A line l is called a *support line* of a planar convex body K if it passes through at least one boundary point of K and if the entire planar convex body K lies in one of the half-planes determined by l. Let $l(\theta)$ be the support line of K in the direction $\mathbf{u}(\theta)=(\cos\theta,\sin\theta)$, where θ is the oriented angle from the positive x axis to a line perpendicular to $l(\theta)$. The *support function* of K is defined by

$$h(K,\theta)=\sup_{x\in K}\langle x,\mathbf{u}(\theta)\rangle, \quad \theta\in[0,2\pi].$$

It is easy to see that $h(K,\theta)$ is the signed distance of the support line $l(\theta)$ of K with exterior normal vector $\mathbf{u}(\theta)$ from the origin.

If the support function $h(K,\theta)$ is differentiable, then the boundary of K, ∂K, can be parameterized by (see Hsiung [10, p. 115])

$$\begin{aligned} x(\theta) &= h(K,\theta)\cos\theta - h'(K,\theta)\sin\theta,\\ y(\theta) &= h(K,\theta)\sin\theta + h'(K,\theta)\cos\theta. \end{aligned}$$

The radius of curvature $\rho(\theta)$ of ∂K at the point $(x(\theta),y(\theta))$ is given by $h(K,\theta)+h''(K,\theta)$ when $h(K,\theta)$ is a twice-differentiable function. Thus, the convexity of K is equivalent to the condition $h(K,\theta)+h''(K,\theta)>0$. The *width function* of K is given by

$$w(K,\theta)=h(K,\theta)+h(K,\theta+\pi).$$

The length L_K and area A_K of a planar convex body K can be expressed as

$$L_K = \int_0^{2\pi} h(K,\theta)\,d\theta,$$
$$A_K = \frac{1}{2}\int_0^{2\pi} h(K,\theta)(h(K,\theta)+h''(K,\theta))\,d\theta = \frac{1}{2}\int_0^{2\pi}(h(K,\theta)^2 - h'(K,\theta)^2)\,d\theta.$$

They are known as *Cauchy's formula* and *Blaschke's formula* (see [10, pp. 115–116]), respectively. Since $h(K,\theta)$ is continuous, bounded, and 2π-periodic, it has a Fourier series expansion

$$h(K,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n \sin n\theta), \tag{7}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} h(K,\theta)\,d\theta$$

and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} h(K,\theta)\cos n\theta\,d\theta, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} h(K,\theta)\sin n\theta\,d\theta, \quad n\in\mathbb{Z}^+.$$

Differentiation of (7) with respect to θ gives

$$h'(K,\theta) = \sum_{n=1}^{\infty} n(-a_n\sin n\theta + b_n\cos n\theta). \tag{8}$$

Thus, by (7), (8), and Parseval's identity (see [9, p. 61 (3.1.4)]), L_K and A_K can also be expressed as

$$L_K = \pi a_0, \tag{9}$$
$$A_K = \frac{\pi a_0^2}{4} + \frac{\pi}{2}\sum_{n=2}^{\infty}(1-n^2)(a_n^2+b_n^2). \tag{10}$$

Let γ represent the locus of the centers of curvature of ∂K. Then $\gamma(\theta) = (\gamma_1(\theta),\gamma_2(\theta))$ can be expressed as

$$\begin{aligned}\gamma(\theta) &= (x(\theta),y(\theta)) - \rho(\theta)\mathbf{u}(\theta)\\ &= (-h'(K,\theta)\sin\theta - h''(K,\theta)\cos\theta, h'(K,\theta)\cos\theta - h''(K,\theta)\sin\theta).\end{aligned}$$

By Green's formula, the *oriented area* $\tilde{A}_K$ of K (i.e., the oriented area enclosed by γ) is given by (see [16, Pro.3.1])

$$\tilde{A}_K = \frac{1}{2}\int_0^{2\pi} h'(K,\theta)(h'(K,\theta)+h'''(K,\theta))\,d\theta = \frac{1}{2}\int_0^{2\pi}(h'(K,\theta)^2 - h''(K,\theta)^2)\,d\theta.$$

Similarly to (10), we obtain

$$\tilde{A}_K = \frac{\pi}{2}\sum_{n=2}^{\infty} n^2(1-n^2)(a_n^2+b_n^2). \tag{11}$$

Obviously, the oriented area $\tilde{A}_K$ of K is non-positive.

Generally, for planar convex bodies K and L with respective support functions $h(K,\theta)$ and $h(L,\theta)$, the functions most frequently used to measure the deviation between K and L are the Hausdorff distance

$$h_1(K,L) = \max_{\theta} |h(K,\theta) - h(L,\theta)|$$

and the L_2 metric

$$h_2(K,L) = \left(\int_0^{2\pi} |h(K,\theta) - h(L,\theta)|^2 d\theta\right)^{\frac{1}{2}}.$$

3. Two Chernoff-type inequalities

In this section, we first show a stronger version of the Chernoff inequality (2) and obtain a reverse Chernoff-type inequality, and then we give another proof of the reverse isoperimetric inequality (3).

THEOREM 1. *If K is a planar convex body with perimeter L_K and area A_K, then*

$$\int_0^{\frac{\pi}{2}} w(K,\theta) w\left(K, \theta + \frac{\pi}{2}\right) d\theta \geqslant \frac{L_K^2 + 2\pi A_K}{3\pi}, \tag{12}$$

where equality holds if and only if the support function of K is of the form (4).

Proof. Since $w(K,\theta) = h(K,\theta) + h(K,\theta+\pi)$, by variable substitution and the fact that $h(K,\theta)$ is periodic with period 2π,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} w(K,\theta) w\left(K, \theta + \frac{\pi}{2}\right) d\theta &= \sum_{i=0}^{3} \int_{\frac{i\pi}{2}}^{\frac{(i+1)\pi}{2}} h(K,\theta) h\left(K, \theta + \frac{\pi}{2}\right) d\theta \\ &= \int_0^{2\pi} h(K,\theta) h\left(K, \theta + \frac{\pi}{2}\right) d\theta. \end{aligned} \tag{13}$$

If $h(K,\theta)$ has the Fourier expansion

$$h(K,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta), \tag{14}$$

then

$$\begin{aligned} h\left(K, \theta + \frac{\pi}{2}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(&\left(a_n \cos\frac{n\pi}{2} + b_n \sin\frac{n\pi}{2}\right) \cos n\theta \\ &+ \left(b_n \cos\frac{n\pi}{2} - a_n \sin\frac{n\pi}{2}\right) \sin n\theta \Big). \end{aligned} \tag{15}$$

From (14), (15), and a version of Parseval's theorem (see [9, p. 61 (3.1.5)]), it follows that

$$\int_0^{2\pi} h(K,\theta)h\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta = \frac{\pi a_0^2}{2} + \sum_{n=2}^{\infty} \pi(a_n^2+b_n^2)\cos\frac{n\pi}{2}, \tag{16}$$

which, together with (9), (10), and (13), yields

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta - \frac{L_K^2+2\pi A_K}{3\pi} = \sum_{n=2}^{\infty} \pi\left(\cos\frac{n\pi}{2}+\frac{n^2-1}{3}\right)(a_n^2+b_n^2). \tag{17}$$

Since $\cos\frac{n\pi}{2}+\frac{n^2-1}{3}$ is equal to 0 for $n=2$ and is larger than 0 for $n=3,4,\ldots$, (12) follows, and equality holds in (12) only when $a_n=b_n=0$ for $n=3,4,\ldots$. □

REMARK 1. By (12) and the improved isoperimetric inequality (see [20, Thm. 3.4])

$$L_K^2 \geqslant 4\pi A_K + 8\pi|\tilde{A}_w(K)|,$$

where $\tilde{A}_w(K)$ is the oriented area of the Wigner caustic of K, we obtain

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta \geqslant 2A_K + \frac{8}{3}|\tilde{A}_w(K)|,$$

which is stronger than the inequality (see [18, Cor. 3.4 (3.14)])

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta \geqslant 2A_K + 2|\tilde{A}_w(K)|.$$

THEOREM 2. *If K is a planar convex body of area A_K, then*

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta \leqslant 2A_K + \frac{1}{3}|\tilde{A}_K|, \tag{18}$$

and equality holds if and only if the support function of K is of the form (4).

Proof. From (10), (11), (13), and (16), it follows that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta - 2A_K - \frac{1}{3}|\tilde{A}_K|\\ = \sum_{n=2}^{\infty} \pi\left(\cos\frac{n\pi}{2} - \frac{(n^2-6)(n^2-1)}{6}\right)(a_n^2+b_n^2).\end{aligned} \tag{19}$$

Since $\cos\frac{n\pi}{2}-\frac{(n^2-6)(n^2-1)}{6}$ is equal to 0 for $n=2$ and is less than 0 for $n=3,4,\ldots$, (18) follows, and equality holds in (18) only when $a_n=b_n=0$ for $n=3,4,\ldots$. □

Using Theorems 1 and 2, we obtain the reverse isoperimetric inequality (3) (see [5, Thm. 1.3]).

COROLLARY 1. *If K is a planar convex body of perimeter L_K and area A_K, then*

$$L_K^2 \leqslant 4\pi A_K + \pi|\tilde{A}_K|,$$

and equality holds if and only if the support function of K is of the form (4).

4. Stability versions of the two Chernoff-type inequalities

Let K be a planar convex body and let $\mathbf{v}(\theta) = (\cos 2\theta, \sin 2\theta)$. A planar convex body $P(K)$ is called a *polynomial body* associated with K if its support function is of the form

$$h(P(K),\theta) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta,$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} h(K,\theta)\,d\theta,$$
$$(a_1,b_1) = \frac{1}{\pi}\int_0^{2\pi} \mathbf{u}(\theta)h(K,\theta)\,d\theta$$

and

$$(a_2,b_2) = \frac{1}{\pi}\int_0^{2\pi} \mathbf{v}(\theta)h(K,\theta)\,d\theta.$$

If we denote by $h_1(K)$ and $h_2(K)$, respectively, the Hausdorff distance and the L_2 metric between K and $P(K)$ then, since

$$|h(K,\theta) - h(P(K),\theta)| = \left|\sum_{n=3}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)\right| \leqslant \sum_{n=3}^{\infty}|a_n\cos n\theta + b_n\sin n\theta|,$$

we get that

$$h_2(K)^2 = \sum_{n=3}^{\infty}\pi(a_n^2+b_n^2) \tag{20}$$

and

$$h_1(K) \leqslant \sum_{n=3}^{\infty}\sqrt{a_n^2+b_n^2}. \tag{21}$$

Next, we shall use the Hausdorff distance and the L_2 metric to prove stability versions of the two Chernoff-type inequalities in Theorems 1 and 2.

THEOREM 3. *Let K be a planar convex body with perimeter L_K and area A_K. Then*

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta - \frac{L_K^2+2\pi A_K}{3\pi} \geqslant \frac{8}{3}h_2(K)^2,$$

where equality holds if and only if the support function of K is of the form

$$h(K,\theta) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta + a_2\cos 2\theta + b_2\sin 2\theta + a_3\cos 3\theta + b_3\sin 3\theta.$$

Proof. By (17) and (20), we obtain

$$\begin{aligned}\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta-\frac{L_K^2+2\pi A_K}{3\pi}\\ =\sum_{n=3}^{\infty}\pi\left(\cos\frac{n\pi}{2}+\frac{n^2-1}{3}\right)(a_n^2+b_n^2)\\ \geqslant\frac{8}{3}\sum_{n=3}^{\infty}\pi(a_n^2+b_n^2)=\frac{8}{3}h_2(K)^2,\end{aligned}$$

where equality holds if and only if $a_n=b_n=0$ for $n\geqslant 4$. $\square$

THEOREM 4. *Let K be a planar convex body with perimeter L_K and area A_K. Then*

$$\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta-\frac{L_K^2+2\pi A_K}{3\pi}\geqslant\frac{16\pi}{25}h_1(K)^2. \tag{22}$$

Proof. From (17), (21), and Hölder's inequality, it follows that

$$\begin{aligned}\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta-\frac{L_K^2+2\pi A_K}{3\pi}\\ =\sum_{n=3}^{\infty}\pi\left(\cos\frac{n\pi}{2}+\frac{n^2-1}{3}\right)(a_n^2+b_n^2)\\ \geqslant\sum_{n=3}^{\infty}\frac{\pi}{3}(n^2-4)(a_n^2+b_n^2)\\ \geqslant\pi\left(\sum_{n=3}^{\infty}\frac{3}{n^2-4}\right)^{-1}\left(\sum_{n=2}^{\infty}\sqrt{a_n^2+b_n^2}\right)^2=\frac{16\pi}{25}h_1(K)^2. \quad\square\end{aligned} \tag{23}$$

THEOREM 5. *Let K be a planar convex body with area A_K. Then*

$$2A_K+\frac{1}{3}|\tilde{A}_K|-\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta\geqslant 4h_2(K)^2,$$

and equality holds if and only if the support function of K is of the form

$$h(K,\theta)=\frac{a_0}{2}+a_1\cos\theta+b_1\sin\theta+a_2\cos2\theta+b_2\sin2\theta+a_3\cos3\theta+b_3\sin3\theta.$$

Proof. By (19) and (20), we have

$$\begin{aligned}2A_K+\frac{1}{3}|\tilde{A}_K|-\int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta\\ =\sum_{n=3}^{\infty}\pi\left(\frac{(n^2-6)(n^2-1)}{6}-\cos\frac{n\pi}{2}\right)(a_n^2+b_n^2)\\ \geqslant 4\sum_{n=3}^{\infty}\pi(a_n^2+b_n^2)=4h_2(K)^2,\end{aligned}$$

where equality holds if and only if $a_n=b_n=0$ for $n\geqslant 4$. $\square$

THEOREM 6. *Let K be a planar convex body with area A_K. Then*

$$2A_K + \frac{1}{3}|\tilde{A}_K| - \int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta \geqslant \frac{7\Lambda\pi}{6}h_1(K)^2, \tag{24}$$

where $\Lambda = \left(\frac{51}{28} - \frac{\pi\cot(\sqrt{7}\pi)}{2\sqrt{7}} - \frac{\pi^2}{6}\right)^{-1} \approx 2.1318.$

Proof. From (19) and (21), it follows that

$$\begin{aligned}
&2A_K + \frac{1}{3}|\tilde{A}_K| - \int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta \\
&= \sum_{n=3}^{\infty}\frac{(n^2-6)(n^2-1)}{6}\pi(a_n^2+b_n^2) - \sum_{l=2}^{\infty}\pi(-1)^l(a_{2l}^2+b_{2l}^2) \\
&\geqslant \sum_{n=3}^{\infty}\left(\frac{(n^2-6)(n^2-1)}{6}-1\right)\pi(a_n^2+b_n^2) \\
&= \sum_{n=3}^{\infty}\frac{(n^2-7)n^2}{6}\pi(a_n^2+b_n^2),
\end{aligned}$$

which, together with Hölder's inequality, yields

$$\begin{aligned}
&2A_K + \frac{1}{3}|\tilde{A}_K| - \int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta \\
&\geqslant \pi\left(\sum_{n=3}^{\infty}\frac{6}{(n^2-7)n^2}\right)^{-1}\left(\sum_{n=3}^{\infty}\sqrt{a_n^2+b_n^2}\right)^2 \\
&= \frac{7\pi}{6}\left(\sum_{n=3}^{\infty}\frac{1}{n^2-7} - \sum_{n=3}^{\infty}\frac{1}{n^2}\right)^{-1} h_1(K)^2.
\end{aligned} \tag{25}$$

Recall that if c is not an integer, then, by the Fourier series expansion of the function $\cot(c\pi)$, we have

$$\pi\cot(c\pi) = \frac{1}{c} - 2c\sum_{n=1}^{\infty}\frac{1}{n^2-c^2}. \tag{26}$$

By (25), (26), and the fact that $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$, we obtain

$$\begin{aligned}
&2A_K + \frac{1}{3}|\tilde{A}_K| - \int_0^{\frac{\pi}{2}} w(K,\theta)w\Big(K,\theta+\frac{\pi}{2}\Big)\,d\theta \\
&\geqslant \frac{7\pi}{6}\left(\frac{51}{28} - \frac{\pi\cot(\sqrt{7}\pi)}{2\sqrt{7}} - \frac{\pi^2}{6}\right)^{-1} h_1(K)^2. \quad \square
\end{aligned}$$

REMARK 2. Since Hölder's inequality is used in the proofs of (22) and (24), the conditions for it to be an equality yield that equality in the last step of (23) and in the first inequality of (25) hold if and only if there exist constants c_1 and c_2 such that

$$\begin{aligned} c_1 &= \frac{n^2-4}{3}\sqrt{a_n^2+b_n^2}, \\ c_2 &= \frac{(n^2-7)n^2}{6}\sqrt{a_n^2+b_n^2} \end{aligned}$$

for $n \geqslant 3$. Thus, we cannot conclude that the support functions of any two planar convex bodies required for equality to hold in (23) and (25) are necessarily of the form (4), and we cannot derive the necessary conditions for equality to hold in (22) and (24). Hence, we do not know whether the constants in (22) and (24) are optimal.

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