Mathematical
Inequalities
& Applications
Volume 25, Number 3 (2022), 839–850 doi:10.7153/mia-2022-25-53

INEQUALITIES FOR TRIGONOMETRIC SUMS IN TWO VARIABLES

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(*Communicated by M. Praljak*)

Abstract. We prove sharp inequalities for the two-variable versions of the sum $\sum_{k=1}^{n} \frac{\sin(kx)}{k}$.

1. Introduction

Inequalities for trigonometric sums and polynomials have many applications in approximation theory, special functions theory, see e.g. [4]. The classical one is

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k} > 0 \qquad (n \geqslant 1, \quad 0 < x < \pi). \tag{1}$$

The validity of (1) was conjectured by Fejér in 1910 and proved one year later by Jackson [5]. Many refinements, generalization can be found in the literature [7].

In 1932, Koschmieder [6] showed that the Fejér-Jackson inequality (1) can be applied to prove an extension for two variables:

$$F_n(x,y) = \sum_{k=1}^{n} \frac{\sin(kx)\sin(ky)}{k^2} > 0 \qquad (n \geqslant 1, \quad 0 < x,y < \pi). \tag{2}$$

Turán [8] published in 1938 an upper bound for $F_n(x,y)$. Let $0 < y < \pi$. Then

$$F_n(x,y) < \begin{cases} (\pi - y)x, & \text{if } 0 \leqslant x \leqslant y, \\ (\pi - x)y, & \text{if } y \leqslant x < \pi. \end{cases} \tag{3}$$

In 2009 Alzer and Shi [3] proved that

$$F_n(x,y) < \pi^2/8 \qquad (n \geqslant 1, \quad x,y \in (0,\pi)), \tag{4}$$

where $\pi^2/8$ is the best possible constant. They also proved for $n \geqslant 1$, $x,y \in (0,\pi)$

$$-1 < G_n(x,y) = \sum_{k=1}^{n} \frac{\cos(kx)\cos(ky)}{k^2} < \frac{\pi^2}{6}, \tag{5}$$

Mathematics subject classification (2020): 26D05, 42A05.
Keywords and phrases: Trigonometric sum, two variables, inequalities.

$$-\frac{\sqrt[4]{3}(3+\sqrt{3})}{4\sqrt{2}} < H_n(x,y) = \sum_{k=1}^{n} \frac{\cos(kx)\sin(ky)}{k^2} < \frac{\sqrt[4]{3}(3+\sqrt{3})}{4\sqrt{2}}, \tag{6}$$

where the constants are the best possible.

In 1932, Koschmieder [6] showed that for $n \geqslant 1$, $0 < x-y < \pi$, $0 < x+y < \pi$

$$-\frac{\pi^2}{12} < G_n(x,y) = \sum_{k=1}^{2n} \frac{\cos(kx)\cos(ky)}{k^2} < \frac{\pi^2}{6}.$$

The aim of this paper is to improve these results and to give some other estimations for trigonometric sums.

We prove the following estimations in Theorem 1-3:

$$F_n(x,y) \leqslant \frac{1}{2}(x(\pi-x)y(\pi-y))^{1/2}.$$

Here $(x(\pi-x)y(\pi-y))^{1/2}/2 \leqslant \pi^2/8$, (see (4)), and for

$$\frac{\pi y}{4\pi-3y} \leqslant x \leqslant y < \pi, \qquad \text{or} \qquad \frac{\pi x}{4\pi-3x} \leqslant y \leqslant x < \pi,$$

it is sharper than (3).

The next one is

$$G_n(x,y) < \left(\frac{\pi^2}{6} - \frac{1}{2}x(\pi-x)\right)^{1/2} \left(\frac{\pi^2}{6} - \frac{1}{2}y(\pi-y)\right)^{1/2},$$

which is sharper than $\pi^2/6$, (see (5)).

At last, we have

$$H_n(x,y) < \left(\frac{\pi^2}{6} - \frac{1}{2}x(\pi-x)\right)^{1/2} \left(\frac{1}{2}y(\pi-y)\right)^{1/2},$$

that is sharper than (6) for certain numbers $x,y \in (0,\pi)$.

In Theorem 4 a refinement of (6) is given. In Theorem 5 a lower estimation is given for a generalization of $F_n(x,y)$. In Theorem 6 we extend the validity of (3) for a generalization of $F_n(x,y)$. In Theorem 7 a lower estimation of $G_n(x,y)$ is derived.

2. Known and new Lemmas

We define for positive integers n and real numbers x the function

$$g_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k^2}.$$

LEMMA 1. *For $n \geqslant 1$ and $x \in (0,\pi)$ we have $g_n(x) > 0$.*

Proof. See [3] Lemma 2.1. $\square$

In the next lemma we give an improvement of [3] Lemma 2.3.

LEMMA 2. *Let s,t be real numbers with $0 \leqslant s \leqslant 2\pi$, $-\pi \leqslant t \leqslant \pi$, and $t \leqslant s$. Then we have for $n \geqslant 1$*

$$\frac{g_n(s)+g_n(t)}{2} \leqslant \begin{cases} \frac{g_n(s)}{2}, & \text{if } -\pi \leqslant t \leqslant 0, 0 \leqslant s \leqslant \pi, \\ 0, & \text{if } -\pi \leqslant t \leqslant 0, \pi \leqslant s \leqslant 2\pi, \\ \frac{g_n(s)+g_n(t)}{2}, & \text{if } 0 \leqslant t \leqslant s \leqslant \pi, \\ \frac{g_n(t)}{2}, & \text{if } 0 \leqslant t \leqslant \pi, \pi \leqslant s \leqslant 2\pi. \end{cases}$$

Proof. We use the idea of [3] Lemma 2.3, modifying it where it is needed.

Case 1. $-\pi \leqslant t \leqslant 0$.
Let $0 \leqslant s \leqslant \pi$. Lemma 1 gives $g_n(t) \leqslant 0$. Thus

$$\frac{g_n(s)+g_n(t)}{2} \leqslant \frac{g_n(s)}{2}.$$

Let $\pi \leqslant s \leqslant 2\pi$. Since

$$g_n(s) = -g_n(2\pi - s) \leqslant 0$$

we have

$$\frac{g_n(s)+g_n(t)}{2} \leqslant 0.$$

Case 2. $0 \leqslant t \leqslant \pi$.
Let $\pi \leqslant s$. Then $\pi \leqslant s \leqslant 2\pi$ and $g_n(s) \leqslant 0$. We obtain

$$\frac{g_n(s)+g_n(t)}{2} \leqslant \frac{g_n(t)}{2}.$$

This completes the proof of lemma. $\square$

LEMMA 3. *For $n \geqslant 1$ and $0 \leqslant x \leqslant \pi$ we have*

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k} \leqslant \alpha_n(\pi - x)$$

with the best possible constant factor

$$\alpha_n = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ 0.663959\ldots, & \text{if } n \text{ is even}. \end{cases}$$

Proof. See [1] $\square$

The next lemma is well-known.

LEMMA 4. *For $n \geqslant 1$ we have*

$$\sum_{k=1}^{n} \frac{1}{k} = \log n + \gamma + \frac{1}{2n} - \sum_{k=1}^{p-1} \frac{B_{2k}}{2kn^{2k}} - \frac{\theta_1 B_{2p}}{2pn^{2p}},$$

where $0 < \theta_1 < 1$, γ is the Euler's constant, and B_j's are the Bernoulli numbers, $B_2 = 1/6$, $B_4 = -1/30$.

LEMMA 5. *For $n \geqslant 1$ and $0 \leqslant x \leqslant \pi$ we have*

$$g_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k^2} \leqslant \beta_n(\pi - x),$$

where

$$\beta_n = \begin{cases} 1, & \text{if } n = 1, \\ 5/6, & \text{if } n = 3, \\ (1-\alpha_2)\log 2 + \alpha_2 = 0.896884\ldots, & \text{if } n \neq 1, 3. \end{cases}$$

Proof. Using summation by parts we obtain

$$g_n(x) = \sum_{k=1}^{n} S_k(x)\left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{S_n(x)}{n+1}.$$

Case 1. $n = 2m$.
Applying Lemma 3 we can write

$$\begin{aligned} g_{2m}(x) &\leqslant \sum_{k=1}^{2m} \alpha_k(\pi - x)\left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{\alpha_{2m}(\pi - x)}{2m+1} \\ &= (\pi - x)\left(\sum_{k=1}^{2m} \alpha_k \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{\alpha_2}{2m+1}\right). \end{aligned}$$

Here

$$\begin{aligned} \sum_{k=1}^{2m} \alpha_k \left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{\alpha_2}{2m+1} &= \alpha_1 \sum_{j=1}^{m} \left(\frac{1}{2j-1} - \frac{1}{2j}\right) \\ &\quad + \alpha_2 \sum_{j=1}^{m} \left(\frac{1}{2j} - \frac{1}{2j+1}\right) + \frac{\alpha_2}{2m+1} \\ &= (\alpha_1 - \alpha_2) \sum_{j=1}^{m} \frac{1}{2j-1} + \alpha_2 + \frac{\alpha_2 - \alpha_1}{2} \sum_{j=1}^{m} \frac{1}{j}. \end{aligned}$$

Since

$$\sum_{j=1}^{m} \frac{1}{2j} + \sum_{j=1}^{m} \frac{1}{2j-1} = \sum_{k=1}^{2m} \frac{1}{k},$$

that is,

$$\sum_{j=1}^{m}\frac{1}{2j-1}=\sum_{k=1}^{2m}\frac{1}{k}-\frac{1}{2}\sum_{j=1}^{m}\frac{1}{j},$$

we obtain

$$\sum_{k=1}^{2m}\alpha_k\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{\alpha_2}{2m+1}=(\alpha_1-\alpha_2)\sum_{k=1}^{2m}\frac{1}{k}+(\alpha_2-\alpha_1)\sum_{j=1}^{m}\frac{1}{j}+\alpha_2.$$

Using Lemma 4 with $p=1$, we get for $m\geqslant 1$

$$\begin{aligned}
&(\alpha_1-\alpha_2)\sum_{k=1}^{2m}\frac{1}{k}+(\alpha_2-\alpha_1)\sum_{j=1}^{m}\frac{1}{j}+\alpha_2\\
&\qquad=(\alpha_1-\alpha_2)\left(\log 2+\log(m)+\gamma+\frac{1}{4m}-\frac{\theta_1 B_2}{2^3m^2}\right)\\
&\qquad\quad-(\alpha_1-\alpha_2)\left(\log(m)+\gamma+\frac{1}{2m}-\frac{\theta_2 B_2}{2m^2}\right)+\alpha_2\\
&\qquad=(\alpha_1-\alpha_2)\left(\log 2-\frac{1}{4m}+\frac{4\theta_2-\theta_1}{48m^2}\right)+\alpha_2\\
&\qquad\leqslant(\alpha_1-\alpha_2)\log 2+\alpha_2=0.896884\ldots.
\end{aligned}$$

Thus

$$g_{2m}(x)\leqslant((\alpha_1-\alpha_2)\log 2+\alpha_2)(\pi-x).$$

Case 2. $n=2m-1$.

Applying Lemma 3 we can write

$$\begin{aligned}
g_{2m-1}(x)&\leqslant\sum_{k=1}^{2m-1}\alpha_k(\pi-x)\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{\alpha_{2m-1}(\pi-x)}{2m}\\
&=(\pi-x)\left(\sum_{k=1}^{2m-1}\alpha_k\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{\alpha_1}{2m}\right).
\end{aligned}$$

Here

$$\begin{aligned}
\sum_{k=1}^{2m-1}\alpha_k\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{\alpha_1}{2m}&=\alpha_1\sum_{j=1}^{m}\left(\frac{1}{2j-1}-\frac{1}{2j}\right)+\frac{\alpha_1}{2m}\\
&\quad+\alpha_2\sum_{j=1}^{m-1}\left(\frac{1}{2j}-\frac{1}{2j+1}\right)\\
&=\alpha_2+(\alpha_1-\alpha_2)\sum_{j=1}^{m}\frac{1}{2j-1}+\frac{\alpha_2-\alpha_1}{2}\sum_{j=1}^{m-1}\frac{1}{j}\\
&=\alpha_2+(\alpha_1-\alpha_2)\sum_{j=1}^{m}\frac{1}{2j-1}-\frac{\alpha_2-\alpha_1}{2m}+\frac{\alpha_2-\alpha_1}{2}\sum_{j=1}^{m}\frac{1}{j}.
\end{aligned}$$

Since

$$\sum_{j=1}^{m} \frac{1}{2j-1} = \sum_{k=1}^{2m} \frac{1}{k} - \frac{1}{2} \sum_{j=1}^{m} \frac{1}{j},$$

we obtain

$$\sum_{k=1}^{2m-1} \alpha_k \left(\frac{1}{k} - \frac{1}{k+1} \right) + \frac{\alpha_1}{2m} = \alpha_2 + \frac{\alpha_1 - \alpha_2}{2m} + (\alpha_1 - \alpha_2) \sum_{k=1}^{2m} \frac{1}{k} + (\alpha_2 - \alpha_1) \sum_{j=1}^{m} \frac{1}{j}.$$

Using Lemma 4 with $p = 1$, we get for $m \geqslant 3$

$$\begin{aligned} &\frac{\alpha_1 - \alpha_2}{2m} + \alpha_2 + (\alpha_1 - \alpha_2) \sum_{k=1}^{2m} \frac{1}{k} + (\alpha_2 - \alpha_1) \sum_{j=1}^{m} \frac{1}{j} \\ &\quad = (\alpha_1 - \alpha_2) \left(\log 2 + \frac{\alpha_1 - \alpha_2}{2m} - \frac{1}{4m} + \frac{4\theta_2 - \theta_1}{48m^2} \right) + \alpha_2 \\ &\quad \leqslant (\alpha_1 - \alpha_2) \log 2 + \alpha_2 = 0.896884\ldots. \end{aligned}$$

For $m = 1$ we obtain

$$\alpha_2 + \frac{\alpha_1 - \alpha_2}{2m} + (\alpha_1 - \alpha_2) \sum_{k=1}^{2m} \frac{1}{k} + (\alpha_2 - \alpha_1) \sum_{j=1}^{m} \frac{1}{j} = \alpha_1 = 1.$$

This constant estimation is sharp, because the function $f : [0, \pi) \to \mathbf{R}$,

$$f(x) := \frac{g_1(x)}{\pi - x} = \frac{\sin(x)}{\pi - x}$$

is strictly increasing and

$$\lim_{x \to \pi - 0} f(x) = 1.$$

For $m = 2$ we obtain

$$g_3(x) = \sum_{k=1}^{3} \frac{\sin(kx)}{k^2}.$$

Define the function $f : [0, \pi) \to \mathbf{R}$,

$$f(x) := \frac{g_3(x)}{\pi - x}.$$

We prove that the function f is strictly increasing.

Let us write

$$f(x) = \frac{1}{36} \frac{\sin(x)}{\pi - x} (32 + 18\cos(x) + 16\cos^2(x)).$$

Then

$$f'(x) = \frac{1}{36(\pi - x)^2} F(x),$$

where

$$F(x) = (\pi - x)(48\cos^3(x) + 36\cos^2(x) - 18) \\ + \sin(x)(32 + 18\cos(x) + 16\cos^2(x)).$$

We have to show that $F(x) > 0$ for $x \in [0,\pi)$. Calculation gives

$$F'(x) = -72(\pi - x)\sin(x)\cos(x)(2\cos(x) + 1).$$

It yields that F is strictly increasing in $[0,\pi/2]$ and $[2\pi/3,\pi]$, strictly increasing in $[\pi/2, 2\pi/3]$. Since $F(0) = 66\pi$, $F(\pi/2) = -9\pi + 32$, $F(2\pi/3) = -5\pi + 27\sqrt{3}/2$ are positive numbers and $F(\pi) = 0$ it follows that F is positive in $[0,\pi)$.

From the strictly monotonicity of f on $[0,\pi)$ we get

$$f(x) < \lim_{x \to \pi - 0} f(x) = \frac{5}{6} = 0.833333\ldots,$$

that is,

$$g_3(x) \leqslant \frac{5}{6}(\pi - x). \quad \square$$

LEMMA 6. *Let a_k $(k = 1,\ldots,n)$ be real numbers such that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$. Then we have for all integers $n \geqslant 1$ and $x \in (0,\pi)$*

$$\sum_{k=1}^{n} a_k \frac{\sin(kx)}{k} \geqslant \frac{a_1}{2} x^2 \left(\cot\left(\frac{x}{2}\right) - \frac{\pi - x}{2}\right).$$

Proof. See [2] Corollary 1. $\square$

LEMMA 7. *The Taylor series of tangent function is*

$$\tan(z) = \sum_{n=1}^{\infty} 2(2^{2n} - 1)\zeta(2n)\pi^{-2n} z^{2n-1}, \quad |z| < \pi/2,$$

where ζ is the zeta function of Riemann.

Proof. See e.g. [4] 1.20(4). $\square$

LEMMA 8. *For $n \geqslant 1$ and $0 \leqslant x \leqslant \pi$ we have*

$$h_n(x) = \sum_{k=1}^{n} \frac{\cos(kx)}{k^2} \geqslant \tau_n - \frac{\alpha_n}{2} x(2\pi - x),$$

where

$$\tau_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Proof. Applying the inequality in Lemma 3 we obtain

$$\begin{aligned} h_n(x) &= \sum_{k=1}^{n} \frac{\cos(kx)-\cos(k0)}{k^2} + \sum_{k=1}^{n} \frac{1}{k^2} \\ &= \tau_n - \int_0^x \sum_{k=1}^{n} \frac{\sin(kt)}{k}\, dt \\ &\geqslant \tau_n - \alpha_n \int_0^x (\pi - t)\, dt \\ &= \tau_n - \frac{\alpha_n}{2} x(2\pi - x). \quad \square \end{aligned}$$

3. New Theorems

In this section we prove our sharp estimations for the trigonometric sums in two variables, which we defined in the first section.

THEOREM 1. *For all integers $n \geqslant 1$ and real numbers $x, y \in (0,\pi)$ we have*

$$F_n(x,y) = \sum_{k=1}^{n} \frac{\sin(kx)\sin(ky)}{k^2} < \frac{1}{2}(x(\pi-x)y(\pi-y))^{1/2}.$$

Proof. By the Cauchy-Schwarz inequality we have

$$\begin{aligned} \sum_{k=1}^{n} \frac{\sin(kx)\sin(ky)}{k^2} &\leqslant \left(\sum_{k=1}^{n} \frac{\sin^2(kx)}{k^2}\right)^{1/2} \left(\sum_{k=1}^{n} \frac{\sin^2(ky)}{k^2}\right)^{1/2} \\ &\leqslant \left(\sum_{k=1}^{\infty} \frac{\sin^2(kx)}{k^2}\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{\sin^2(ky)}{k^2}\right)^{1/2} \\ &= \frac{1}{2}(x(\pi-x)y(\pi-y))^{1/2}, \end{aligned}$$

because

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\sin^2(kx)}{k^2} &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1-\cos(2kx)}{k^2} \\ &= \frac{1}{2}\left(\frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{1}{k^2}\cos(2kx)\right). \quad \square \end{aligned}$$

THEOREM 2. *For all integers $n \geqslant 1$ and real numbers $x, y \in (0,\pi)$ we have*

$$G_n(x,y) = \sum_{k=1}^{n} \frac{\cos(kx)\cos(ky)}{k^2} < \left(\frac{\pi^2}{6} - \frac{1}{2}x(\pi-x)\right)^{1/2} \left(\frac{\pi^2}{6} - \frac{1}{2}y(\pi-y)\right)^{1/2}.$$

Proof. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{\cos(kx)\cos(ky)}{k^2} &\leqslant \left(\sum_{k=1}^{\infty}\frac{\cos^2(kx)}{k^2}\right)^{1/2}\left(\sum_{k=1}^{\infty}\frac{\cos^2(ky)}{k^2}\right)^{1/2}\\ &= \left(\sum_{k=1}^{\infty}\frac{1-\sin^2(kx)}{k^2}\right)^{1/2}\left(\sum_{k=1}^{\infty}\frac{1-\sin^2(ky)}{k^2}\right)^{1/2}\\ &= \left(\frac{\pi^2}{6}-\frac{1}{2}x(\pi-x)\right)^{1/2}\left(\frac{\pi^2}{6}-\frac{1}{2}y(\pi-y)\right)^{1/2}. \quad \square\end{aligned}$$

THEOREM 3. *For all integers $n \geqslant 1$ and real numbers $x, y \in (0, \pi)$ we have*

$$H_n(x,y)=\sum_{k=1}^{n}\frac{\cos(kx)\sin(ky)}{k^2} < \left(\frac{\pi^2}{6}-\frac{1}{2}x(\pi-x)\right)^{1/2}\left(\frac{1}{2}y(\pi-y)\right)^{1/2}.$$

Proof. By the Cauchy-Schwarz inequality we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{\cos(kx)\sin(ky)}{k^2} &\leqslant \left(\sum_{k=1}^{\infty}\frac{\cos^2(kx)}{k^2}\right)^{1/2}\left(\sum_{k=1}^{\infty}\frac{\sin^2(kx)}{k^2}\right)^{1/2}\\ &= \left(\frac{\pi^2}{6}-\frac{1}{2}x(\pi-x)\right)^{1/2}\left(\frac{1}{2}y(\pi-y)\right)^{1/2}. \quad \square\end{aligned}$$

THEOREM 4. *For all integers $n \geqslant 1$ and real numbers $x, y \in (0, \pi)$ we have*

$$H_n(x,y)=\sum_{k=1}^{n}\frac{\cos(kx)\sin(ky)}{k^2} \leqslant \begin{cases} \frac{\beta_n}{2}(\pi-x-y), & \{\pi/2 \leqslant x,\, y \leqslant \pi-x\},\ or\\ & \{y \leqslant x \leqslant \pi/2\};\\ 0, & \{\pi/2 \leqslant x,\, \pi-x \leqslant y \leqslant x\};\\ \beta_n(\pi-y), & \{x \leqslant \pi/2,\, x \leqslant y \leqslant \pi-x\};\\ \frac{\beta_n}{2}(\pi-y-x), & \{x \leqslant \pi/2,\, \pi-x \leqslant y\},\ or\\ & \{\pi/2 \leqslant x \leqslant y\};\end{cases}$$

and

$$H_n(x,y) \geqslant \begin{cases} -\frac{\beta_n}{2}(\pi+y-x), & \{\pi/2 \leqslant x,\, y \leqslant \pi-x\},\ or\\ & \{y \leqslant x \leqslant \pi/2\};\\ -\beta_n y, & \{\pi/2 \leqslant x,\, \pi-x \leqslant y \leqslant x\};\\ 0, & \{x \leqslant \pi/2,\, x \leqslant y \leqslant \pi-x\};\\ -\frac{\beta_n}{2}(x+y-\pi), & \{x \leqslant \pi/2,\, \pi-x \leqslant y\},\ or\\ & \{\pi/2 \leqslant x \leqslant y\}.\end{cases}$$

Proof. First consider the upper estimation. Let $0 < x, y < \pi$, $s = x + y$, and $t = y - x$. Then we get

$$H_n(x,y) = \frac{g_n(s) + g_n(t)}{2}.$$

Since $0 \leqslant s \leqslant 2\pi$, $-\pi \leqslant t \leqslant \pi$, and $t \leqslant s$ we can apply Lemma 2. Solving the inequalities for x and y we have the following cases.

Case 1. $\pi/2 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi - x$, or $0 \leqslant y \leqslant x \leqslant \pi/2$.
In this case

$$H_n(x,y) \leqslant \frac{1}{2} g_n(x+y) \leqslant \frac{\beta_n}{2}(\pi - x - y).$$

Case 2. $\pi/2 \leqslant x \leqslant \pi$, and $\pi - x \leqslant y \leqslant x$.
In this case

$$H_n(x,y) \leqslant 0.$$

Case 3. $0 \leqslant x \leqslant \pi/2$ and $x \leqslant y \leqslant \pi - x$.
In this case

$$H_n(x,y) \leqslant \beta_n(\pi - y).$$

Case 4. $0 \leqslant x \leqslant \pi/2$ and $\pi - x \leqslant y \leqslant \pi$, or $\pi/2 \leqslant x \leqslant y \leqslant \pi$.
In this case

$$H_n(x,y) \leqslant \frac{\beta_n}{2}(\pi - y + x).$$

Now consider the lower estimation. Since

$$H_n(x,y) = -H_n(\pi - x, \pi - y),$$

from the upper estimation we immediately obtain the lower estimation. □

THEOREM 5. *Let a_k $(k = 1, \ldots, n)$ be real numbers such that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$. Then we have for all integers $n \geqslant 1$ and $x, y \in (0, \pi)$, $x \geqslant y$*

$$\sum_{k=1}^{n} a_k \frac{\sin(kx)\sin(ky)}{k^2} \geqslant \begin{cases} a_1 p(x,y), & x + y \leqslant \pi; \\ a_1 p(\pi - x, \pi - y), & x + y > \pi, \end{cases}$$

where

$$p(x,y) = \frac{1}{180} y \left(15x^4(3\pi - x) - (45\pi^2 + 50y^2)x^3 \right.$$
$$\left. + 15\pi(\pi^2 + 6y^2)x^2 - 15y^2(3\pi^2 + y^2)x + 5\pi^3 y^2 + 9\pi y^4\right).$$

Proof. Since $\sin(k(\pi - x))\sin(k(\pi - y)) = \sin(kx)\sin(ky)$ we may assume that $x + y \leqslant \pi$. Obviously

$$2\frac{\sin(kx)\sin(ky)}{k^2} = \frac{\cos(k(x-y))}{k^2} - \frac{\cos(k(x+y))}{k^2}$$
$$= \int_{x-y}^{x+y} \frac{\sin(kt)}{k}\, dt.$$

It yields

$$\sum_{k=1}^{n} a_k \frac{\sin(kx)\sin(ky)}{k^2} = \frac{1}{2}\int_{x-y}^{x+y} \sum_{k=1}^{n} a_k \frac{\sin(kt)}{k}\, dt.$$

Applying Lemma 6 we obtain

$$\sum_{k=1}^{n} a_k \frac{\sin(kx)\sin(ky)}{k^2} \geqslant \frac{a_1}{4}\int_{x-y}^{x+y} t^2 \left(\cot\left(\frac{t}{2}\right) - \frac{\pi - t}{2}\right) dt.$$

Here $0 \leqslant x+y \leqslant \pi$, $0 \leqslant x-y < \pi$. Using Lemma 7 it implies

$$\begin{aligned}\int_{x-y}^{x+y} t^2 \left(\cot\left(\frac{t}{2}\right) - \frac{\pi - t}{2}\right) dt &= \int_{x-y}^{x+y} t^2 \left(\tan\left(\frac{\pi - t}{2}\right) - \frac{\pi - t}{2}\right) dt \\ &\geqslant \int_{x-y}^{x+y} t^2 \frac{(\pi - t)^3}{24}\, dt \\ &= \frac{1}{180} y \left(15x^4(3\pi - x) - (45\pi^2 + 50y^2)x^3 \right. \\ &\quad \left. + 15\pi(\pi^2 + 6y^2)x^2 - 15y^2(3\pi^2 + y^2)x + 5\pi^3 y^2 + 9\pi y^4\right).\end{aligned}$$

Now the proof is complete. □

COROLLARY 1. *If we assume that $x \leqslant \pi/2$ in Theorem 5, then the sum is positive. By numerical investigation we can extend the restriction beyond $\pi/2$.*

THEOREM 6. *Let a_k $(k = 1, \ldots, n)$ be real numbers such that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$. Then we have for all integers $n \geqslant 1$ and $x, y \in [0, \pi]$*

$$\sum_{k=1}^{n} a_k \frac{\sin(kx)\sin(ky)}{k^2} \leqslant \begin{cases} a_1 y(\pi - x), & y \leqslant x; \\ a_1 x(\pi - y), & x \leqslant y. \end{cases}$$

Proof. Since $\sin(k(\pi - x))\sin(k(\pi - y)) = \sin(kx)\sin(ky)$ we may assume that $x + y \leqslant \pi$. In addition, we may assume that $x \geqslant y$. As we have seen in the proof of previous theorem it holds

$$\sum_{k=1}^{n} a_k \frac{\sin(kx)\sin(ky)}{k^2} = \frac{1}{2}\int_{x-y}^{x+y} \sum_{k=1}^{n} a_k \frac{\sin(kt)}{k}\, dt,$$

where $0 \leqslant x-y \leqslant x+y \leqslant \pi$. Here we have the identity (see also [2] (3.13))

$$\sum_{k=1}^{n} a_k \frac{\sin(kt)}{k} = \sum_{k=1}^{n} \left((a_k - a_{k+1}) \sum_{j=1}^{k} \frac{\sin(jt)}{j}\right), \quad (a_{n+1} = 0).$$

Using the estimation of Turán [8]

$$\sum_{j=1}^{k} \frac{\sin(jt)}{j} \leqslant \pi - t,$$

and $a_k - a_{k+1} \geqslant 0 \ (k = 1, \dots, n)$ it follows

$$\sum_{k=1}^{n} a_k \frac{\sin(kx)\sin(ky)}{k^2} \leqslant \frac{1}{2}\int_{x-y}^{x+y} a_1(\pi - t)\,dt = a_1 y(\pi - x).$$

The other cases, $x \leqslant y$, $x + y > \pi$ can be handled similarly, see [8], p. 281. □

THEOREM 7. *For all integers $n \geqslant 1$ and real numbers $x, y \in [0, \pi]$, $x \geqslant y$, $x + y \leqslant \pi$ we have*

$$G_n(x,y) = \sum_{k=1}^{n} \frac{\cos(kx)\cos(ky)}{k^2} \geqslant \tau_n + \frac{\alpha_n}{2}(-2\pi x + x^2 + y^2).$$

Proof. Since $2\cos(kx)\cos(ky) = \cos(x-y) + \cos(x+y)$, we can write

$$G_n(x,y) = \frac{h_n(x-y) + h_n(x+y)}{2}.$$

Applying Lemma 8 we get

$$\begin{aligned} G_n(x,y) &\geqslant \frac{1}{2}\left(\tau_n - \frac{\alpha_n}{2}(x-y)(2\pi - x + y) + \tau_n - \frac{\alpha_n}{2}(x+y)(2\pi - x - y)\right) \\ &= \tau_n + \frac{\alpha_n}{2}(-2\pi x + x^2 + y^2). \quad \square \end{aligned}$$

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(Received April 14, 2022)

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