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LYAPUNOV–TYPE INEQUALITIES FOR A NONLINEAR SEQUENTIAL FRACTIONAL BVP IN THE FRAME OF GENERALIZED HILFER DERIVATIVES

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Abstract. We consider a nonlinear sequential fractional boundary value problem (BVP) in the frame of generalized ψ-Hilfer derivatives. We obtain the Green function and some of its properties, from which we derive a new Lyapunov-type inequality for our problem. As a consequence, we present a lower bound for the eigenvalues of the problem. We give some existence results. We emphasize that our results are still valid for some other classes of source functions having some singularities.

1. Introduction

The well-known Lyapunov inequality was firstly proposed and proved by Lyapunov [10], in which the author showed that the necessary condition for the following problem

$$\begin{cases} u''(t)+q(t)u(t)=0 \ \ a<t<b \\ u(a)=u(b)=0 \end{cases}$$

to have a nontrivial classical solution is

$$\int_a^b |q(s)| \, \mathrm{d}s > \frac{4}{b-a}.$$

This result is one of the most significant inequalities and has many practical applications in the theory of differential equations such as oscillation theory, stability criteria, disconjugacy, eigenvalue problems, etc., we refer to [2, 12, 14, 15, 16] and the references therein.

In recent years, there are numerous versions of Lyapunov inequality that have been investigated for differential equations with various types of fractional derivatives. In fact, Ferreira [3, 4] derived some Lyapunov-type inequalities for the fractional BVPs with Riemann-Liouville and Caputo fractional derivatives. Ma et al [11] investigated Lyapunov-type inequality for the fractional BVP with Hadamard fractional derivative.

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Wang et al [17] considered Lyapunov-type inequalities for the fractional differential equation involving Hilfer fractional derivative with multi-point boundary conditions. Very recently, Zohra et al [19] established Lyapunov and Hartman-Wintner-type inequalities for the nonlinear fractional BVP with generalized ψ-Hilfer fractional derivative. Besides, the authors also obtained some existence results for this problem.

Lyapunov-type inequalities for BVPs involving sequential fractional derivatives were also considered in some recent papers. Ferreira [5] and Zhang et al [18] investigated Lyapunov-type inequality for the linear fractional sequential BVPs involving Caputo fractional derivatives and Hilfer fractional derivatives, respectively. In 2020, Ferreira [6] derived Lyapunov-type inequality for the nonlinear sequential fractional BVP with mixed Riemann-Liouville fractional derivative and Caputo fractional derivative. Recently, some Lyapunov-type inequalities for nonlinear fractional hybird equations with sequential Riemann-Liouville fractional derivatives were also considered in [8] and other existing references therein.

Motivated by [5, 6, 18], we consider the following nonlinear sequential fractional BVP in the frame of generalized ψ-Hilfer fractional derivatives

$$\left({}^{H}D_{a+}^{\alpha_1,\beta_1,\psi}\,{}^{H}D_{a+}^{\alpha_2,\beta_2,\psi}u\right)(t)+f(t,u(t))=0,\quad a<t<b \tag{1}$$

subject to the conditions

$$u(a)={}^{H}D_{a+}^{\alpha_3,\beta_3,\psi}u(b)=0, \tag{2}$$

where $0<\alpha_i\leqslant 1$, $0\leqslant\beta_i\leqslant 1$ $(i=1,2,3)$, $\alpha_1+\alpha_2>1$, and ${}^{H}D_{a+}^{\alpha_i,\beta_i,\psi}$ $(i=1,2,3)$ stand for the ψ-Hilfer fractional derivatives (see definitions in section 2). Here we obtain a Lyapunov-type inequality and some existence results for our problem. As a result of Lyapunov-type inequality, we also obtain a lower bound for the potential eigenvalues of our problem.

It is worth noting that the results closest to our work were obtained in [5, 6, 18]. However, in the work of Ferreira [5] and Zhang et al [18], Lyapunov-type inqualities were only investigated for the linear sequential fractional BVPs in the frame of Caputo fractional derivatives ${}^{C}D_{a+}^{\alpha}\,{}^{C}D_{a+}^{\beta}(\cdot)$ and Hilfer fractional derivatives $D_{a+}^{\alpha_1,\beta_1}\,D_{a+}^{\alpha_2,\beta_2}(\cdot)$, respectively. Particularly, Ferreira [6] investigated Lyapunov inequality for the nonlinear sequential fractional BVP with Riemann-Liouville and Caputo fractional derivatives $D_{0+}^{\alpha}\,{}^{C}D_{0+}^{\beta}(\cdot)$. In our work, we consider a nonlinear sequential fractional BVP (1) and (2) in the frame of generalized ψ-Hilfer fractional derivatives ${}^{H}D_{a+}^{\alpha_1,\beta_1,\psi}\,{}^{H}D_{a+}^{\alpha_2,\beta_2,\psi}(\cdot)$.

The paper is organized as follows. In section 2, we set up some notations, present the concept of ψ-Hilfer fractional derivative and some of its properties. We also introduce some lemmas that will be used in the proofs of the main results. In section 3, we discuss a Lyapunov inequality and investigate some existence results for our problem.

2. Mathematical preliminaries

This section is devoted to presenting definitions and some basic properties involving ψ-Hilfer fractional derivative. Moreover, some auxiliary lemmas are given prior to proceeding to the main results of this paper. We start by setting some notations.

For $a<b$, let us define

$$H_+^1[a,b]=\left\{\psi\in C^1[a,b]:\ \psi'(t)>0 \ \text{ for all } \ t\in[a,b]\right\}.$$

For $\varphi\in C([a,b],\mathbb{R})$, we denote $\|\varphi\|=\sup_{t\in[a,b]}|\varphi(t)|$. We also denote $\mathbb{R}_+=\{x\in\mathbb{R}: x\geqslant 0\}$.

We now present the concepts of fractional integral and fractional derivative of a function with respect to another function.

DEFINITION 1. (see [9, 13]) For $\alpha>0$, $\psi\in H_+^1[a,b]$, and $f\in L^1[a,b]$, the fractional integral of a function f with respect to the function ψ is defined by

$$I_{a+}^{\alpha,\psi}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t \psi'(\tau)(\psi(t)-\psi(\tau))^{\alpha-1}f(\tau)\,\mathrm{d}\tau,$$

where $\Gamma(\cdot)$ is the classic Gamma function.

DEFINITION 2. (see [13]) For $n-1<\alpha\leqslant n$, and $f,\psi\in C^n[a,b]$ with $\psi'(t)>0$ for all $t\in[a,b]$, the left-side ψ-Hilfer fractional derivative ${}^HD_{a+}^{\alpha,\beta,\psi}(\cdot)$ of function of order α and type $0\leqslant\beta\leqslant 1$, is defined by

$${}^HD_{a+}^{\alpha,\beta,\psi}f(t)=I_{a+}^{\beta(n-\alpha),\psi}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n I_{a+}^{(1-\beta)(n-\alpha),\psi}f(t). \tag{3}$$

REMARK 1. The ψ-Hilfer fractional derivative is generalized from well-know fractional derivatives such as Caputo, Caputo-Katugampola, Hadamard, Riemann-Liouville, etc. For example

(i). Taking the limit $\beta\to 1$ on both side of Eq. (3), we get

$${}^HD_{a+}^{\alpha,1,\psi}f(t)=I_{a+}^{\beta(n-\alpha),\psi}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t)={}^CD_{a+}^{\alpha,\psi}f(t)$$

the ψ-Caputo fractional derivative with respect to another function.

(ii). Taking the limit $\beta\to 0$ on both side of Eq. (3), we get

$${}^HD_{a+}^{\alpha,0,\psi}f(t)=\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right)^n I_{a+}^{(1-\beta)(n-\alpha),\psi}f(t)=D_{a+}^{\alpha,\psi}f(t)$$

the ψ-Riemann-Liouville fractional derivative with respect to another function.

For complete surveys of basic properties of the fractional operators $I_{a+}^{\alpha,\psi}(\cdot)$ and ${}^HD_{a+}^{\alpha,\beta,\psi}(\cdot)$, we refer to [9, 13]. In this paper, we will use only the following properties.

LEMMA 1. (see [9, 13]) *We have*

(i). If $f\in C^n[a,b]$, $n-1<\alpha<n$, and $0\leqslant\beta\leqslant 1$, then

$$I_{a+}^{\alpha,\psi}{}^HD_{a+}^{\alpha,\beta,\psi}f(t)=f(t)-\sum_{k=1}^{n}c_k(\psi(t)-\psi(a))^{\gamma-k},$$

with $\gamma=\alpha+\beta(n-\alpha)$ *and* $c_k=\frac{1}{\Gamma(\gamma-k+1)}f_\psi^{[n-k]}I_{a+}^{(1-\beta)(n-\alpha)}f(a)$. *Herein* $f_\psi^{[n]}f(x)=\left(\frac{1}{\psi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x)$.

(ii). For $\alpha,\beta>0$, *we have* $I_{a+}^{\alpha,\psi}I_{a+}^{\beta,\psi}f(t)=I_{a+}^{\alpha+\beta,\psi}f(t)$.

Now, we present some lemmas which play an importance role in the proof of main results of the paper.

LEMMA 2. (see [1]) *Let* α,β *be two non-negative numbers. Then, the function*

$$p(x,\alpha,\beta)=\frac{\Gamma(x)\Gamma(\alpha+\beta+x)}{\Gamma(\alpha+x)\Gamma(\beta+x)}$$

is completely monotonic on $(0,+\infty)$. *Moreover, we have*

$$\frac{\Gamma(x)\Gamma(\alpha+\beta+x)}{\Gamma(\alpha+x)\Gamma(\beta+x)}\geqslant 1$$

for any $x>0$.

LEMMA 3. (Jensen's inequality) *Let* μ *be a positive measure and let* Ω *be a measurable set with* $\mu(\Omega)=1$. *If* u *is a real function in* $L^1(\mu)$, *if* $a<u(t)<b$ *for all* $t\in\Omega$, *and if* f *is a convex on* (a,b), *then*

$$f\left(\int_\Omega u\,\mathrm{d}\mu\right)\leqslant\int_\Omega (f\circ u)\,\mathrm{d}\mu. \tag{4}$$

If f *is concave on* (a,b), *then the inequality* (4) *holds with* $\leqslant$ *reversed.*

LEMMA 4. (The nonlinear Leray-Schauder alternatives fixed point theorem [7]) *Let* $\mathbb{B}$ *be a Banach space, and let* W *be a closed convex subset of* $\mathbb{B}$. *Let* V *be a relatively open subset of* W *and* $0\in V$. *Suppose that* $Q:\overline{V}\to W$ *is a continuous compact mapping. Then we have either*

(i). Q *has a fixed point in* $\overline{V}$

or

(ii). There exist $\lambda\in(0,1)$ *and* $u\in\partial V$ *such that* $u=\lambda Qu$.

3. Main results

This section is divided in three parts. In the first part, we investigate the Green function of our problem and some of its properties. In the second part, we present a Lyapunov inequality and give a lower bound estimate for the possible eigenvalues of the problem. In the third part, based on the properties of the Green function, we discuss the existence results for our problem.

3.1. Green's function

We begin by transforming the problem (1) and (2) to integral equation.

LEMMA 5. *Let* $0 < \alpha_i \leqslant 1$, $0 \leqslant \beta_i \leqslant 1$ $(i = 1,2,3)$ *and* $\alpha_1 + \alpha_2 > 1$, *and let* $\psi \in H_+^1[a,b]$. *Let* u *be a solution of the problem* (1) *and* (2). *If* $f(\cdot,u(\cdot)) \in L^1[a,b]$ *then* u *is a solution of the following integral equation*

$$u(t) = \int_a^b G(s,t)\psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1} f(s,u(s))\,\mathrm{d}s \tag{5}$$

where

$$G(s,t) = \begin{cases} C_1(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1} - C_2\frac{(\psi(t)-\psi(s))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}}, & a \leqslant s \leqslant t \leqslant b, \\ C_1(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1} & a \leqslant t \leqslant s \leqslant b, \end{cases} \tag{6}$$

with $\gamma_1 = \alpha_1 + \beta_1(1-\alpha_1)$ *and* $C_1 = \Gamma(\alpha_2+\gamma_1-\alpha_3)/\Big[\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-\alpha_3-1}\Big]$, $C_2 = 1/\Gamma(\alpha_1+\alpha_2)$.

REMARK 2. The Green function G in Lemma 5 is independent of the parameters β_2, β_3. Consequently, if the problem (1) and (2) admit a mild solution then, it may not depend continuously on these parameters.

Proof. We have

$$I_{a+}^{\alpha_1,\psi}\left({}^H D_{a+}^{\alpha_1,\beta_1,\psi}\, {}^H D_{a+}^{\alpha_2,\beta_2,\psi} u\right)(t) = -I_{a+}^{\alpha_1,\psi} f(t,u(t)). \tag{7}$$

On the other hand, we invoke Lemma 2.4 to deduce that

$$I_{a+}^{\alpha_1,\psi}\left({}^H D_{a+}^{\alpha_1,\beta_1,\psi}\, {}^H D_{a+}^{\alpha_2,\beta_2,\psi} u\right)(t) = {}^H D_{a+}^{\alpha_2,\beta_2,\psi} - d_1(\psi(t)-\psi(a))^{\gamma_1-1}. \tag{8}$$

Combining (7) and (8), we obtain

$${}^H D_{a+}^{\alpha_2,\beta_2,\psi} u(t) = -I_{a+}^{\alpha_1,\psi} f(t,u(t)) + d_1(\psi(t)-\psi(a))^{\gamma_1-1}.$$

Again, using Lemma 1, we get

$$\begin{aligned} u(t) &= d_2(\psi(t)-\psi(a))^{\gamma_2-1} + I_{a+}^{\alpha_2,\psi}\left(d_1(\psi(t)-\psi(a))^{\gamma_1-1}\right) - I_{a+}^{\alpha_1+\alpha_2,\psi} f(t,u(t)) \\ &= d_2(\psi(t)-\psi(a))^{\gamma_2-1} + d_1\frac{\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1} - I_{a+}^{\alpha_1+\alpha_2,\psi} f(t,u(t)) \end{aligned} \tag{9}$$

due to $I_{a+}^{\alpha_2,\psi}(\psi(t)-\psi(a))^{\gamma_1-1} = \frac{\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1)}(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1}$, where $\gamma_i = \alpha_i + \beta_i(1-\alpha_i)$ for $i = 1,2$. Using the condition $u(a) = 0$ and $\gamma_2 - 1 = (1-\alpha_2)(\beta_1 - 1) \leqslant 0$, we figure out that $d_2 = 0$. On the other hand, we have

$$\begin{aligned} {}^H D_{a+}^{\alpha_3,\beta_3,\psi} I_{a+}^{\alpha_1+\alpha_2,\psi} f(t,u(t)) &= {}^H D_{a+}^{\alpha_3,\beta_3,\psi} I_{a+}^{\alpha_3,\psi}\left(I_{a+}^{\alpha_1+\alpha_2-\alpha_3,\psi} f(t,u(t))\right) \\ &= I_{a+}^{\alpha_1+\alpha_2-\alpha_3,\psi} f(t,u(t)) \end{aligned}$$

and ${}^{H}D_{a+}^{\alpha_3,\beta_3,\psi}(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1} = \Gamma(\alpha_2+\gamma_1)(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-\alpha_3-1}/\Gamma(\alpha_2+\gamma_1-\alpha_3)$. It follows from (9) together with $d_2=0$ that

$$ {}^{H}D_{a+}^{\alpha_3,\beta_3,\psi}u(t) = d_1\frac{\Gamma(\gamma_1)}{\Gamma(\alpha_2+\gamma_1-\alpha_3)}(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-\alpha_3-1} - I_{a+}^{\alpha_1+\alpha_2-\alpha_3,\psi}f(t,u(t)) $$

From the condition ${}^{H}D_{a+}^{\alpha_3,\beta_3,\psi}u(b)=0$, we obtain

$$ \begin{aligned} &d_1\frac{\Gamma(\gamma_1)(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-\alpha_3-1}}{\Gamma(\alpha_2+\gamma_1-\alpha_3)} \\ &\quad -\frac{1}{\Gamma(\alpha_1+\alpha_2-\alpha_3)}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}f(s,u(s))\,\mathrm{d}s = 0. \end{aligned} $$

Or,

$$ \begin{aligned} d_1 = {}& \frac{\Gamma(\alpha_2+\gamma_1-\alpha_3)}{\Gamma(\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-\alpha_3-1}} \\ &\times\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}f(s,u(s))\,\mathrm{d}s. \end{aligned} $$

Finally, pushing the obtained coefficients d_1 and d_2 into (9), we have

$$ \begin{aligned} u(t) = {}& \frac{\Gamma(\alpha_2+\gamma_1-\alpha_3)(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1}}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-\alpha_3-1}} \\ &\times\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}f(s,u(s))\,\mathrm{d}s \\ &-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha_1+\alpha_2-1}f(s,u(s))\,\mathrm{d}s \\ ={}& \int_a^b G(s,t)\psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-1}f(s,u(s))\,\mathrm{d}s. \end{aligned} $$

The proof of Lemma is completed. □

DEFINITION 3. The function $G(\cdot,\cdot)$ given by (6) is called the Green function of the problem (1) and (2).

In the following propositions, we investigate some properties of the Green function.

PROPOSITION 1. Let $0<\alpha_i\leqslant 1$, $0\leqslant\beta_i\leqslant 1$ $(i=1,2,3)$ and $\alpha_1+\alpha_2>1$, and $\gamma_1=\alpha_1+\beta_1(1-\alpha_1)$. Let $\psi\in H_+^1[a,b]$ and the Green function G be defined as in Lemma 5. Then, for any $a\leqslant t_1\leqslant t_2\leqslant b$, we have

$$ |G(s,t_1)-G(s,t_2)|\leqslant C(\psi(t_2)-\psi(t_1))^{\sigma}, $$

where $\sigma=\min\{\alpha_1+\alpha_2-1,\alpha_3\}$ and $C=C_1(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-\sigma-1}+C_2(\psi(b)-\psi(a))^{\alpha_3}$.

Proof. For $a \leqslant t_1 \leqslant t_2 \leqslant s \leqslant b$, using the fact that $|A^x - B^x| \leqslant |A-B|^x$ for any $A,B \geqslant 0$, $x \in [0,1]$, we have

$$\begin{aligned}|G(s,t_2)-G(s,t_1)| &= C_1\left((\psi(t_2)-\psi(a))^{\alpha_2+\gamma_1-1}-(\psi(t_1)-\psi(a))^{\alpha_2+\gamma_1-1}\right)\\ &\leqslant C_1(\psi(t_2)-\psi(t_1))^{\alpha_2+\gamma_1-1}\\ &\leqslant C_1(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-1-\sigma}(\psi(t_2)-\psi(t_1))^{\sigma} \end{aligned}\tag{10}$$

due to $0<\alpha_2+\gamma_1-1\leqslant 1$, where $\sigma=\min\{\alpha_1+\alpha_2-1,\alpha_3\}$.

For $a \leqslant s \leqslant t_1 \leqslant t_2 \leqslant b$, we also use the fact that $|A^x - B^x| \leqslant |A-B|^x$ for any $A,B \geqslant 0$, $x \in [0,1]$ together with $0<\alpha_1+\alpha_2-1\leqslant\alpha_2+\gamma_1-1\leqslant 1$ to obtain

$$\begin{aligned}&|G(s,t_2)-G(s,t_1)|\\ &\leqslant C_1\left((\psi(t_2)-\psi(a))^{\alpha_2+\gamma_1-1}-(\psi(t_1)-\psi(a))^{\alpha_2+\gamma_1-1}\right)\\ &\quad +C_2\frac{(\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1}-(\psi(t_1)-\psi(s))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}}\\ &\leqslant C_1(\psi(t_2)-\psi(t_1))^{\alpha_2+\gamma_1-1}+C_2\frac{(\psi(t_2)-\psi(t_1))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}}\\ &\leqslant C_1(\psi(t_2)-\psi(t_1))^{\alpha_2+\gamma_1-1}+C_2(\psi(b)-\psi(s))^{\alpha_3}(\psi(t_2)-\psi(t_1))^{\sigma}\\ &=\left[C_1(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-\sigma-1}+C_2(\psi(b)-\psi(a))^{\alpha_3}\right](\psi(t_2)-\psi(t_1))^{\sigma}, \end{aligned}\tag{11}$$

where $\sigma=\min\{\alpha_1+\alpha_2-1,\alpha_3\}$. Here we have used the following estimate

$$\begin{aligned}&\frac{(\psi(t_2)-\psi(t_1))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}}\\ &=\left(\frac{\psi(t_2)-\psi(t_1)}{\psi(b)-\psi(s)}\right)^{\alpha_1+\alpha_2-\sigma-1}(\psi(b)-\psi(s))^{\alpha_3}(\psi(t_2)-\psi(t_1))^{\sigma}\\ &\leqslant(\psi(b)-\psi(a))^{\alpha_3}(\psi(t_2)-\psi(t_1))^{\sigma}.\end{aligned}$$

For $a \leqslant t_1 \leqslant s \leqslant t_2 \leqslant b$, we have

$$\begin{aligned}&|G(s,t_2)-G(s,t_1)|\\ &\leqslant C_1\left((\psi(t_2)-\psi(a))^{\alpha_2+\gamma_1-1}-(\psi(t_1)-\psi(a))^{\alpha_2+\gamma_1-1}\right)\\ &\quad +C_2\frac{(\psi(t_2)-\psi(s))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}}\\ &\leqslant C_1(\psi(t_2)-\psi(t_1))^{\alpha_2+\gamma_1-1}+C_2\frac{(\psi(t_2)-\psi(t_1))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}}\\ &\leqslant\left[C_1(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-\sigma-1}+C_2(\psi(b)-\psi(a))^{\alpha_3}\right](\psi(t_2)-\psi(t_1))^{\sigma}, \end{aligned}\tag{12}$$

where $\sigma=\min\{\alpha_1+\alpha_2-1,\alpha_3\}$. Combining (10), (11) and (12), we obtain the desired result of Proposition. □

PROPOSITION 2. Let $0 < \alpha_i \leqslant 1$, $0 \leqslant \beta_i \leqslant 1$ $(i = 1,2,3)$ and $\alpha_1 + \alpha_2 > 1$, and $\gamma_1 = \alpha_1 + \beta_1(1-\alpha_1)$. Let $\psi \in H_+^1[a,b]$ and the Green function G be defined as in Lemma 5. Then

$$\max_{a \leqslant s,t \leqslant b} |G(s,t)| = C_{\max},$$

where

$$C_{\max} = \max\Bigg\{ \frac{\Gamma(\alpha_2+\gamma_1-\alpha_3)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)} (\psi(b)-\psi(a))^{\alpha_3}, \\ \frac{\gamma_1-\alpha_1}{\alpha_2+\gamma_1-1} \frac{(\psi(b)-\psi(a))^{\alpha_3}}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{\Gamma(\alpha_2+\gamma_1-1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+\gamma_1-\alpha_3)} \right)^{\frac{\alpha_1+\alpha_2-1}{\gamma_1-\alpha_1}} \Bigg\}$$

with $\gamma_1 > \alpha_1$, and

$$C_{\max} = \frac{(\psi(b)-\psi(a))^{\alpha_3}}{\Gamma(\alpha_1+\alpha_2)}$$

with $\gamma_1 = \alpha_1$.

Proof. We define the function

$$\Phi(s,t) = C_1(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1}$$

for $a \leqslant t \leqslant s \leqslant b$. Using the fact that $C_1 > 0$, $\alpha_2 + \gamma_1 \geqslant \alpha_2 + \alpha_1 > 1$, and ψ is the increasing function, it follows that

$$\max_{a \leqslant s,t \leqslant b} \Phi(s,t) = C_1(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-1} = \frac{\Gamma(\alpha_2+\gamma_1-\alpha_3)(\psi(b)-\psi(a))^{\alpha_3}}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}. \tag{13}$$

We continue by defining the function

$$\Upsilon(s,t) = C_1(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1} - C_2 \frac{(\psi(t)-\psi(s))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}}$$

for $a \leqslant s \leqslant t \leqslant b$.

Let us fix $t \in [a,b]$ and consider the function Υ with respect to second variable, then

$$\begin{aligned} \Upsilon_s(s,t) &= -C_2 \frac{(\psi(t)-\psi(s))^{\alpha_1+\alpha_2-2}(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-2}}{(\psi(b)-\psi(s))^{2(\alpha_1+\alpha_2-\alpha_3-1)}} \\ &\quad \times [(\alpha_1+\alpha_2-1)(\psi(b)-\psi(s)) - (\alpha_1+\alpha_2-\alpha_3-1)(\psi(t)-\psi(s))] \\ &\leqslant 0 \end{aligned}$$

due to $(\alpha_1+\alpha_2-1)(\psi(b)-\psi(s)) - (\alpha_1+\alpha_2-\alpha_3-1)(\psi(t)-\psi(s)) \geqslant (\alpha_1+\alpha_2-1)(\psi(t)-\psi(s)) - (\alpha_1+\alpha_2-\alpha_3-1)(\psi(t)-\psi(s)) = \alpha_3(\psi(t)-\psi(s)) \geqslant 0$. Therefore, $\Upsilon(s,t)$ is a decreasing function of s, which implies

$$\max_{a \leqslant s \leqslant t \leqslant b} |\Upsilon(s,t)| = \max_{a \leqslant t \leqslant b} \{|\Upsilon(a,t)|, |\Upsilon(t,t)|\}.$$

It is clear to see that $|\Upsilon(t,t)| = \Phi(s,t)$. Hence, we only find maximum of the function $|\Upsilon(a,t)|$. To this aim, let us consider the following function

$$\begin{aligned}\Theta(t) &= C_1(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1} - C_2\frac{(\psi(t)-\psi(a))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3-1}}\\ &= \frac{\Gamma(\alpha_2+\gamma_1-\alpha_3)(\psi(t)-\psi(a))^{\alpha_2+\gamma_1-1}}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)(\psi(b)-\psi(a))^{\alpha_2+\gamma_1-\alpha_3-1}}\\ &\quad - \frac{1}{\Gamma(\alpha_1+\alpha_2)}\frac{(\psi(t)-\psi(a))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3-1}}.\end{aligned}$$

We can easily see that $\Theta(t) = 0$ for $\gamma_1 = \alpha_1$. So, we consider only the case $\gamma_1 > \alpha_1$.

Firstly, for $\gamma_1 > \alpha_1$, we will prove that $\Theta(t) \leqslant 0$ for all $t \in [a,b]$. Indeed, it is clear to see that $\Theta(a) = 0$. Therefore, we only show that $\Theta(t) \leqslant 0$ for all $t \in (a,b]$. Applying Lemma 2 with $x := \alpha_1+\alpha_2-\alpha_3$, $\alpha := \alpha_3$, $\beta := \gamma_1-\alpha_1$, one has

$$\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\gamma_1-\alpha_3)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)} = \frac{\Gamma(x+\alpha)\Gamma(x+\beta)}{\Gamma(x)\Gamma(x+\alpha+\beta)} \leqslant 1.$$

This leads to

$$\begin{aligned}\Theta(t) &= \frac{1}{\Gamma(\alpha_1+\alpha_2)}\frac{(\psi(t)-\psi(a))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3-1}}\\ &\quad\times\left(\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\gamma_1-\alpha_3)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}\left(\frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}\right)^{\gamma_1-\alpha_1} - 1\right)\\ &\leqslant \frac{1}{\Gamma(\alpha_1+\alpha_2)}\frac{(\psi(t)-\psi(a))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3-1}}\left(\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\gamma_1-\alpha_3)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)} - 1\right)\\ &\leqslant 0\end{aligned}$$

due to $\gamma_1-\alpha_1 > 0$ and $(\psi(t)-\psi(a))/(\psi(b)-\psi(a)) \leqslant 1$ for all $t \in (a,b]$.

Secondly, for $\gamma_1 > \alpha_1$, by direct computations, we have

$$\begin{aligned}\Theta'(t) &= \frac{1}{\Gamma(\alpha_1+\alpha_2-1)}\frac{(\psi(t)-\psi(a))^{\alpha_1+\alpha_2-2}}{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3-1}}\\ &\quad\times\left(\frac{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+\gamma_1-\alpha_3)}{\Gamma(\alpha_2+\gamma_1-1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}\left(\frac{\psi(t)-\psi(a)}{\psi(b)-\psi(a)}\right)^{\gamma_1-\alpha_1} - 1\right).\end{aligned}$$

Thus, $\Theta'(t_0) = 0$ if and only if

$$\left(\frac{\psi(t_0)-\psi(a)}{\psi(b)-\psi(a)}\right)^{\gamma_1-\alpha_1} = \frac{\Gamma(\alpha_2+\gamma_1-1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+\gamma_1-\alpha_3)}. \tag{14}$$

Or,

$$\psi(t_0) = \psi(a) + (\psi(b)-\psi(a))\left(\frac{\Gamma(\alpha_2+\gamma_1-1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+\gamma_1-\alpha_3)}\right)^{1/(\gamma_1-\alpha_1)}.$$

Applying Lemma 2 with $x := \alpha_1 + \alpha_2 - 1$ and $\alpha := \gamma_1 - \alpha_1$, $\beta := 1 - \alpha_3$, we have

$$0 < \frac{\Gamma(\alpha_2+\gamma_1-1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+\gamma_1-\alpha_3)} = \frac{\Gamma(x+\alpha)\Gamma(x+\beta)}{\Gamma(x)\Gamma(x+\alpha+\beta)} \leqslant 1. \tag{15}$$

It follows $\psi(a) < \psi(t_0) \leqslant \psi(a) + (\psi(b) - \psi(a)) = \psi(b)$, or $a = \psi^{-1}(\psi(a)) < t_0 < \psi^{-1}(\psi(b)) = b$. Thus, from the facts that $\Theta(a) = 0$ and $\Theta(t) \leqslant 0$ for all $t \in (a,b]$, we deduce that $\max_{a\leqslant t\leqslant b} |\Theta(t)| = |\Theta(t_0)|$. Using (14), we have

$$\begin{aligned}
\Theta(t_0) &= \frac{1}{\Gamma(\alpha_1+\alpha_2)} \frac{(\psi(t_0)-\psi(a))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3-1}} \\
&\quad \times \left(\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\gamma_1-\alpha_3)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)} \left(\frac{\psi(t_0)-\psi(a)}{\psi(b)-\psi(a)}\right)^{\gamma_1-\alpha_1} - 1 \right) \\
&= \frac{1}{\Gamma(\alpha_1+\alpha_2)} \frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-1}}{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3-1}} \\
&\quad \times \left(\frac{\Gamma(\alpha_2+\gamma_1-1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+\gamma_1-\alpha_3)}\right)^{\frac{\alpha_1+\alpha_2-1}{\gamma_1-\alpha_1}} \left(\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\gamma_1-1)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-1)} - 1 \right) \\
&= \frac{\alpha_1-\gamma_1}{\alpha_2+\gamma_1-1} \frac{(\psi(b)-\psi(a))^{\alpha_3}}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{\Gamma(\alpha_2+\gamma_1-1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+\gamma_1-\alpha_3)}\right)^{\frac{\alpha_1+\alpha_2-1}{\gamma_1-\alpha_1}}
\end{aligned}$$

due to

$$\frac{\Gamma(\alpha_1+\alpha_2)\Gamma(\alpha_2+\gamma_1-1)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-1)} - 1 = \frac{\alpha_1+\alpha_2-1}{\alpha_2+\gamma_1-1} - 1 = \frac{\alpha_1-\gamma_1}{\alpha_2+\gamma_1-1}.$$

Combining the above equality and (13), we obtain the desired result of Proposition. □

REMARK 3. It is clear to see that $\gamma_1 - \alpha_1 \to 0$ if and only if $\alpha_1 \to 1$ or $\beta_1 \to 0$. Moreover, we find from (15) that

$$\begin{aligned}
0 &\leqslant \frac{\gamma_1-\alpha_1}{\alpha_2+\gamma_1-1} \frac{(\psi(b)-\psi(a))^{\alpha_3}}{\Gamma(\alpha_1+\alpha_2)} \left(\frac{\Gamma(\alpha_2+\gamma_1-1)\Gamma(\alpha_1+\alpha_2-\alpha_3)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+\gamma_1-\alpha_3)}\right)^{\frac{\alpha_1+\alpha_2-1}{\gamma_1-\alpha_1}} \\
&\leqslant \frac{\gamma_1-\alpha_1}{\alpha_2+\gamma_1-1} \frac{(\psi(b)-\psi(a))^{\alpha_3}}{\Gamma(\alpha_1+\alpha_2)} \to 0 \text{ as } \gamma_1-\alpha_1 \to 0^+.
\end{aligned} \tag{16}$$

Therefore, we deduce from (16) that if the values of β_1 sufficiently close to zero or α_1 sufficiently close to one, then

$$\max_{a\leqslant s,t\leqslant b} |G(s,t)| = \frac{\Gamma(\alpha_2+\gamma_1-\alpha_3)}{\Gamma(\alpha_2+\gamma_1)\Gamma(\alpha_1+\alpha_2-\alpha_3)} (\psi(b)-\psi(a))^{\alpha_3}.$$

3.2. Lyapunov-type inequality

We present a generalized Lyapunov-type for our problem. We start by making an assumption and a definition.

- **Assumption** $(\mathscr{A}1)$: There exist $q:(a,b)\to\mathbb{R}$, and a positive, non-decreasing and concave function $h:\mathbb{R}\to\mathbb{R}$ such that

$$|f(t,u)| \leqslant |q(t)||h(u)|$$

for any $t\in(a,b)$ and $u\in\mathbb{R}$.

DEFINITION 4. Solution u of the Eq. (5) is called mild solution of the problem (1) and (2).

Based on the above assumption and definition, we can state and prove the main result of this part.

THEOREM 3. *Let* $0<\alpha_i\leqslant 1,\ 0\leqslant\beta_i\leqslant 1\ \ (i=1,2,3)$ *and* $\alpha_1+\alpha_2>1$, *and* $\gamma_1=\alpha_1+\beta_1(1-\alpha_1)$. *Assume that Assumption* $(\mathscr{A}1)$ *holds. If* $\psi\in H_+^1[a,b]$ *and* $g_\psi(\cdot):=\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}q(\cdot)\in L^1(a,b)$ *and the problem (1) and (2) has a nontrivial mild solution, then*

$$\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}|q(s)|\ \mathrm{d}s \geqslant \frac{1}{C_{\max}}\frac{||u||}{h(||u||)},$$

where $C_{\max}$ *defined in Proposition 2.*

Proof. If the problem (1) and (2) has a nontrivial mild solution, then we obtain from Eq. (5) and Lemma 3 that

$$\begin{aligned}
|u(t)| &\leqslant C_{\max}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}|f(s,u(s))|\ \mathrm{d}s\\
&\leqslant C_{\max}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}|q(s)||h(u(s))|\ \mathrm{d}s\\
&= C_{\max}||g_\psi||_{L^1(a,b)}\int_a^b \frac{|g_\psi(s)|}{||g_\psi||_{L^1(a,b)}}|h(u(s))|\ \mathrm{d}s\\
&\leqslant C_{\max}||g_\psi||_{L^1(a,b)}h\left(\int_a^b \frac{|g_\psi(s)|}{||g_\psi||_{L^1(a,b)}}|u(s)|\ \mathrm{d}s\right)\\
&\leqslant C_{\max}||g_\psi||_{L^1(a,b)}h(\|u\|),
\end{aligned}$$

where $g_\psi(\cdot):=\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}q(\cdot)$. This implies

$$||g_\psi||_{L^1(a,b)} \geqslant \frac{1}{C_{\max}}\frac{||u||}{h(||u||)}.$$

The proof of Theorem is completed. □

COROLLARY 1. *Suppose that there exists $q:(a,b)\to\mathbb{R}_+$ such that*

$$|f(t,u)|\leqslant q(t)|u|$$

for all $t\in(a,b)$. If $\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}q(\cdot)\in L^1(a,b)$ and

$$\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}q(s)\,\mathrm{d}s<1/C_{\max},$$

where $C_{\max}$ defined in Proposition 2, then the problem (1) and (2) has no non-trivial mild solution.

If $\beta_1=\beta_2=\beta_3=1$ then ψ-Hilfer fractional derivatives become the ψ-Caputo fractional derivatives (see Remark 1) and we obtain the following result which is a general result of Ferreira [5] as follows.

COROLLARY 2. *Let the assumptions in Theorem 3 hold for $\beta_1=\beta_2=\beta_3=1$. If the following problem*

$$\begin{cases}\left({}^CD_{a+}^{\alpha_1,\psi}\,{}^CD_{a+}^{\alpha_2,\psi}u\right)(t)+f(t,u(t))=0, & a<t<b,\\ u(a)={}^CD_{a+}^{\alpha_3,\psi}u(b)=0\end{cases}$$

has a nontrivial mild solution. Then

$$\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}|q(s)|\,\mathrm{d}s\geqslant\frac{1}{C^*_{\max}}\frac{||u||}{h(||u||)},$$

where

$$C^*_{\max}=(\psi(b)-\psi(a))^{\alpha_3}\max\left\{\frac{\Gamma(\alpha_2+1-\alpha_3)}{\Gamma(\alpha_2+1)\Gamma(\alpha_1+\alpha_2-\alpha_3)},\right.$$
$$\left.\frac{1-\alpha_1}{\alpha_2}\frac{1}{\Gamma(\alpha_1+\alpha_2)}\left(\frac{\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_2-\alpha_3)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_2+1-\alpha_3)}\right)^{\frac{\alpha_1+\alpha_2-1}{1-\alpha_1}}\right\}$$

with $\alpha_1<1$, and

$$C^*_{\max}=\frac{(\psi(b)-\psi(a))^{\alpha_3}}{\Gamma(1+\alpha_2)}$$

with $\alpha_1=1$.

Proof. Apply Theorem 3 with remark that if $\beta_1=1$ then $\gamma_1=1$ and $C_{\max}\equiv C^*_{\max}$. $\square$

The obtained Lyapunov inequality gives us a lower bound for the sequential fractional BVP as below.

COROLLARY 3. *Let* $0<\alpha_i\leqslant 1$, $0\leqslant \beta_i\leqslant 1$ $(i=1,2,3)$, *and* $\psi\in H_+^1[a,b]$. *Suppose that* λ *is an eigenvalue of the following problem*

$$\begin{cases}\left({}^HD_{a+}^{\alpha_1,\beta_1,\psi}\,{}^HD_{a+}^{\alpha_2,\beta_2,\psi}u\right)(t)=\lambda u(t), \quad a<t<b,\\ u(a)={}^HD_{a+}^{\alpha_3,\beta_3,\psi}u(b)=0.\end{cases}$$

Then

$$|\lambda|\geqslant \frac{1}{C_{\max}}\frac{\alpha_1+\alpha_2-\alpha_3}{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3}},$$

where $C_{\max}$ *defined in Proposition 2.*

REMARK 4. If $\alpha_1=\alpha_2=\alpha_3=1$, we have $1/C_{\max}=\psi(b)-\psi(a)$ and obtain $|\lambda|\geqslant 1$.

Proof. Apply Theorem 3 with $q(t)=-\lambda$, $h(u)=u$, we obtain

$$|\lambda|\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}\,\mathrm{d}s\geqslant 1/C_{\max}.$$

Or,

$$|\lambda|\frac{(\psi(b)-\psi(a))^{\alpha_1+\alpha_2-\alpha_3}}{\alpha_1+\alpha_2-\alpha_3}\geqslant 1/C_{\max}.$$

This leads to the desired result of Corollary. □

3.3. Existence and non-existence results

This part, based on properties of the Green function, we obtain some existence results for our problem. It is worth noting that our results hold for some source functions having some singularities.

In the following result, we use the following assumption.

- **Assumption** $(\mathscr{A}2)$**:** There exist two functions $k_f, l_f:(a,b)\to\mathbb{R}_+$, and a positive, non-decreasing function $\vartheta:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$|f(t,u)|\leqslant k_f(t)\vartheta(|u|), \quad a<t<b,\ u\in\mathbb{R},$$
$$|f(t,u)-f(t,v)|\leqslant l_f(t)|\varsigma(u,v)|, \quad a<t<b,\ u,v\in\mathbb{R},$$

  where $\varsigma\in C(\mathbb{R}\times\mathbb{R};\mathbb{R})$ and $\varsigma(u,v)\to 0$ as $|u-v|\to 0$.

We now state and prove the existence result for our problem.

THEOREM 4. *Let* $0<\alpha_i\leqslant 1$, $0\leqslant\beta_i\leqslant 1$ $(i=1,2,3)$ *and* $\alpha_1+\alpha_2>1$, *and* $\gamma_1=\alpha_1+\beta_1(1-\alpha_1)$. *Assume that Assumption* $(\mathscr{A}2)$ *holds. Suppose further that* $\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)$

$\in L^1(a,b)$ and $\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}l_f(\cdot) \in L^1(a,b)$. If there exist $M>0$ such that

$$M > C_{\max}\vartheta(M)\big\|\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)\big\|_{L^1(a,b)},$$

where $C_{\max}$ defined in Proposition 2, then the problem (1) and (2) has at least one mild solution.

Proof. Define the operator $Q : C[a,b] \to C[a,b]$ by

$$Qu(t) = \int_a^b G(s,t)\psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1} f(s,u(s))\,\mathrm{d}s. \tag{17}$$

We firstly verify that Q is a compact operator. Put $\Omega_R = \{u \in C[a,b] : \|u\| \leqslant R\}$.

Claim 1. $Q(\Omega_R)$ maps bounded sets into bounded sets in $C[a,b]$. For $u \in \Omega_R$, in view of Proposition 2 and Assumption $(\mathscr{A}2)$, we have

$$\begin{aligned}
\|Qu\| &= \max_{a\leqslant t\leqslant b}\left|\int_a^b G(s,t)\psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1} f(s,u(s))\,\mathrm{d}s\right| \\
&\leqslant C_{\max}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(s)\vartheta(|u(s)|)\,\mathrm{d}s \\
&\leqslant C_{\max}\vartheta(R)\big\|\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)\big\|_{L^1(a,b)},
\end{aligned}$$

where $C_{\max}$ defined in Proposition 2.

Claim 2. Q is continuous operator, i.e., $|Qu(t)-Qv(t)| \to 0$ as $u \to v$ in $C[a,b]$. Without lost of generality, we assume that $u, v \in \Omega_R$ for some $R>0$. Using Assumption $(\mathscr{A}2)$, we have

$$\begin{aligned}
&\|Qu-Qv\| \\
&= \max_{a\leqslant t\leqslant b}\left|\int_a^b G(s,t)\psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}(f(s,u(s))-f(s,v(s)))\,\mathrm{d}s\right| \\
&\leqslant C_{\max}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}l_f(t)|\varsigma(u(s),v(s))|\,\mathrm{d}s \\
&\leqslant C_{\max}\big\|\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}l_f(\cdot)\big\|_{L^1(a,b)}\|\varsigma(u(\cdot),v(\cdot))\| \\
&\quad \to 0 \text{ as } |u-v| \to 0,
\end{aligned}$$

where $C_{\max}$ defined in Proposition 2.

Claim 3. $Q(\Omega_R)$ maps bounded sets into equicontinuous sets of $C[a,b]$. For $u \in \Omega_R$ and $a \leqslant t_1 \leqslant t_2 \leqslant b$, using Assumption $(\mathscr{A}2)$ and Proposition 1, one has

$$\begin{aligned}
&|Qu(t_2)-Qu(t_1)| \\
&\leqslant \int_a^b |G(s,t_2)-G(s,t_1)|\psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1} f(s,u(s))\,\mathrm{d}s \\
&\leqslant C(\psi(t_2)-\psi(t_1))^{\sigma}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(s)\vartheta(|u(s)|)\,\mathrm{d}s \\
&\leqslant C(\psi(t_2)-\psi(t_1))^{\sigma}\vartheta(R)\big\|\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)\big\|_{L^1(a,b)},
\end{aligned}$$

where C and σ defined in Proposition 1. The latter inequality shows that $|Qu(t_2) - u(t_1)| \to 0$ uniformly as $|t_2 - t_1| \to 0$.

We now at a position to prove the result of Theorem. Put

$$\Omega = \{u \in C[a,b] : \|u\| < M\}.$$

Since $|f(t,u)| \leqslant \vartheta(M)k_f(t)$ for any $u \in \Omega$ and $t \in (a,b)$, by an argument analogous to that used for the proof of *Claim 1*, one has

$$\|Qu\| \leqslant C_{\max}\vartheta(M)\big\|\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)\big\|_{L^1(a,b)}, \tag{18}$$

where $C_{\max}$ defined in Proposition 2. If there exists $\lambda \in (0,1)$ and $u \in \partial\Omega$ such that $u = \lambda Qu$, then we obtain from (18) that

$$M = \|u\| = \lambda\|Qu\| \leqslant \|Qu\| \leqslant C_{\max}\vartheta(M)\big\|\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)\big\|_{L^1(a,b)}.$$

This contradicts the hypothesis. Therefore, by virtue of Lemma 4, we conclude that Q has a fixed point in Ω, which is a mild solution of the problem (1) and (2). The proof of Theorem is completed. □

COROLLARY 4. *Suppose that there exist $\kappa_1, \kappa_2 > 0$ and $\gamma_1, \gamma_2 < \alpha_1 + \alpha_2 - \alpha_3$, $\theta_1, \theta_2 < 1$, and a positive, non-decreasing function $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$|f(t,u)| \leqslant \kappa_1(\psi(b)-\psi(t))^{-\gamma_1}(\psi(t)-\psi(a))^{-\theta_1}\vartheta(|u|), \quad a<t<b,\ u \in \mathbb{R},$$

$$|f(t,u)-f(t,v)| \leqslant \kappa_2(\psi(b)-\psi(t))^{-\gamma_2}(\psi(t)-\psi(a))^{-\theta_2}|\varsigma(u,v)|, \quad a<t<b,\ u,v \in \mathbb{R},$$

where $\varsigma \in C(\mathbb{R}\times\mathbb{R};\mathbb{R})$ and $\varsigma(u,v) \to 0$ as $|u-v| \to 0$. If κ_1 sufficiently close to zero then the problem (1) and (2) has at least one mild solution.

Proof. By change the variable of integration from t to $z = (\psi(t)-\psi(a))/(\psi(b)-\psi(a))$, we can prove that

$$\int_a^b (\psi(b)-\psi(t))^{x-1}(\psi(t)-\psi(a))^{-y}\,\mathrm{d}t = (\psi(b)-\psi(a))^{x-y}B(x,1-y), \tag{19}$$

where $x > 0$ and $y < 1$. Using (19), we can check directly that the assumptions in Theorem 4 hold. □

To close this paper, we present a uniqueness result for our problem.

THEOREM 5. *Let $0 < \alpha_i \leqslant 1$, $0 \leqslant \beta_i \leqslant 1$ $(i = 1,2,3)$ and $\alpha_1 + \alpha_2 > 1$, and $\gamma_1 = \alpha_1 + \beta_1(1-\alpha_1)$. Assume that there exists $k_f : (a,b) \to \mathbb{R}_+$ such that*

$$|f(t,u)-f(t,v)| \leqslant k_f(t)|u-v|,\ \ a<t<b,\ u,v \in \mathbb{R}.$$

If $\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}f(\cdot,0)$, $\psi'(\cdot)(\psi(b)-\psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot) \in L^1(a,b)$, and satisfying

$$\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(s)\,\mathrm{d}s < 1/C_{\max},$$

where $C_{\max}$ defined in Proposition 2, then the problem (1) and (2) has a unique mild solution.

Proof. Regarding the operator Q given by (17), we firstly verify that Q is well-defined. It is clear to see that $|f(t,u)| \leqslant k_f(t)|u| + |f(t,0)|$. For any $u \in C[a,b]$ and $a \leqslant t_1 \leqslant t_2 \leqslant b$, we obtain from Proposition 1 that

$$
\begin{aligned}
&|Qu(t_2) - Qu(t_1)| \\
&\leqslant \int_a^b |G(s,t_2) - G(s,t_1)|\psi'(s)(\psi(b) - \psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}|f(s,u(s))| \,\mathrm{d}s \\
&\leqslant C(\psi(t_2) - \psi(t_1))^{\sigma} \int_a^b \psi'(s)(\psi(b) - \psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}(k_f(t)|u(t)| + |f(t,0))\,\mathrm{d}s \\
&\leqslant C(\psi(t_2) - \psi(t_1))^{\sigma}\|u\| \left\|\psi'(\cdot)(\psi(b) - \psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)\right\|_{L^1(a,b)} \\
&\quad + C(\psi(t_2) - \psi(t_1))^{\sigma}\left\|\psi'(\cdot)(\psi(b) - \psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}f(\cdot,0)\right\|_{L^1(a,b)} \\
&\quad \to 0 \ \text{ as } \ |t_2 - t_1| \to 0.
\end{aligned}
$$

Herein C and σ defined in Proposition 1. This shows that Q is well-defined. For $u, v \in C[a,b]$, by virtue of Proposition 2, one has

$$
\begin{aligned}
\|Qu - Qv\| &\leqslant C_{\max} \int_a^b \psi'(s)(\psi(b) - \psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}|f(s,u(s)) - f(s,v(s))| \,\mathrm{d}s \\
&\leqslant C_{\max} \int_a^b \psi'(s)(\psi(b) - \psi(s))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(s)|u(s) - v(s)| \,\mathrm{d}s \\
&\leqslant C_{\max}\left\|\psi'(\cdot)(\psi(b) - \psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)\right\|_{L^1(a,b)}\|u - v\|.
\end{aligned}
$$

Since $C_{\max}\left\|\psi'(\cdot)(\psi(b) - \psi(\cdot))^{\alpha_1+\alpha_2-\alpha_3-1}k_f(\cdot)\right\|_{L^1(a,b)} < C_{\max}(1/C_{\max}) = 1$, we conclude that Q is a contraction. Consequently, Q admits a unique fixed point in $C[a,b]$, which is a mild solution of the problem (1) and (2). The proof of Theorem is done. $\square$

COROLLARY 5. *Suppose that there exist $\kappa_1, \kappa_2 > 0$, $\gamma_1, \gamma_2 < \alpha_1 + \alpha_2 - \alpha_3$, and $\theta_1, \theta_2 < 1$ such that*

$$
\begin{aligned}
|f(t,0)| &\leqslant \kappa_1(\psi(b) - \psi(t))^{-\gamma_1}(\psi(t) - \psi(a))^{-\theta_1}, \quad a < t < b, \\
|f(t,u) - f(t,v)| &\leqslant \kappa_2(\psi(b) - \psi(t))^{-\gamma_2}(\psi(t) - \psi(a))^{-\theta_2}|u - v|, \quad a < t < b,\ u,v \in \mathbb{R}.
\end{aligned}
$$

If κ_2 is small then the problem (1) and (2) has a unique mild solution.

Proof. We can use (19) to verify that the assumptions in Theorem 5 hold. $\square$

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