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ON MINMAX AND MAXMIN INEQUALITIES FOR CENTERED CONVEX BODIES

ZOKHRAB MUSTAFAEV

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Abstract. One of the challenging problems from the geometry of (normed or) Minkowski spaces is the question of whether the unit ball must be an ellipsoid if it is a solution of the corresponding isoperimetric problem. The inner and outer radii of the unit ball with respect to the corresponding isoperimetrix (represented in terms of cross-section measures) will be used to establish a result on this problem for a specific measure. Some new minmax and maxmin inequalities for centered convex bodies will also be established.

1. Introduction

Let $\mathbb{M}^d := (\mathbb{R}^d, ||\cdot||)$ be a d-dimensional *Minkowski space* (i.e., a finite-dimensional real Banach space). When $d = 2$, it is called a *Minkowski plane*. The setting for this paper will be both the standard Euclidean space $\mathbb{R}^d$ and a d-dimensional Minkowski space $\mathbb{M}^d$. Some definitions and notations from both spaces will be used.

There are various ways of introducing the notion of measure in a Minkowski space. Two notions of measure, one due to Busemann and the other due to Holmes-Thompson (see Section 2 below), have been widely used in the literature. Once the notion of measure is introduced in a given Minkowski space, the question of whether the unit ball B of $\mathbb{M}^d$ must be an ellipsoid (i.e. $\mathbb{M}^d$ is an Euclidean space) if it is a solution of the isoperimetric problem is a challenging open problem for $d \geqslant 3$ (see [3], [4], and [22]). For the Holmes-Thompson measure this question is equivalent to asking whether B must be an ellipsoid if B and the projection body of its polar ΠB° are homothetic (or B° and ΠB are homothetic). With Busemann's definition of measure the question becomes whether B and the polar body of its intersection body $(IB)^\circ$ are homothetic (or B° and IB are homothetic). For $d = 2$, the unit ball B has this property if and only if ∂B is a *Radon curve*. These curves were introduced by Radon [15] (see [14] and the references therein for more about Radon curves). In $\mathbb{M}^2$ the following statement holds: if the unit circle ∂B is a Radon curve, then B is a solution of the isoperimetric problem.

The purpose of this manuscript is to establish some minmax and maxmin inequalities for centered convex bodies. We also show that if the unit ball B satisfies a certain

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property, then B and ΠB° cannot be homothetic unless B is an ellipsoid. To derive the results conveniently, the inner radius and outer radius of the unit ball with respect to its isoperimetrix (for Holmes-Thompson and Busemann measures) will be represented in terms of cross-section measures.

2. Notations and background materials

A *convex body* K in $\mathbb{R}^d$, $d \geqslant 2$, is a compact, convex set with nonempty interior. K is said to be *centered* if it is symmetric with respect to the origin o of $\mathbb{R}^d$. S^{d-1} will denote the standard Euclidean unit sphere in $\mathbb{R}^d$. We write $\lambda_i(\cdot)$ for the i*-dimensional Lebesgue measure (volume)* in $\mathbb{R}^d$, where $1 \leqslant i \leqslant d$, and when $i = d$ we omit the subscript. For a given direction $u \in S^{d-1}$, we use $u^\perp$ to denote the $(d-1)$-dimensional hyperplane (passing through the origin) orthogonal to u, and by l_u the 1-subspace parallel to u. Furthermore, $\lambda_1(K|l_u)$ denotes the *width* of K at u, and $\lambda_{d-1}(K|u^\perp)$ the $(d-1)$*-dimensional outer cross-section measure* or *brightness* of K at $u \in S^{d-1}$, where $K|u^\perp$ is the orthogonal projection of K onto $u^\perp$ (see [4] for these notations).

For a convex body K in $\mathbb{R}^d$, the *polar body* K° of K is defined by

$$K^\circ = \{y \in \mathbb{R}^d : \langle x, y\rangle \leqslant 1, x \in K\},$$

where $\langle \cdot,\cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$.

The following properties of the centered convex bodies will be used here: $(K^\circ)^\circ = K$, $(\alpha K)^\circ = (1/\alpha)K^\circ$ for $\alpha > 0$, and if $K_1 \subseteq K_2$, then $K_2^\circ \subseteq K_1^\circ$.

We will use the standard basis to identify $\mathbb{R}^d$ and its *dual space* $\mathbb{R}^{d*}$. In that case, $\lambda_i(\cdot)$ and $\lambda_i^*(\cdot)$ coincide in $\mathbb{R}^d$. For the i-dimensional volume of the unit ball in $\mathbb{R}^i$, we write ε_i.

The *support function* $h_K : S^{d-1} \to \mathbb{R}$ of a convex body K is defined as $h_K(u) = \sup\{\langle u, y\rangle : y \in K\}$. It is well known that h_K is monotone with respect to inclusion (i.e. if $K \subseteq L$, then $h_K \leqslant h_L$), and positive homogeneous (i.e. $h_{\alpha K}(u) = h_K(\alpha u) = \alpha h_K$ for all $\alpha > 0$). Furthermore, if $0 \in K$, then $h_K(u)$ is the distance from the origin to the supporting hyperplane of K with outer unit normal vector u (see [19] for more about support functions). When the origin is an interior point of K its *radial function* $\rho_K(u)$ is defined by $\rho_K(u) = \max\{\alpha \geqslant 0 : \alpha u \in K\}$. The following relation between these two functions is well known:

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)}, \; u \in S^{d-1}. \tag{1}$$

Note that if K is a centered convex body, then $2\rho_K(u) = \lambda_1(K \cap l_u)$, and $2h_K(u) = \lambda_1(K|l_u)$ for any $u \in S^{d-1}$.

For a given convex body K in $\mathbb{R}^d$, the *projection body* ΠK of K is defined as the convex body whose supporting hyperplane in a given direction u has a distance $\lambda_{d-1}(K|u^\perp)$ from the origin, i.e., $h_{\Pi K}(u) = \lambda_{d-1}(K|u^\perp)$ for each $u \in S^{d-1}$ (see [4, Chapter 4]). Note that any projection body is a *zonoid* (i.e., a limit of vector sums of segments) centered at the origin (see [20] and [5] for properties and applications of zonoids). The *intersection body* IK of a convex body $K \subset \mathbb{R}^d$ is defined by its radial function $\rho_{IK}(u) = \lambda_{d-1}(K \cap u^{\perp})$ for each $u \in S^{d-1}$ (cf. [7] and [4, Chapter 8]). If K is a symmetric convex body, then the result of Busemann (see [2]) states that IK is also convex and symmetric.

A Minkowski space $\mathbb{M}^d$ with unit ball B possesses a Haar measure μ_B, and this measure is unique up to multiplication of the Lebesgue measure by a constant, i.e., $\mu_B(\cdot) = \sigma_B \lambda(\cdot)$. These two measures μ_B and λ must also agree in the standard Euclidean space. For a given convex body K in $\mathbb{M}^d$, its *d-dimensional Busemann volume* is defined by

$$\mu_B^{Bus}(K) = \frac{\varepsilon_d}{\lambda(B)}\lambda(K), \text{ i.e., } \sigma_B = \frac{\varepsilon_d}{\lambda(B)},$$

and its *d-dimensional Holmes-Thompson volume* is defined by

$$\mu_B^{HT}(K) = \frac{\lambda(K)\lambda(B^\circ)}{\varepsilon_d}, \text{ i.e., } \sigma_B = \frac{\lambda(B^\circ)}{\varepsilon_d},$$

see [22, Chapter 5]. In order to define the Minkowski surface area of a convex body K in $\mathbb{M}^d$, $\mu_B(\partial K)$, one has to define σ_B similarly in $\mathbb{M}^{d-1}$. This area generating function $\sigma_B(u)$ is invariant under linear transformations of $\mathbb{R}^d$, continuous with respect to Hausdorff metric, and normalized by $\sigma(E) = \varepsilon_{d-1}$ if E is an $(d-1)$-dimensional ellipsoid. For the Holmes-Thompson measure, this function is defined to be $\sigma_B(u) = \lambda_{d-1}((B \cap u^{\perp})^\circ)/\varepsilon_{d-1}$, and for the Busemann measure it is $\sigma_B(u) = \varepsilon_{d-1}/\lambda_{d-1}(B \cap u^{\perp})$ (see [22, pp. 150–151]).

$\sigma_B(u)$ is the support function of a convex body I_B which is related to isoperimetric problems in Minkowski spaces. Namely, among all convex bodies of a given volume (area), a homothetical copy of I_B has minimal surface area (perimeter). In a Minkowski plane, I_B is the polar body of the unit disc B, rotated by an angle of 90°.

For the Holmes-Thompson measure, I_B is given by $I_B^{HT} = \Pi B^\circ/\varepsilon_{d-1}$ (cf. [22, p. 150 and p. 157] for detailed explanation), and therefore it is a centered zonoid. For the Busemann measure, I_B is defined by $I_B^{Bus} = \varepsilon_{d-1}(IB)^\circ$ (see again [22, pp. 150-151]).

For a given Minkowski space $\mathbb{M}^d$ with unit ball B, $\hat{I}_B = \sigma_B^{-1} I_B$ is called the *isoperimetrix* of the space. This body has the property of $\mu_B(\partial \hat{I}_B) = d\mu_B(\hat{I}_B)$. Thus, for the Holmes-Thompson measure, we have

$$\hat{I}_B^{HT} = \frac{\varepsilon_d}{\lambda(B^\circ)} I_B^{HT} = \frac{\varepsilon_d}{\varepsilon_{d-1}} \frac{1}{\lambda(B^\circ)} \Pi B^\circ, \tag{2}$$

and for the Busemann measur, we have

$$\hat{I}_B^{Bus} = \frac{\lambda(B)}{\varepsilon_d} I_B^{Bus} = \frac{\varepsilon_{d-1}}{\varepsilon_d} \lambda(B)(IB)^\circ. \tag{3}$$

For convex bodies K and L, we define *the (relative) inner radius* of K with respect to L to be the largest value of $\alpha \geqslant 0$ such that a translate of K contains αL, i.e.

$$r(K,L) := \max\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \subseteq K + x\},$$

and the *(relative) outer radius* of K with respect to L to be the smallest value of $\alpha \geqslant 0$ such that a translate of K is contained in αL, i.e.

$$R(K,L) := \min\{\alpha : \exists x \in \mathbb{M}^d \text{ with } \alpha L \supseteq K + x\}.$$

When K and L are centered convex bodies, the quantities $r(K,L)$ and $R(K,L)$ can also be defined in terms of the support functions of the involved sets. That is $r(K,L)$ is the maximum value of α such that $\alpha \leqslant h_K(u)/h_L(u)$ for all $u \in S^{d-1}$. Similarly, $R(K,L)$ is the minimum value of α such that $\alpha \geqslant h_K(u)/h_L(u)$ for all $u \in S^{d-1}$ (cf. [18] and [23]).

3. Representations of radii using cross-section measures

For a convex body K in $\mathbb{M}^d$, let $w_B(K)$ and $D_B(K)$ be the *Minkowskian width* (i.e., $w_B(K) = \min_{u \in S^{d-1}} \frac{2w(K,u)}{w(B,u)}$, where $w(K,u)$ is the Euclidean width of K in the direction u) and the respective maximum, namely the *Minkowskian diameter*, respectively. One can easily see that $r(\hat{I}_B, B) = R(B, \hat{I}_B)^{-1}$ and $R(\hat{I}_B, B) = r(B, \hat{I}_B)^{-1}$ hold for the Holmes-Thompon and also for the Busemann measures. Also, it is easy to establish that if K is a centered convex body in $\mathbb{M}^d$, then $2r(K,B) = w_B(K)$ and $2R(K,B) = D_B(K)$.

Some sharp bounds for $r(B, \hat{I}_B^{HT})$, $R(B, \hat{I}_B^{HT})$, $r(B, \hat{I}_B^{Bus})$, and $R(B, \hat{I}_B^{Bus})$ have been already established (see [10], [22]). It turns out that the inner and outer radii of the unit ball with respect to its isoperimetrix can also be represented in terms of cross-section measures in Minkowski spaces. These representations are given below, and the confirmation of them presented here is different (and simpler) than the one given in [12] and [13].

PROPOSITION 3.1. *Let B be the unit ball of $\mathbb{M}^d$. Then*

$$r(B, \hat{I}_B^{HT}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u \in S^{d-1}} \frac{\lambda(B^\circ)}{\lambda_{d-1}(B^\circ | u^\perp)\lambda_1(B^\circ \cap l_u)},$$

$$R(B, \hat{I}_B^{HT}) = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \max_{u \in S^{d-1}} \frac{\lambda(B^\circ)}{\lambda_{d-1}(B^\circ | u^\perp)\lambda_1(B^\circ \cap l_u)}.$$

Proof. As mentioned above,

$$r(B, \hat{I}_B^{HT}) = \min_{u \in S^{d-1}} \frac{h_B(u)}{h_{\hat{I}_B^{HT}}}.$$

Using (1) and (2), the right side can be written as

$$r(B, \hat{I}_B^{HT}) = \frac{\lambda(B^\circ)\varepsilon_{d-1}}{\varepsilon_d} \min_{u \in S^{d-1}} \frac{h_B(u)}{h_{\Pi B^\circ}} = \frac{2\varepsilon_{d-1}}{\varepsilon_d} \min_{u \in S^{d-1}} \frac{\lambda(B^\circ)}{\lambda_{d-1}(B^\circ | u^\perp)\lambda_1(B^\circ \cap l_u)}.$$

For $R(B, \hat{I}_B^{HT})$, we use

$$R(B, \hat{I}_B^{HT}) = \max_{u \in S^{d-1}} \frac{h_B(u)}{h_{\hat{I}_B^{HT}}}.$$

Hence,

$$R(B,\hat{I}_B^{HT}) = \frac{\lambda(B^\circ)\varepsilon_{d-1}}{\varepsilon_d}\max_{u\in S^{d-1}}\frac{h_B(u)}{h_{\Pi B^\circ}} = \frac{2\varepsilon_{d-1}}{\varepsilon_d}\max_{u\in S^{d-1}}\frac{\lambda(B^\circ)}{\lambda_{d-1}(B^\circ|u^\perp)\lambda_1(B^\circ\cap l_u)}. \qquad \square$$

PROPOSITION 3.2. *Let B be the unit ball of $\mathbb{M}^d$. Then*

$$r(B,\hat{I}_B^{Bus}) = \frac{\varepsilon_d}{2\varepsilon_{d-1}}\min_{u\in S^{d-1}}\frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)},$$

$$R(B,\hat{I}_B^{Bus}) = \frac{\varepsilon_d}{2\varepsilon_{d-1}}\max_{u\in S^{d-1}}\frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)}.$$

Proof. From the definition of the inner radius, we have

$$r(B,\hat{I}_B^{Bus}) = \min_{u\in S^{d-1}}\frac{h_B(u)}{h_{\hat{I}_B^{Bus}}}.$$

Applying (1) and (3), we get

$$\begin{aligned} r(B,\hat{I}_B^{Bus}) &= \frac{\varepsilon_d}{\lambda(B)\varepsilon_{d-1}}\min_{u\in S^{d-1}}\frac{h_B(u)}{h_{(IB)^\circ}} = \frac{\varepsilon_d}{\varepsilon_{d-1}}\min_{u\in S^{d-1}}\frac{h_B(u)\rho_{IB}(u)}{\lambda(B)} \\ &= \frac{\varepsilon_d}{2\varepsilon_{d-1}}\min_{u\in S^{d-1}}\frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)}. \end{aligned}$$

For $R(B,\hat{I}_B^{Bus})$, we have

$$R(B,\hat{I}_B^{Bus}) = \max_{u\in S^{d-1}}\frac{h_B(u)}{h_{\hat{I}_B^{Bus}}}.$$

The result is obtained by expanding the right side of this identity similar to $r(B,\hat{I}_B^{Bus})$. $\square$

We mention the following well-known sharp inequalities for centered convex bodies in $\mathbb{R}^d$ (see [16], [21], and [9] for general results).

$$1 \leqslant \frac{\lambda_{d-1}(B|u^\perp)\lambda_1(B\cap l_u)}{\lambda(B)} \leqslant d,$$

$$1 \leqslant \frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)} \leqslant d.$$

Using these inequalities and Propositions 3.1 and 3.2, one can easily establish some sharp bounds for the inner and outer radii of the unit ball B with respect to its isoperimetrix. Establishing some other exact bounds are challenging minmax/maxmin problems. For example, for centered convex bodies of B in $\mathbb{R}^d$, the minimum value of

$$\max_{u\in S^{d-1}}\frac{\lambda_{d-1}(B\cap u^\perp)\lambda_1(B|l_u)}{\lambda(B)}$$

is still unknown.

4. Inequalities for cross-section measures and radii

If B and ΠB° are homothetic, then $r(B,\hat{I}_B^{HT}) = R(B,\hat{I}_B^{HT})$. Due to Proposition 3.1, it is equivalent to the fact that $\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)/\lambda(B)$ is a constant for all $u \in S^{d-1}$. The quantity $\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)$ is the volume of a cylinder circumscribed about B with generators parallel to u and bounded by the two parallel hyperplanes that support B at $\partial B \cap l_u$.

THEOREM 4.1. *If B is a centered convex body in $\mathbb{R}^d$, $d \geqslant 3$, satisfying*

$$\min_{u\in S^{d-1}} \frac{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)}{\lambda(B)} \leqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

then B and ΠB° are homothetic if and only if B is an ellipsoid.

Proof. It is well known that $r(B,\hat{I}_B^{HT}) \leqslant 1$ with equality if and only if B is an ellipsoid. Using Proposition 3.1, we get

$$\max_{u\in S^{d-1}} \frac{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)}{\lambda(B)} \geqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d},$$

with equality if and only if B is an ellipsoid (see also [10], [6], and [17] for all convex bodies). By Proposition 3.1, the property

$$\min_{u\in S^{d-1}} \frac{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)}{\lambda(B)} \leqslant \frac{2\varepsilon_{d-1}}{\varepsilon_d}$$

is equvalent to $R(B,\hat{I}_B^{HT}) \geqslant 1$. Therefore $r(B,\hat{I}_B^{HT}) = R(B,\hat{I}_B^{HT})$ if and only if

$$\frac{\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)}{\lambda(B)} = \frac{2\varepsilon_{d-1}}{\varepsilon_d}$$

for all $u \in S^{d-1}$. This is the true if and only if B is an ellipsoid. $\square$

The quantity $(1/d)\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)$ is the maximum volume of a double-cone inscribed in B with base $B\cap u^{\perp}$. We mention that in their paper [3], Busemann and Petty posed ten problems about centrally symmetric convex bodies. So far only Problem 1 from there (called *the Busemann-Petty problem*) has been solved completely (see [4] and the references therein). Problem 5 of that paper asks the following: For a given unit vector u we construct the cone of maximal volume $C(u)$ with base $\lambda_{d-1}(B\cap u^{\perp})$ and apex in B. The apex of such a cone is any point of B on a supporting hyperplane parallel to $u^{\perp}$. Are the ellipsoids characterized by the property that $C(u)$ is constant when $d \geqslant 3$? In [1], the authors proved that if B is sufficiently close to the Euclidean ball in the Banach-Mazur metric, then B is an ellipsoid. In [8], Lutwak proved the following result for the volume of double-cones inscribed in a centered convex body B in $\mathbb{R}^d$ with $d \geqslant 3$:

$$\min_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u) \leqslant \frac{2\varepsilon_d\varepsilon_{d-1}}{\lambda(B^{\circ})},$$

with equality if and only if B is an ellipsoid.

One can also see that the quantities $\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)/\lambda(B)$ and $\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)/\lambda(B)$ are invariant under a dilatation. Therefore, one could set $\lambda(B)=\varepsilon_d$. In [11], it was proved that if B is a centered convex body in $\mathbb{R}^d$ with $\lambda(B)=\varepsilon_d$, then

$$\min_{u\in S^{d-1}}\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B^{\circ}|l_u)\leqslant 2\varepsilon_{d-1}, \tag{4}$$

with equality if and only if B is an ellipsoid.

We also mention that $\lambda_{d-1}(B\cap u^{\perp})\leqslant\lambda_{d-1}(B|u^{\perp})$ and $\lambda_1(B\cap l_u)\leqslant\lambda_1(B|l_u)$ for all $u\in S^{d-1}$. Furthermore, $\min\limits_{u\in S^{d-1}}\lambda_1(B\cap l_u)=\min\limits_{u\in S^{d-1}}(B|l_u)$.

We present here the proof of the following exact inequalities (see also [12]).

THEOREM 4.2. *If B is the unit ball of $\mathbb{M}^d$, $d\geqslant 2$, then*

$$R(B,\hat{I}_B^{HT})r(B^{\circ},\hat{I}_{B^{\circ}}^{Bus})\leqslant 1,$$

$$R(B,\hat{I}_B^{Bus})r(B^{\circ},\hat{I}_{B^{\circ}}^{HT})\leqslant 1.$$

Proof. It is well known that $IB^{\circ}\subseteq\Pi B^{\circ}$, with equality for $d\geqslant 3$ if and only if B is an ellipsoid (see [4]). Using (2) and (3), this inclusion can be written as

$$(\hat{I}_{B^{\circ}}^{Bus})^{\circ}\subseteq\hat{I}_B^{HT} \tag{5}$$

with equality for $d\geqslant 3$ if and only if B is an ellipsoid. From the definition of inner and outer radii of the polar of B for the Busemann measure, we have

$$r(B^{\circ},\hat{I}_{B^{\circ}}^{Bus})\hat{I}_{B^{\circ}}^{Bus}\subseteq B^{\circ}\subseteq R(B^{\circ},\hat{I}_{B^{\circ}}^{Bus})\hat{I}_{B^{\circ}}^{Bus}.$$

Thus,

$$\frac{1}{R(B^{\circ},\hat{I}_{B^{\circ}}^{Bus})}(\hat{I}_{B^{\circ}}^{Bus})^{\circ}\subseteq B\subseteq\frac{1}{r(B^{\circ},\hat{I}_{B^{\circ}}^{Bus})}(\hat{I}_{B^{\circ}}^{Bus})^{\circ}\subseteq\frac{1}{r(B^{\circ},\hat{I}_{B^{\circ}}^{Bus})}\hat{I}_B^{HT}.$$

Hence

$$R(B,\hat{I}^{HT})\leqslant\frac{1}{r(B^{\circ},\hat{I}_{B^{\circ}}^{Bus})}.$$

To obtain the second inequality one needs to use the inner and outer radii of B° for the Holmes-Thompson measure and the dual of (5). One can easily establish that equality holds for both cases when B is a centered Euclidean ball. □

COROLLARY 4.3. *If B is a centered convex body in $\mathbb{R}^d$, then*

$$\min_{u\in S^{d-1}}\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)\leqslant\min_{u\in S^{d-1}}\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u),$$

$$\max_{u\in S^{d-1}}\lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)\leqslant\max_{u\in S^{d-1}}\lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u).$$

Proof. The results follow from Theorem 4.2, Propositions 3.1 and 3.2. □

The significance of Corollary 4.3 is given by the fact that if

$$\min_{u\in S^{d-1}} \lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)/\lambda(B) \leqslant 2\varepsilon_{d-1}/\varepsilon_d,$$

then

$$\min_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u)/\lambda(B) \leqslant 2\varepsilon_{d-1}/\varepsilon_d.$$

In [13], it was proved that for a Minkowski plane with unit ball B

$$r(B,\hat{I}_B^{HT})r(B^{\circ},\hat{I}_{B^{\circ}}^{Bus}) = 1,$$

$$(B,\hat{I}_B^{Bus})r(B^{\circ},\hat{I}_{B^{\circ}}^{HT}) = 1$$

if and only if ∂B is a Radon curve.

One might conjecture that for $d\geqslant 3$, $r(B,\hat{I}_B^{HT})r(B^{\circ},\hat{I}_{B^{\circ}}^{Bus})=1$ and $r(B,\hat{I}_B^{Bus})r(B^{\circ},\hat{I}_{B^{\circ}}^{HT})$ $=1$ if and only if B is a centered ellipsoid.

QUESTION. Is it true that in $\mathbb{M}^d$, $d\geqslant 3$,

$$R(B,\hat{I}_B^{Bus})R(B^{\circ},\hat{I}_{B^{\circ}}^{HT}) \geqslant 1$$

with equality if and only if B is an ellipsoid? This question is equivalent to asking whether

$$\max_{u\in S^{d-1}} \lambda_{d-1}(B\cap u^{\perp})\lambda_1(B|l_u) \geqslant \min_{u\in S^{d-1}} \lambda_{d-1}(B|u^{\perp})\lambda_1(B\cap l_u)$$

with equality if and only if B is an ellipsoid?

Establishing the exact upper bound of $r(B,\hat{I}_B^{Bus})$ is a challenging open maxmin problem (cf. Problem 6 in [3]). We prove the following related inequality.

THEOREM 4.4. *Let B be the unit ball of $\mathbb{M}^d$ with $\lambda(B)=\varepsilon_d$. Then*

$$r(B^{\circ},\hat{I}_B^{Bus}) \leqslant 1,$$

with equality if and only if B is an ellipsoid.

Proof. From the definition of the inner radius, we have

$$r(B^{\circ},\hat{I}_B^{Bus}) = \min_{u\in S^{d-1}} \frac{h_{B^{\circ}}(u)}{h_{\hat{I}_B^{Bus}}(u)}.$$

The right side of this identity can be expanded as

$$\min_{u\in S^{d-1}} \frac{h_{B^{\circ}}(u)}{h_{\hat{I}_B^{Bus}}(u)} = \frac{\lambda(B)}{\varepsilon_d\varepsilon_{d-1}} \min_{u\in S^{d-1}} \frac{h_{B^{\circ}}(u)}{h_{(IB)^{\circ}}(u)} = \frac{\lambda(B)}{\varepsilon_d\varepsilon_{d-1}} \min_{u\in S^{d-1}} h_{B^{\circ}}(u)\rho_{IB}(u).$$

The result follows from $\lambda(B)=\varepsilon_d$, (4), and Proposition 3.2. □

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Zokhrab Mustafaev
Department of Mathematics
University of Houston-Clear Lake
Houston, TX 77058 USA
mustafaev@uhcl.edu