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ON STOCHASTIC ORDERS DEFINED BY OTHER STOCHASTIC ORDERS AND TRANSFORMATIONS OF PROBABILITIES

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Abstract. Significant stochastic orders can be characterized by other stochastic orders when probabilities are transformed by means of measurable mappings which fulfils certain conditions. The main aim of this manuscript is to provide a unified approach of those orders. Our study shows how to obtain important results of such orders by means of conditions on the underlying stochastic orders. Those results are mainly in relation to maximal generators, to transition kernels between ordered probabilities, and to probabilistic operators of kernels.

1. Introduction

Relevant stochastic orders share a common mathematical pattern. They can be characterized by well-known stochastic orders when probabilities are transformed by appropriate mappings. Basically, an order $\preceq_1$ of that class satisfies that $P' \preceq_1 P''$ when $P' \circ m^{-1} \preceq_2 P'' \circ m^{-1}$, where $\preceq_2$ is a known stochastic order and m is a measurable mapping which fulfils certain conditions. In this paper, we focus our attention on the analysis of stochastic orders which can be characterized by means of other stochastic orders when random elements are transformed adequately, giving a unified approach to those orderings.

The structure of the paper is the following. Section 2 contains the concepts and notation that we need for our analysis. The results and applications of the manuscript are included in Section 3. Section 3.1 contains examples of stochastic orders given by other orders when probabilities are transformed by some mappings. Among other examples, we see that the linear stochastic order can be characterized by a Scarsini's order when probabilities are transformed by a specific mapping and vice versa. Section 3.2 provides how to derive maximal generators of those orders when the corresponding underlying orders are integral. Easy characterizations of the maximal generators of some important orders are derived in Section 3.3. Section 3.4 approaches different questions

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about such stochastic orders in relation to transition kernels, kernels and probabilistic operators of kernels. Namely, it is proved the existence of a transition kernel between probabilities ordered in $\preceq_1$ under mild conditions on the maximal generator of $\preceq_2$. Moreover, such a kernel is obtained by means of a transition kernel between probabilities ordered in $\preceq_2$. It is shown that for the probabilistic operators associated with the above kernels, the monotone condition with respect to the corresponding stochastic order is equivalent. We also relate the property of being smaller with respect to the orders $\preceq_1$ and $\preceq_2$ for probabilistic operators associated with kernels.

2. Preliminaries

A stochastic order is a pre-order on a set of probabilities associated with a measurable space. A detailed and rigorous analysis of stochastic orders, from both applied and theoretical approaches, can be found, for instance, in the books [10], [14] and [1].

Throughout the paper, $\mathscr{S}$ and $\mathscr{S}'$ will stand for Polish spaces, and $\mathscr{B}_{\mathscr{S}}$ and $\mathscr{B}_{\mathscr{S}'}$ for the corresponding Borel σ-algebras.

From now on, the term measurable mapping will refer to Borel σ-algebras (Borel measurability).

A stochastic order $\preceq$ on a set of probabilities on the measurable space $(\mathscr{S},\mathscr{B}_{\mathscr{S}})$ is said to be integral, when there exists a set $\mathscr{F}$ of measurable mappings from $\mathscr{S}$ to $\mathbb{R}$ ($\mathbb{R}$ endowed with the usual Borel σ-algebra) such that

$$P \preceq Q \quad \text{when} \quad \int_{\mathscr{S}} f\,dP \leqslant \int_{\mathscr{S}} f\,dQ \tag{1}$$

for any $f \in \mathscr{F}$ such that the integrals exist. Any set of mappings $\mathscr{F}$ satisfying (1) is said to be a generator of the order.

A key concept in the manuscript is the so-called maximal generator of an integral stochastic order. Roughly speaking, the maximal generator is the largest generator of that order inside an appropriate class of measurable mappings.

The maximal generator of an integral stochastic order depends on the weight function. A weight function is a measurable mapping $b:\mathscr{S}\to[1,\infty)$. That mapping defines the b-norm of a function $f:\mathscr{S}\to\mathbb{R}$, denoted by $\|f\|_b$, as

$$\|f\|_b = \sup_{x\in\mathscr{S}} \frac{|f(x)|}{b(x)}.$$

Given b a weight function, we will denote by $\mathscr{B}_b$ the set of measurable mappings from $\mathscr{S}$ to $\mathbb{R}$ with bounded b-norm, and by $\mathscr{P}_b$ the set of probabilities on $\mathscr{B}_{\mathscr{S}}$ such that

$$\int_{\mathscr{S}} b\,dP < \infty.$$

If $\preceq$ is an integral stochastic order on $\mathscr{P}_b$, the maximal generator of $\preceq$ is the set of all functions $f \in \mathscr{B}_b$ such that

$$P \preceq Q \quad \text{implies} \quad \int_{\mathscr{S}} f\,dP \leqslant \int_{\mathscr{S}} f\,dQ.$$

Note that since $f \in \mathscr{B}_b$ and $P, Q \in \mathscr{P}_b$, the integrals exist.

The reader is referred to [9] and Chapter 2 of [10] for a rigorous and detailed analysis of integral stochastic orders.

Let $m : \mathscr{S} \to \mathscr{S}'$ be a measurable mapping. Let P be a probability on $\mathscr{B}_{\mathscr{S}}$, $P \circ m^{-1}$ will stand for the probability defined as $P \circ m^{-1}(B) = P(m^{-1}(B))$ for any $B \in \mathscr{B}_{\mathscr{S}'}$.

It is worth mentioning that the inverse of a bijective (Borel) measurable mapping is also (Borel) measurable (see, for instance, [5] or [15]). Thus, if $m : \mathscr{S} \to \mathscr{S}'$ is a bijective measurable mapping and P is a probability on $\mathscr{B}_{\mathscr{S}'}$, $P \circ m$ is a probability on $\mathscr{B}_{\mathscr{S}}$. Observe that this is not true in general if we consider other σ-algebras.

Given $\preceq$ a partial order on $\mathscr{S}$, a mapping $f : \mathscr{S} \to \mathbb{R}$ is said to be $\preceq$-preserving if for any $x, y \in \mathscr{S}$ with $x \preceq y$, we have that $f(x) \leqslant f(y)$ (see, for instance, [11] or [13]).

Briefly, we summarize the concepts of kernel and transition kernel (see, for instance, [4]).

Let $(\Omega_1, \mathscr{A}_1)$ and $(\Omega_2, \mathscr{A}_2)$ be measurable spaces. A kernel Q from Ω_1 to Ω_2 is a mapping $Q : \Omega_1 \times \mathscr{A}_2 \to \mathbb{R}$ satisfying

i) $Q(\omega_1, \cdot) : \mathscr{A}_2 \to \mathbb{R}$ is a probability on $(\Omega_2, \mathscr{A}_2)$ for all $\omega_1 \in \Omega_1$,

ii) $Q(\cdot, E) : \Omega_1 \to \mathbb{R}$ is measurable (with respect to $\mathscr{A}_1$ and the usual Borel σ-algebra on $\mathbb{R}$) for any $E \in \mathscr{A}_2$.

If Q is a kernel from Ω_1 to Ω_2 and P_1 is a probability on $(\Omega_1, \mathscr{A}_1)$, Q "transforms" the probability P_1 into a probability on $(\Omega_2, \mathscr{A}_2)$. That probability, denoted by QP_1, is given by

$$QP_1(E) = \int_{\Omega_1} Q(\omega_1, E)\, dP_1$$

for any $E \in \mathscr{A}_2$.

If P_2 is a probability on $(\Omega_2, \mathscr{A}_2)$ and there exists Q, a kernel from Ω_1 to Ω_2, such that $QP_1 = P_2$, then Q is said to be a transition kernel between P_1 and P_2.

Let T be an operator on the set of probabilities associated with the measurable space $(\mathscr{S}, \mathscr{B}_{\mathscr{S}})$, that is, a mapping from such a set to itself. Let $\preceq$ be a stochastic order on that set of probabilities.

The operator T is said to be $\preceq$-monotone if $TP_1 \preceq TP_2$ for any probabilities P_1 and P_2 with $P_1 \preceq P_2$.

Given the operators T_1 and T_2, T_1 is said to be smaller than T_2 for the stochastic order $\preceq$, if $T_1 P \preceq T_2 P$ for any probability P. It will be denoted by $T_1 \preceq T_2$.

Given Q a kernel from $\mathscr{S}$ to $\mathscr{S}$, we will denote by T_Q the associated probabilistic operator, that is, $T_Q P = QP$.

If A is a subset of $\mathbb{R}^n$, I_A will stand for the indicator mapping of the set A.

From now on, e_i will stand for the i^{th} unit vector of $\mathbb{R}^n$, $1 \leqslant i \leqslant n$.

To conclude this section, we include some stochastic orders which will appear in the manuscript. Those orders are for probabilities associated with the measurable space $(\mathbb{R}^n, \mathscr{B}_{\mathbb{R}^n})$, where $\mathscr{B}_{\mathbb{R}^n}$ stands for the usual Borel σ-algebra on $\mathbb{R}^n$.

Let X and Y be random vectors, X is said to be smaller than Y in the *i*) usual stochastic order, denoted by $X \preceq_{st} Y$, if $E(f(X)) \leqslant E(f(Y))$ for all increasing mappings $f:\mathbb{R}^n \to \mathbb{R}$ such that the expectations exist, that is, for all mappings $f:\mathbb{R}^n \to \mathbb{R}$ with $f(x) \leqslant f(y)$ when $x \leqslant y$,

ii) upper orthant order, denoted by $X \preceq_{uo} Y$, if $\overline{F}_X(z) \leqslant \overline{F}_Y(z)$ for all $z \in \mathbb{R}^n$, where $\overline{F}_W$ stands for the multivariate survival function of the random vector W,

iii) extremality stochastic order in the direction $u \in S^{n-1}$, denoted by $X \preceq_{E_u} Y$, when $P(\mathscr{R}_u(X-t) \geqslant 0) \leqslant P(\mathscr{R}_u(Y-t) \geqslant 0)$ for all $t \in \mathbb{R}^n$, where $\mathscr{R}_u$ is a rotation matrix such that $\mathscr{R}_u u = \frac{1}{\sqrt{n}}\mathbf{1}$, with $\mathbf{1} = (1,\ldots,1)^t \in \mathbb{R}^n$ (see [6]),

iv) strong extremality stochastic order in the direction $u \in S^{n-1}$, denoted by $X \preceq_{SE_u} Y$, when $E(f(X)) \leqslant E(f(Y))$ for any $\preceq^u$-preserving mapping $f:\mathbb{R}^n \to \mathbb{R}$ such that the expectations exist, where the partial order $\preceq^u$ on $\mathbb{R}^n$ is given by $x \preceq^u y$ when $f(x) \leqslant f(y)$ for all $f \in \mathscr{G}^u$, with $x,y \in \mathbb{R}^n$, $\mathscr{G}^u = \{I_{C_t^u} \mid t \in \mathbb{R}^n\}$ and $C_t^u = \{x \in \mathbb{R}^n \mid \mathscr{R}_u(x-t) \geqslant 0\}$ (see [7]),

v) time value of money stochastic ordering, denoted by $X \preceq_{tvm} Y$, if $E(f(X)) \leqslant E(f(Y))$ for any $f \in \mathscr{F}$ such that the expectations exist, where $\mathscr{F} = \{f:\mathbb{R}^n \to \mathbb{R} \mid f(x+\varepsilon_i e_i) \geqslant f(x+\varepsilon_{i+1}e_{i+1})$ for all $x \in \mathbb{R}^n, 0 \leqslant \varepsilon_{i+1} \leqslant \varepsilon_i, 1 \leqslant i \leqslant n-1$, and $f(x+\varepsilon_n e_n) \geqslant f(x)$ for all $x \in \mathbb{R}^n$ and $0 \leqslant \varepsilon_n\}$ (see [8]),

vi) order $\preceq^1$, denoted by $X \preceq^1 Y$, if $a^t X \preceq_{st} a^t Y$ for any $a \in \mathbb{R}^n$ with $1 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$ (finite dimensional version, see [12]),

vii) linear stochastic order, when $s^t X \preceq_{st} s^t Y$ for any $s \in \mathbb{R}^n$ such that $0 \leqslant s$. It is denoted as $X \preceq_{l-st} Y$ (see [2]).

3. Main results

From now on, X and Y will denote random elements (measurable mappings) defined on a certain probability space, which take values on $\mathscr{S}$, P_X and P_Y will stand for the corresponding induced probabilities on $\mathscr{B}_{\mathscr{S}}$ and $m:\mathscr{S} \to \mathscr{S}'$ will be a bijective measurable mapping.

Let $\preceq_2$ be a stochastic order on the set of probabilities associated with the measurable space $(\mathscr{S}',\mathscr{B}_{\mathscr{S}'})$. Let $\preceq_1$ be the stochastic order on the set of probabilities associated with the measurable space $(\mathscr{S},\mathscr{B}_{\mathscr{S}})$, given by $P' \preceq_1 P''$ when $P' \circ m^{-1} \preceq_2 P'' \circ m^{-1}$, or in terms of random elements, $X \preceq_1 Y$ ($P_X \preceq_1 P_Y$) when $m(X) \preceq_2 m(Y)$ ($P_{m(X)} \preceq_2 P_{m(Y)}$).

Observe that the relation $\preceq_1$ inherits the reflexivity and transitivity of $\preceq_2$. Moreover, if $\preceq_2$ is antisymmetric, so is $\preceq_1$.

3.1. Some examples of stochastic orders characterized by other stochastic orders

Firstly, we include some examples of stochastic orders which can be characterized by means of other stochastic orders when probabilities are transformed adequately.

The following result relates the linear stochastic order $\preceq_{l-st}$ and the Scarsini's order $\preceq^1$. Basically, it says that the linear stochastic order $\preceq_{l-st}$ can be characterized by means of the order $\preceq^1$ and an appropriate transformation of random vectors, and vice versa.

PROPOSITION 3.1. *There exists a bijective mapping* $h : \mathbb{R}^n \to \mathbb{R}^n$ *such that for any random vectors* X *and* Y, $X \preceq_{l-st} Y$ *is the same as* $h(X) \preceq^1 h(Y)$.

Proof. Note that $X \preceq_{l-st} Y$ when $s^t X \preceq_{st} s^t Y$ for any $s \in \mathbb{R}^n$ such that $0 \leqslant s$. Observe that this is $\sum_{i=1}^n s_i X_i \preceq_{st} \sum_{i=1}^n s_i Y_i$ for any $0 \leqslant s_i$, with $1 \leqslant i \leqslant n$. On the other hand, $X \preceq^1 Y$ when $a^t X \preceq_{st} a^t Y$ for any $a \in \mathbb{R}^n$ with $1 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$. The condition $1 \geqslant a_1$ is irrelevant for the definition of the order, since the usual stochastic order is preserved under the product by positive scalars. So, it is sufficient to consider $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n \geqslant 0$.

Note that $\sum_{i=1}^n a_i X_i = (a_1 - a_2)X_1 + (a_2 - a_3)(X_1 + X_2) + (a_3 - a_4)(X_1 + X_2 + X_3) + \ldots + (a_{n-1} - a_n)(X_1 + X_2 + \ldots + X_{n-1}) + a_n(X_1 + X_2 + \ldots + X_n)$. As a consequence, it holds that $X \preceq^1 Y$ if and only if $\sum_{i=1}^n s_i(X_1 + X_2 + \ldots + X_i) \preceq_{st} \sum_{i=1}^n s_i(Y_1 + Y_2 + \ldots + Y_i)$ for any $s \in \mathbb{R}^n$ with $0 \leqslant s$.

That is, $X \preceq^1 Y$ is the same as $h^{-1}(X) \preceq_{l-st} h^{-1}(Y)$, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is the bijective linear map such that $h(e_i) = e_i - e_{i+1}$ for all $1 \leqslant i \leqslant n-1$ and $h(e_n) = e_n$. Alternatively, $X \preceq_{l-st} Y$ is equivalent to $h(X) \preceq^1 h(Y)$. □

A statistical test for the order $\preceq_{l-st}$ can be found in [2]. The relation between the linear stochastic order $\preceq_{l-st}$ and the Scarsini's order $\preceq^1$ stated in the present manuscript, permits to test on the order $\preceq^1$ by means of the test on $\preceq_{l-st}$.

EXAMPLE 3.2. The family of extremality stochastic orders allows the comparison of random vectors in different directions determined by unit vectors. This family of orders is motivated by important applications in the research of optimal allocations of wealth among risks in single period portfolio problems (see [6]). In that reference, it is proved that such an order can be characterized as follows, for any $u \in S^{n-1}$, $X \preceq_{E_u} Y$ when $\mathscr{R}_u X \preceq_{uo} \mathscr{R}_u Y$.

EXAMPLE 3.3. A stronger family of stochastic orders, the so-called strong extremality orders, denoted by $\preceq_{SE_u}$ for each $u \in S^{n-1}$, is introduced in [7]. In that manuscript, it is proved that two random vectors X and Y satisfy $X \preceq_{SE_u} Y$ when $\mathscr{R}_u X \preceq_{st} \mathscr{R}_u Y$.

EXAMPLE 3.4. Motivated by another problem in relation to financial mathematics, the so-called time value of money stochastic order, denoted by $\preceq_{tvm}$, is introduced in [8]. That order permits to compare long-term investments affected by the time value of money issue. After a detailed analysis of the order, the following characterization is obtained, given two random vectors X and Y, $X \preceq_{tvm} Y$ when $h^{-1}(X) \preceq_{st} h^{-1}(Y)$, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is the bijective linear map such that $h(e_i) = e_i - e_{i+1}$ for all $1 \leqslant i \leqslant n-1$ and $h(e_n) = e_n$.

The manuscript [2] also contains a test for the order $\preceq_{st}$. Than can be used to test on the orders $\preceq_{tvm}$ and $\preceq_{SE_u}$.

3.2. On the maximal generator of $\preceq_1$

Now, we study how to derive a maximal generator of the stochastic order $\preceq_1$ by means of a maximal generator of the stochastic order $\preceq_2$, when $\preceq_2$ is integral.

PROPOSITION 3.5. *Let $\preceq_2$ be an integral stochastic order and let $\mathscr{F}_2$ be a generator of such an order. Then, $\preceq_1$ is an integral stochastic order and the class of mappings $\mathscr{F}_2(m) = \{f \circ m \mid f \in \mathscr{F}_2\}$ is a generator of the order $\preceq_1$.*

Proof. By definition, $X \preceq_1 Y$ when $m(X) \preceq_2 m(Y)$, thus $X \preceq_1 Y$ when

$$\int_{\mathscr{S}'} f\, dP_{m(X)} \leqslant \int_{\mathscr{S}'} f\, dP_{m(Y)}$$

for any $f \in \mathscr{F}_2$ such that the integrals exist. Note that for any $B \in \mathscr{B}_{\mathscr{S}'}$, we have that $P_{m(X)}(B) = P_X \circ m^{-1}(B)$. By a change of variable (see, for instance, [3]), we obtain that

$$\int_{\mathscr{S}'} f\, dP_{m(X)} = \int_{\mathscr{S}} f \circ m\, dP_X.$$

Therefore, $X \preceq_1 Y$ when

$$\int_{\mathscr{S}} f \circ m\, dP_X \leqslant \int_{\mathscr{S}} f \circ m\, dP_Y$$

for any $f \in \mathscr{F}_2$ such that the integrals exist. □

PROPOSITION 3.6. *Let $b : \mathscr{S}' \to [1,\infty)$ be a weight function and let $f \in \mathscr{B}_b$. Then, $b \circ m : \mathscr{S} \to [1,\infty)$ is a weight function and $f \circ m \in \mathscr{B}_{b\circ m}$.*

Proof. Note that $b \circ m : \mathscr{S} \to [1,\infty)$ is measurable, hence $b \circ m$ is a weight function. On the other hand,

$$\|f \circ m\|_{b\circ m} = \sup_{x\in\mathscr{S}} \frac{|f \circ m(x)|}{b \circ m(x)} = \sup_{y\in\mathscr{S}'} \frac{|f(y)|}{b(y)} = \|f\|_b,$$

which proves the result. □

PROPOSITION 3.7. *Let $\mathscr{MG}_2$ be the maximal generator of the integral stochastic order $\preceq_2$ for the weight function b. Then, the maximal generator of the integral stochastic order $\preceq_1$ for the weight function $b \circ m$, denoted by $\mathscr{MG}_1$, is $\mathscr{MG}_1 = \{f \circ m \mid f \in \mathscr{MG}_2\}$.*

Proof. By Proposition 3.5, $\mathscr{MG}_2(m) = \{f \circ m \mid f \in \mathscr{MG}_2\}$ is a generator of $\preceq_1$. By Proposition 3.6, the mappings in $\mathscr{MG}_2(m)$ are of bounded $b \circ m$-norm. Therefore, $\mathscr{MG}_2(m) \subset \mathscr{MG}_1$.

Let $g : \mathscr{S} \to \mathbb{R}$ belong to $\mathscr{MG}_1$. Let W and Z be random elements with values in $\mathscr{S}'$ such that $W \preceq_2 Z$. This is the same as $m(m^{-1}(W)) \preceq_2 m(m^{-1}(Z))$, that is, $m^{-1}(W) \preceq_1 m^{-1}(Z)$. As a consequence,

$$\int_{\mathscr{S}} g\, dP_{m^{-1}(W)} \leqslant \int_{\mathscr{S}} g\, dP_{m^{-1}(Z)},$$

equivalently,

$$\int_{\mathscr{S}'} g \circ m^{-1}\, dP_W \leqslant \int_{\mathscr{S}'} g \circ m^{-1}\, dP_Z.$$

Note that $g \circ m^{-1} \in \mathscr{B}_b$ since

$$\|g \circ m^{-1}\|_b = \sup_{s' \in S'} \frac{|g \circ m^{-1}(s')|}{b(s')} = \sup_{s \in S} \frac{|g(s)|}{b \circ m(s)} = \|g\|_{b \circ m},$$

and this value is finite because $g \in \mathscr{MG}_1$. Therefore, $g \circ m^{-1} \in \mathscr{MG}_2$, and so $g \in \mathscr{MG}_2(m)$. Thus, $\mathscr{MG}_1 = \mathscr{MG}_2(m)$. $\square$

3.3. Maximal generators of some stochastic orders

In this section, we obtain easy expressions of the maximal generators of some well-known multivariate stochastic orders.

EXAMPLE 3.8. *On the maximal generator of the strong extremality order in the direction $u \in S^{n-1}$.*

Consider $\preceq_{SE_u}$, the strong extremality order in the direction $u \in S^{n-1}$. In [7], it is proved that two random vectors X and Y satisfy $X \preceq_{SE_u} Y$ when $\mathscr{R}_u X \preceq_{st} \mathscr{R}_u Y$. Note that $\mathscr{R}_u$ is a regular matrix and so the map $m : \mathbb{R}^n \to \mathbb{R}^n$, given by $m(x) = \mathscr{R}_u x$ for any $x \in \mathbb{R}^n$, is bijective and measurable.

For the weight function $b = 1$, the maximal generator of the stochastic order $\preceq_{st}$ is the set $\{f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ is bounded and increasing}\}$.

Proposition 3.7 proves that for the weight function $b \circ m = 1$, the maximal generator of the stochastic order $\preceq_{SE_u}$ is given by

$$\{f(\mathscr{R}_u(\cdot)) : \mathbb{R}^n \to \mathbb{R} \mid f \text{ is bounded and increasing}\}.$$

We should note that in [7], such a maximal generator was obtained involving a mathematical formulation which is difficult to use and apply. Namely, given $u \in S^{n-1}$, the class of bounded measurable $\preceq^u$-preserving mappings is the maximal generator of $\preceq_{SE_u}$, where the binary relation $\preceq^u$ on $\mathbb{R}^n$ is given by $x \preceq^u y$ when $f(x) \leqslant f(y)$ for all $f \in \mathscr{G}^u$, with $x, y \in \mathbb{R}^n$, $\mathscr{G}^u = \{I_{C_t^u} \mid t \in \mathbb{R}^n\}$ and $C_t^u = \{x \in \mathbb{R}^n \mid \mathscr{R}_u(x - t) \geqslant 0\}$, where $t \in \mathbb{R}^n$.

Observe that with the new characterization, no partial orders on $\mathbb{R}^n$ and preserving mappings of those partial orders are needed for the formulation of the maximal generator.

EXAMPLE 3.9. *On the maximal generator of the extremality order in the direction $u \in S^{n-1}$.*

In the case of $\preceq_{E_u}$, the extremality order in the direction $u \in S^{n-1}$, it holds that $X \preceq_{E_u} Y$ when $\mathscr{R}_u X \preceq_{uo} \mathscr{R}_u Y$ (see [6]).

A map $f : \mathbb{R}^n \to \mathbb{R}$ is called Δ-monotone if for every subset $J = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$ and for every $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k > 0$, it holds that $\Delta_{i_1}^{\varepsilon_1} \Delta_{i_2}^{\varepsilon_2} \dots \Delta_{i_k}^{\varepsilon_k} f(x) \geqslant 0$ for all $x \in \mathbb{R}^n$, where $\Delta_i^{\varepsilon} f(x) = f(x + \varepsilon e_i) - f(x)$.

It is known that for the weight function $b = 1$, the maximal generator of the upper orthant order $\preceq_{uo}$ is given by the class $\{f : \mathbb{R}^n \to \mathbb{R} \mid f$ is bounded and Δ-monotone$\}$.

By means of Proposition 3.7, we obtain that for the weight function $b \circ m = 1$, where m is the mapping in Example 3.8, the maximal generator of the stochastic order $\preceq_{E_u}$ is given by

$$\{f(\mathscr{R}_u(\cdot)) : \mathbb{R}^n \to \mathbb{R} \mid \ f \text{ is bounded and } \Delta\text{-monotone}\}.$$

It is interesting to point out that in [7], that maximal generator of $\preceq_{E_U}$ was obtained in a harder way.

EXAMPLE 3.10. *On the maximal generator of the time value of money stochastic order.*

The time value of money stochastic order satisfies that $X \preceq_{tvm} Y$ when $h^{-1}(X) \preceq_{st} h^{-1}(Y)$, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map such that $h(e_i) = e_i - e_{i+1}$ for all $1 \leqslant i \leqslant n-1$ and $h(e_n) = e_n$.

The maximal generator of this order when the weight function is $b \circ h^{-1} = 1$, can be obtained by means of Proposition 3.7. Thus, the set of mappings

$$\{f(h^{-1}(\cdot)) : \mathbb{R}^n \to \mathbb{R} \mid \ f \text{ is bounded and increasing}\}$$

is that maximal generator.

We should indicate that the maximal generator of this order was obtained in [8] more laboriously. In such a paper, it was proved that the class $\mathscr{F}_b$, where $\mathscr{F}_b = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \in \mathscr{F}$ and f is bounded$\}$ and

$\mathscr{F} = \{ f : \mathbb{R}^n \to \mathbb{R} \mid$

i) $f(x + \varepsilon_i e_i) \geqslant f(x + \varepsilon_{i+1} e_{i+1})$ for all $x \in \mathbb{R}^n$, $0 \leqslant \varepsilon_{i+1} \leqslant \varepsilon_i$, $1 \leqslant i \leqslant n-1$,

ii) $f(x + \varepsilon_n e_n) \geqslant f(x)$ for all $x \in \mathbb{R}^n$ and $0 \leqslant \varepsilon_n \}$,

is the maximal generator of the stochastic order $\preceq_{tvm}$.

It is worth noting that the new characterizations of the above maximal generators permits to check if a mapping belongs to those generators very easily. Previous characterizations were not so appropriate for such a purpose.

It is not hard to see that the linear stochastic order $\preceq_{l-st}$ and the Scarsini's order $\preceq^1$ are integral. To the best of the authors' knowledge, the problems of finding easy characterizations of the maximal generators of those orders remain unsolved. The results of this section imply that the solution of one of them provides the solution of the remaining maximal generator.

3.4. Applications to kernels

In this section, we obtain conditions on the maximal generator of $\preceq_2$ to guarantee the existence of a transition kernel between probabilities ordered in $\preceq_1$. That kernel is characterized by means of a transition kernel between probabilities ordered in $\preceq_2$. Moreover, we study relations between probabilistic operators given by kernels, with respect to the order $\preceq_1$ and to the order $\preceq_2$.

LEMMA 3.11. *If $\mathscr{MG}_2$ is closed under maximization, so is $\mathscr{MG}_1$.*

Proof. Let $g_1, g_2 \in \mathscr{MG}_1$. By Proposition 3.7, there exist $f_1, f_2 \in \mathscr{MG}_2$ such that $g_1 = f_1 \circ m$ and $g_2 = f_2 \circ m$. Thus, $\max\{g_1, g_2\} = \max\{f_1 \circ m, f_2 \circ m\} = \max\{f_1, f_2\} \circ m$. We have that $\mathscr{MG}_2$ is closed under maximization, $\max\{f_1, f_2\} \in \mathscr{MG}_2$, and so $\max\{g_1, g_2\} \in \mathscr{MG}_1$. □

The following result provides a condition on $\mathscr{MG}_2$ to guarantee the existence of transition kernels between probabilities ordered in $\preceq_1$.

PROPOSITION 3.12. *If $\mathscr{MG}_2$ is closed under maximization, then conditions $i)$ and $ii)$ are equivalent,*

$i)$ *$P' \preceq_1 P''$, with P' and P'' probabilities on $(\mathscr{S}, \mathscr{B}_{\mathscr{S}})$,*

$ii)$ *there exists a transition kernel Q^1 from $\mathscr{S}$ to $\mathscr{S}$ such that $Q^1 P' = P''$, and for any $x \in \mathscr{S}$,*

$$\int_{\mathscr{S}} f \, dQ^1(x, \cdot) \geqslant f(x)$$

for any $f \in \mathscr{MG}_1$.

Proof. It follows from Theorem 2.6.1 in [10], and Lemma 3.11. □

As a consequence, we obtain the following results about the strong extremality order in any direction and the time value of money order.

PROPOSITION 3.13. *Let X and Y be random vectors and $u \in S^{n-1}$. The following conditions are equivalent,*

$i)$ *$P_X \preceq_{SE_u} P_Y$,*

$ii)$ *there exists a transition kernel Q^{SE_u} from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $Q^{SE_u} P_X = P_Y$, and for any $x \in \mathbb{R}^n$, it holds that*

$$\int_{\mathbb{R}^n} f(\mathscr{R}_u z) \, dQ^{SE_u}(x, \cdot) \geqslant f(\mathscr{R}_u x)$$

for any increasing and bounded mapping $f : \mathbb{R}^n \to \mathbb{R}$.

Proof. It is a consequence of Proposition 3.12 and Example 3.8. Observe that the maximal generator of the usual stochastic order ($b = 1$) is closed under maximization. □

PROPOSITION 3.14. *Let X and Y be random vectors. Conditions i) and ii) are equivalent,*

i) $P_X \preceq_{tvm} P_Y$,

ii) there exists a transition kernel Q^{tvm} from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $Q^{tvm}P_X = P_Y$, and for any $x \in \mathbb{R}^n$, it holds that

$$\int_{\mathbb{R}^n} f(h^{-1}(z))\,dQ^{tvm}(x,\cdot) \geqslant f(h^{-1}(x))$$

for any increasing and bounded mapping $f : \mathbb{R}^n \to \mathbb{R}$, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is the linear map such that $h(e_i) = e_i - e_{i+1}$ for all $1 \leqslant i \leqslant n-1$ and $h(e_n) = e_n$.

Proof. The result follows from Proposition 3.12 and Example 3.10. □

Note that Proposition 3.12 cannot be applied to the family of extremality orders since the class of bounded and Δ-monotone functions is not closed under maximization.

Observe that when $\mathscr{MG}_2$ is closed under maximization, Theorem 2.6.1 in [10] reads that the following conditions are equivalent

1.– $P' \preceq_2 P''$, with P' and P'' probabilities on $(\mathscr{S}', \mathscr{B}_{\mathscr{S}'})$

2.– there exists a transition kernel Q^2 from $\mathscr{S}'$ to $\mathscr{S}'$ such that $Q^2P' = P''$, and for any $x \in \mathscr{S}'$,

$$\int_{\mathscr{S}'} f\,dQ^2(x,\cdot) \geqslant f(x)$$

for any $f \in \mathscr{MG}_2$.

Under the conditions of Proposition 3.12, we aim to obtain a kernel Q^1 satisfying *ii*) in that proposition, by means of the above kernel Q^2. For such a purpose, we state the following result.

PROPOSITION 3.15. *Let Q be a kernel from $\mathscr{S}'$ to $\mathscr{S}'$. Then, $Q_m : \mathscr{S} \times \mathscr{B}_{\mathscr{S}} \to \mathbb{R}$, defined by $Q_m(x,B) = Q(m(x), m(B))$ for any $x \in \mathscr{S}$ and any $B \in \mathscr{B}_{\mathscr{S}}$, is a kernel from $\mathscr{S}$ to $\mathscr{S}$.*

Proof. Note that since m is bijective and measurable, so is m^{-1}, and thus, $m(B) \in \mathscr{B}_{\mathscr{S}'}$ for any $B \in \mathscr{B}_{\mathscr{S}}$. Therefore, Q_m is well defined.

Observe that for any $x \in \mathscr{S}$, it holds that $Q_m(x,\cdot) = Q(m(x), m(\cdot)) : \mathscr{B}_{\mathscr{S}} \to \mathbb{R}$ is a probability since m is bijective and measurable.

On the other hand, if $B \in \mathscr{B}_{\mathscr{S}}$, then $Q_m(\cdot, B) = Q(m(\cdot), m(B)) : \mathscr{S} \to \mathbb{R}$ is measurable since it is a composition of measurable mappings. □

PROPOSITION 3.16. *Let X and Y be random elements with values in $\mathscr{S}$. Let Q be a transition kernel from $\mathscr{S}'$ to $\mathscr{S}'$ between $P_{m(X)}$ and $P_{m(Y)}$, such that for any $x \in \mathscr{S}'$, it holds that*

$$\int_{\mathscr{S}'} f\,dQ(x,\cdot) \geqslant f(x)$$

for any $f \in \mathscr{MG}_2$. Then $Q_m : \mathscr{S} \times \mathscr{B}_{\mathscr{S}} \to \mathbb{R}$, defined by $Q_m(x,B) = Q(m(x),m(B))$ for any $x \in \mathscr{S}$ and any $B \in \mathscr{B}_{\mathscr{S}}$, is a transition kernel from $\mathscr{S}$ to $\mathscr{S}$ between P_X and P_Y, such that for any $x \in \mathscr{S}$, it holds that

$$\int_{\mathscr{S}} g\,dQ_m(x,\cdot) \geqslant g(x)$$

for any $g \in \mathscr{MG}_1$.

Proof. Proposition 3.15 reads that Q_m is a kernel from $\mathscr{S}$ to $\mathscr{S}$. Let us see that Q_m is a transition kernel between P_X and P_Y.

For any $B \in \mathscr{B}_{\mathscr{S}}$, we have that

$$\begin{aligned} P_Y(B) = P_{m(Y)}(m(B)) &= \int_{\mathscr{S}'} Q(x,m(B))\,dP_{m(X)} \\ &= \int_{\mathscr{S}'} Q(x,m(B))\,dP_X \circ m^{-1} = \int_{\mathscr{S}} Q(m(x),m(B))\,dP_X = \int_{\mathscr{S}} Q_m(x,B)\,dP_X, \end{aligned}$$

which proves that Q_m is a transition kernel between P_X and P_Y.

Now, let us prove that for any $x \in \mathscr{S}$,

$$\int_{\mathscr{S}} g(s)\,dQ_m(x,\cdot) \geqslant g(x) \tag{2}$$

for any $g \in \mathscr{MG}_1$.

In accordance with Proposition 3.7, $\mathscr{MG}_1 = \mathscr{MG}_2(m)$. Thus, condition (2) is equivalent to: for any $x \in \mathscr{S}$,

$$\int_{\mathscr{S}} f \circ m(s)\,dQ_m(x,\cdot) \geqslant f \circ m(x) \tag{3}$$

for any $f \in \mathscr{MG}_2$. We should observe that

$$\int_{\mathscr{S}} f \circ m(s)\,dQ_m(x,\cdot) = \int_{\mathscr{S}'} f(s)\,dQ_m(x,\cdot) \circ m^{-1},$$

and for any $B \in \mathscr{B}_{\mathscr{S}'}$, $Q_m(x,\cdot) \circ m^{-1}(B) = Q_m(x,m^{-1}(B)) = Q(m(x),B) = Q(m(x),\cdot)(B)$. Therefore, condition (3) is the same as: for any $x \in \mathscr{S}$,

$$\int_{\mathscr{S}'} f(s)\,dQ(m(x),\cdot) \geqslant f \circ m(x)$$

for any $f \in \mathscr{MG}_2$.

Since the map m is bijective, this is the same as: for any $y \in \mathscr{S}'$, it holds that

$$\int_{\mathscr{S}'} f(s)\, dQ(y,\cdot) \geqslant f(y)$$

for any $f \in \mathscr{M}\mathscr{G}_2$, which proves the result. □

Now, we analyze some results in relation to operators of probability associated with kernels and the orders $\preceq_1$ and $\preceq_2$. The following result connects those operators with the monotone property.

PROPOSITION 3.17. *Let Q be a kernel from $\mathscr{S}'$ to $\mathscr{S}'$, and T_Q the associated operator. Then, T_Q is $\preceq_2$-monotone if and only if T_{Q_m} is $\preceq_1$-monotone.*

Proof. Assume that T_Q is $\preceq_2$-monotone. Theorem 5.2.3 in [10] assures that T_Q is $\preceq_2$-monotone if and only if for any $f \in \mathscr{M}\mathscr{G}_2$, the mapping $f_{T_Q} : \mathscr{S}' \to \mathbb{R}$, with

$$f_{T_Q}(y) = \int_{\mathscr{S}'} f(t)\, dQ(y,\cdot)$$

for any $y \in \mathscr{S}'$, belongs to $\mathscr{M}\mathscr{G}_2$.

Let $f \in \mathscr{M}\mathscr{G}_2$. Take $f_{T_Q} \circ m : \mathscr{S} \to \mathbb{R}$, which is given by

$$f_{T_Q} \circ m(x) = \int_{\mathscr{S}'} f(t)\, dQ(m(x),\cdot)$$

for any $x \in \mathscr{S}$.

In accordance with Proposition 3.7, the family of mappings $\{f \circ m \mid f \in \mathscr{M}\mathscr{G}_2\}$ is the set $\mathscr{M}\mathscr{G}_1$. Therefore, $f_{T_Q} \circ m$ belongs to $\mathscr{M}\mathscr{G}_1$.

By a change of variable,

$$f_{T_Q} \circ m(x) = \int_{\mathscr{S}} f \circ m(t)\, dQ(m(x), m(\cdot)) = \int_{\mathscr{S}} f \circ m(t)\, dQ_m(x,\cdot).$$

Thus, the mapping

$$x \to \int_{\mathscr{S}} g(t)\, dQ_m(x,\cdot)$$

belongs to $\mathscr{M}\mathscr{G}_1$ for any $g \in \mathscr{M}\mathscr{G}_1$. Applying again Theorem 5.2.3 in [10], we obtain that the operator T_{Q_m} is $\preceq_1$-monotone.

The converse is a consequence of the proven part. □

COROLLARY 3.18. *Let T_Q be an operator associated with a kernel Q from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then,*

i) T_Q is $\preceq_{st}$-monotone if and only if T_{Q_m} is $\preceq_{SE_u}$-monotone, with $m : \mathbb{R}^n \to \mathbb{R}^n$ given by $m(x) = \mathscr{R}_u x$ for any $x \in \mathbb{R}^n$,

ii) T_Q is $\preceq_{uo}$-monotone if and only if T_{Q_m} is $\preceq_{E_u}$-monotone, with $m : \mathbb{R}^n \to \mathbb{R}^n$ given by $m(x) = \mathscr{R}_u x$ for any $x \in \mathbb{R}^n$, *iii)* T_Q *is* $\preceq_{st}$*-monotone if and only if* T_{Q_m} *is* $\preceq_{tvm}$*-monotone, with* $m : \mathbb{R}^n \to \mathbb{R}^n$ *given by* $m(x) = h^{-1}(x)$ *for any* $x \in \mathbb{R}^n$*, where* $h : \mathbb{R}^n \to \mathbb{R}^n$ *is the linear map such that* $h(e_i) = e_i - e_{i+1}$ *for all* $1 \leqslant i \leqslant n-1$ *and* $h(e_n) = e_n$*,*

iv) T_Q *is* $\preceq^1$*-monotone if and only if* T_{Q_m} *is* $\preceq_{l-st}$*-monotone, with* $m : \mathbb{R}^n \to \mathbb{R}^n$ *given by* $m(x) = h(x)$ *for any* $x \in \mathbb{R}^n$*, where* h *is the mapping described in point iii).*

PROPOSITION 3.19. *Let* T_Q *be an operator associated with a kernel* Q *from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$*. Then,*

i) T_Q *is* $\preceq_{SE_u}$*-monotone if and only the mapping* $x \mapsto Q(\mathscr{R}_u^{-1}x, \mathscr{R}_u^{-1}(B))$ *is increasing for any closed upper set (with respect to the usual componentwise order)* $B \subset \mathbb{R}^n$*,*

ii) T_Q *is* $\preceq_{E_u}$*-monotone if and only the mapping* $x \mapsto Q(\mathscr{R}_u^{-1}x, \mathscr{R}_u^{-1}(B_y))$ *is* Δ*-monotone for any* $y \in \mathbb{R}^n$*, where* $B_y = \{z \in \mathbb{R}^n \mid y_i < z_i,\, 1 \leqslant i \leqslant n\}$*,*

iii) T_Q *is* $\preceq_{tvm}$*-monotone if and only the mapping* $x \mapsto Q(h(x), h(B))$ *is increasing for any closed upper set (with respect to the usual componentwise order)* $B \subset \mathbb{R}^n$*.*

Proof. By Corollary 3.18, T_Q is $\preceq_{SE_u}$-monotone if and only $T_{Q_{m^{-1}}}$ is $\preceq_{st}$-monotone. Theorem 5.2.3 in [10] reads that $T_{Q_{m^{-1}}}$ is $\preceq_{st}$-monotone if and only if for any f of a generator of the order $\preceq_{st}$, the mapping

$$x \mapsto \int_{\mathbb{R}^n} f(t)\, dQ_{m^{-1}}(x, \cdot)$$

belongs to the maximal generator of the order $\preceq_{st}$. It is sufficient to take as a generator of $\preceq_{st}$ the class of mappings $\{I_B \mid B \subset \mathbb{R}^n \text{ is an upper closed set}\}$, to derive *i)*.

The cases *ii)* and *iii)* are analogous. For the case *ii)*, it is sufficient to take as a generator of the order $\preceq_{uo}$ the class $\{I_{B_y} \mid y \in \mathbb{R}^n\}$. □

Now we analyze relations between the properties of being smaller for the order $\preceq_1$ and being smaller for the order $\preceq_2$, for operators associated with kernels.

PROPOSITION 3.20. *Let* T_{Q^a} *and* T_{Q^b} *be the probabilistic operators associated with kernels* Q^a *and* Q^b *from* $\mathscr{S}'$ *to* $\mathscr{S}'$*, respectively. It holds that*

$$T_{Q^a} \preceq_2 T_{Q^b} \quad \text{if and only if} \quad T_{Q^a_m} \preceq_1 T_{Q^b_m}.$$

Proof. In accordance with Theorem 5.2.5 of [10], $T_{Q^a} \preceq_2 T_{Q^b}$ if and only if

$$Q^a(y, \cdot) \preceq_2 Q^b(y, \cdot) \text{ for any } y \in \mathscr{S}'.$$

Since m is bijective, this is the same as

$$Q^a(m(x), \cdot) \preceq_2 Q^b(m(x), \cdot) \text{ for any } x \in \mathscr{S},$$

equivalently,

$$Q^a(m(x), m\circ m^{-1}(\cdot)) \preceq_2 Q^b(m(x), m\circ m^{-1}(\cdot)) \text{ for any } x\in\mathscr{S},$$

or,

$$Q^a(m(x), m(\cdot))\circ m^{-1}(\cdot) \preceq_2 Q^b(m(x), m(\cdot))\circ m^{-1}(\cdot) \text{ for any } x\in\mathscr{S}.$$

By the definition of $\preceq_1$, this is the same as

$$Q^a(m(x), m(\cdot)) \preceq_1 Q^b(m(x), m(\cdot)) \text{ for any } x\in\mathscr{S},$$

that is,

$$Q^a_m(x,\cdot) \preceq_1 Q^b_m(x,\cdot) \text{ for any } x\in\mathscr{S},$$

equivalently, $T_{Q^a_m} \preceq_1 T_{Q^b_m}$. □

COROLLARY 3.21. *Let T_{Q^a} and T_{Q^b} be operators with kernels Q^a and Q^b from $\mathbb{R}^n$ to $\mathbb{R}^n$, respectively. It holds that*

i) $T_{Q^a} \preceq_{st} T_{Q^b}$ *if and only if* $T_{Q^a_m} \preceq_{SE_u} T_{Q^b_m}$, *with* $m:\mathbb{R}^n\to\mathbb{R}^n$ *given by* $m(x)=\mathscr{R}_u x$ *for any* $x\in\mathbb{R}^n$,

ii) $T_{Q^a} \preceq_{uo} T_{Q^b}$ *if and only if* $T_{Q^a_m} \preceq_{E_u} T_{Q^b_m}$, *with* $m:\mathbb{R}^n\to\mathbb{R}^n$ *given by* $m(x)=\mathscr{R}_u x$ *for any* $x\in\mathbb{R}^n$,

iii) $T_{Q^a} \preceq_{st} T_{Q^b}$ *if and only if* $T_{Q^a_m} \preceq_{tvm} T_{Q^b_m}$, *with* $m:\mathbb{R}^n\to\mathbb{R}^n$ *given by* $m(x) = h^{-1}(x)$ *for any* $x\in\mathbb{R}^n$, *where* $h:\mathbb{R}^n\to\mathbb{R}^n$ *is the linear map such that* $h(e_i) = e_i - e_{i+1}$ *for all* $1\leqslant i\leqslant n-1$ *and* $h(e_n)=e_n$,

iv) $T_{Q^a} \preceq^1 T_{Q^b}$ *if and only if* $T_{Q^a_m} \preceq_{l-st} T_{Q^b_m}$, *with* $m:\mathbb{R}^n\to\mathbb{R}^n$ *given by* $m(x)=h(x)$ *for any* $x\in\mathbb{R}^n$, *where* h *is the mapping described in point* $iii)$.

PROPOSITION 3.22. *Let Q^a and Q^b be kernels from $\mathbb{R}^n$ to $\mathbb{R}^n$. It holds that*

i) $T_{Q^a} \preceq_{SE_u} T_{Q^b}$, *if and only if* $Q^a(x,\mathscr{R}_u^{-1}(B)) \leqslant Q^b(x,\mathscr{R}_u^{-1}(B))$ *for any* $x\in\mathbb{R}^n$ *and for any closed upper set (with respect to the usual componentwise order)* $B\subset\mathbb{R}^n$,

ii) $T_{Q^a} \preceq_{E_u} T_{Q^b}$, *if and only if* $Q^a(x,\mathscr{R}_u^{-1}(B_y)) \leqslant Q^b(x,\mathscr{R}_u^{-1}(B_y))$ *for any* $x\in\mathbb{R}^n$ *and for any* $y\in\mathbb{R}^n$, *where* $B_y=\{z\in\mathbb{R}^n \mid y_i<z_i,\ 1\leqslant i\leqslant n\}$.

iii) $T_{Q^a} \preceq_{tvm} T_{Q^b}$, *if and only if* $Q^a(x,h(B)) \leqslant Q^b(x,h(B))$ *for any* $x\in\mathbb{R}^n$ *and for any closed upper set (with respect to the usual componentwise order)* $B\subset\mathbb{R}^n$.

Proof. By Corollary 3.21, $T_{Q^a} \preceq_{SE_u} T_{Q^b}$ if and only if $T_{Q^a_{m^{-1}}} \preceq_{st} T_{Q^b_{m^{-1}}}$. Theorem 5.2.5 in [10] reads that $T_{Q^a_{m^{-1}}} \preceq_{st} T_{Q^b_{m^{-1}}}$ if and only if $Q^a_{m^{-1}}(x,\cdot) \preceq_{st} Q^b_{m^{-1}}(x,\cdot)$ for any $x\in\mathbb{R}^n$, which is the same as $Q^a(x,\mathscr{R}_u^{-1}(B)) \leqslant Q^b(x,\mathscr{R}_u^{-1}(B))$ for any $x\in\mathbb{R}^n$ and for any closed upper set (with respect to the usual componentwise order) $B\subset\mathbb{R}^n$.

The cases $ii)$ and $iii)$ are analogous. □

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