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GENERALIZED GAGLIARDO–NIRENBERG INEQUALITIES VIA MURAMATU'S INTEGRAL FORMULA

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Abstract. We derive three generalized Gagliardo-Nirenberg inequalities in Lorentz, BMO and homogeneous Lipschitz spaces. They have the forms

$$\|\nabla^k f\|_{L^{p,\alpha}(\mathbb{R}^d)} \lesssim \|f\|^{\theta}_{L^{q,\infty}(\mathbb{R}^d)} \|\nabla^m f\|^{1-\theta}_{L^{r,\infty}(\mathbb{R}^d)},$$

$$\|\nabla^k f\|_{L^{p,\alpha_1}(\mathbb{R}^d)} \lesssim \|f\|^{\theta}_{L^{q,\alpha_2}(\mathbb{R}^d)} \|\nabla^m f\|^{1-\theta}_{BMO(\mathbb{R}^d)},$$

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^d)} \lesssim \|f\|^{\theta}_{L^{q,\infty}(\mathbb{R}^d)} \|f\|^{1-\theta}_{\dot{\Lambda}_\eta(\mathbb{R}^d)},$$

whose parameters satisfy specific conditions. We use the so-called Muramatu's integral formula as the main approach throughout the paper.

1. Introduction to the main results

The original inequality

$$\|\nabla^k f\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|^{\theta}_{L^q(\mathbb{R}^d)} \|\nabla^m f\|^{1-\theta}_{L^r(\mathbb{R}^d)} \tag{1.1}$$

was investigated independently by Gagliardo in [6] and Nirenberg in [14]. Due to its widespread applications in partial differential equations, it is of fundamental interest to generalize the inequality to other function spaces. In this paper we prove interpolation inequalities analogous to (1.1) in the setting of Lorentz, BMO and Lipschitz spaces. We follow a somewhat non-traditional approach, making use of the so-called Muramatu's integral formula. Our work is inspired by the paper [9], in which Miyazaki thoroughly investigated the integral representation formula originated in [13] by Muramatu and then applied it to study various aspects of Sobolev spaces.

To describe our main results, define

$$\mathbb{N} := \{0,1,2,\ldots\} \quad \text{and} \quad \mathbb{N}^* := \{1,2,3,\ldots\}.$$

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For each function f defined on $\mathbb{R}^d$ and $k \in \mathbb{N}^*$, the notation $\nabla^k f$ is understood as a tuple of all k-th order derivatives of f. Given a normed space F, we write $\nabla^k f \in F$ to mean that every component of $\nabla^k f$ belongs to F. Moreover, we set

$$||\nabla^k f||_F = \sum_{\gamma \in \mathbb{N}^d : |\gamma| = k} ||\partial^\gamma f||_F.$$

Let $p, q, r \in (1, \infty)$, $\alpha \in (0, \infty]$ and $k, m \in \mathbb{N}$ be such that $0 \leqslant k < m$. We consider

$$||\nabla^k f||_{L^{p,\alpha}(\mathbb{R}^d)} \lesssim ||f||^\theta_{L^{q,\infty}(\mathbb{R}^d)} \, ||\nabla^m f||^{1-\theta}_{L^{r,\infty}(\mathbb{R}^d)} \tag{1.2}$$

and ask under which conditions on the parameters (1.2) holds for all $f \in C_c^\infty(\mathbb{R}^d)$. Using a scaling argument and appropriate test functions, we derive a necessary condition for (1.2) to hold, which is

$$\theta = \frac{m - k - d\left(\frac{1}{r} - \frac{1}{p}\right)}{m - d\left(\frac{1}{r} - \frac{1}{q}\right)} \in \left[0, 1 - \frac{k}{m}\right]. \tag{1.3}$$

The main difficulty lies in finding a sufficient condition. McCormick et al. proved in [11] that in the case $k = 0$, $r = 2$, $q < p$, $\alpha = p$ and $m - \frac{d}{2} > -\frac{d}{p}$, there holds

$$||f||_{L^p(\mathbb{R}^d)} = ||f||_{L^{p,p}(\mathbb{R}^d)} \lesssim ||f||^\theta_{L^{q,\infty}(\mathbb{R}^d)} \, ||\nabla^m f||^{1-\theta}_{L^2(\mathbb{R}^d)}. \tag{1.4}$$

This result was later strengthened in [4, Corollary 2.2], where the condition $\alpha = p$ is dropped and one arrives at

$$||f||_{L^{p,\alpha}(\mathbb{R}^d)} \lesssim ||f||^\theta_{L^{q,\infty}(\mathbb{R}^d)} \, ||\nabla^m f||^{1-\theta}_{L^2(\mathbb{R}^d)}.$$

Moreover, [4, Corollary 2.5] asserts that if $k = 0$, $q < p$ and $m - \frac{d}{r} > 0$, then

$$||f||_{L^{p,\alpha}(\mathbb{R}^d)} \lesssim ||f||^\theta_{L^{q,\infty}(\mathbb{R}^d)} \, ||f||^{1-\theta}_{W^{m,r}(\mathbb{R}^d)}. \tag{1.5}$$

On the other hand, if $\alpha = p$, $q = r < p$ and $m - \frac{d}{r} > k - \frac{d}{p}$, then we know from the Sobolev embedding theorem (cf. [9, Theorem 3.1]) that

$$||\nabla^k f||_{L^p(\mathbb{R}^d)} \lesssim ||f||^\theta_{L^r(\mathbb{R}^d)} \, ||\nabla^m f||^{1-\theta}_{L^r(\mathbb{R}^d)}. \tag{1.6}$$

In this paper, we will give a sufficient condition for (1.2) which extends all the aforementioned estimates (1.4), (1.5) and (1.6). Specifically, we will show that in addition to (1.3) if the conditions

$$q < p \quad \text{and} \quad m - \frac{d}{r} > k - \frac{d}{p} \tag{1.7}$$

are also satisfied, then (1.2) holds for all $f \in L^{q,\infty}(\mathbb{R}^d)$ with $\nabla^m f \in L^{r,\infty}(\mathbb{R}^d)$. It is noted that (1.3) and (1.7) together imply either

$$p > r \quad \text{or} \quad q \leqslant p_* < p \leqslant r,$$

where

$$\frac{1}{p_*} := \frac{1-\frac{k}{m}}{q} + \frac{\frac{k}{m}}{r}. \tag{1.8}$$

Indeed, suppose (1.3) and (1.7) hold. Suppose further that $p \leqslant r$. Using (1.3), we arrive at

$$\theta = \frac{m-k-d\left(\frac{1}{r}-\frac{1}{p}\right)}{m-d\left(\frac{1}{r}-\frac{1}{q}\right)} < 1-\frac{k}{m}.$$

Equivalently,

$$m-k-d\left(\frac{1}{r}-\frac{1}{p}\right) < \left(1-\frac{k}{m}\right)\left[m-d\left(\frac{1}{r}-\frac{1}{q}\right)\right]$$

since $q<p$ and $m-\frac{d}{r}>k-\frac{d}{p}$ together guarantee that $m-d\left(\frac{1}{r}-\frac{1}{q}\right)>0$. In turn, we obtain

$$\left(1-\frac{k}{m}\right)\left(\frac{1}{r}-\frac{1}{q}\right) < \frac{1}{r}-\frac{1}{p}$$

and hence

$$\frac{1}{p} < \frac{1-\frac{k}{m}}{q} + \frac{\frac{k}{m}}{r} = \frac{1}{p_*}.$$

This implies $p_* < p$.

To see that $q \leqslant p_*$, we argue as follows. First, $q<r$ due to the assumptions that $q<p$ and $p \leqslant r$. Secondly,

$$\frac{1}{p_*} = \frac{1-\frac{k}{m}}{q} + \frac{\frac{k}{m}}{r} \leqslant \frac{1-\frac{k}{m}}{q} + \frac{\frac{k}{m}}{q} = \frac{1}{q},$$

which means $q \leqslant p_*$. Thus the claim follows.

The whole discussion above is now summarized in our first main theorem as follows.

THEOREM 1. *Let $1<q<p<\infty$, $1<r<\infty$ and $0<\alpha\leqslant\infty$. Let $k,m\in\mathbb{N}$ be such that*

$$0 \leqslant k < m \quad \textit{and} \quad m-\frac{d}{r} > k-\frac{d}{p}.$$

Assume either

$$p>r \quad \textit{or} \quad q \leqslant p_* < p \leqslant r,$$

where p_* is given by (1.8). Let $f\in L^{q,\infty}(\mathbb{R}^d)$ satisfy $\nabla^m f\in L^{r,\infty}(\mathbb{R}^d)$. Then $\nabla^k f\in L^{p,\alpha}(\mathbb{R}^d)$. Moreover, there exists a constant $C=C(p,q,r,k,m,d,\alpha)>0$ such that

$$\|\nabla^k f\|_{L^{p,\alpha}(\mathbb{R}^d)} \leqslant C\|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{\theta}\|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)}^{1-\theta}, \tag{1.9}$$

where

$$\theta = \frac{m-k-d\left(\frac{1}{r}-\frac{1}{p}\right)}{m-d\left(\frac{1}{r}-\frac{1}{q}\right)} \in \left(0, 1-\frac{k}{m}\right).$$

A remark is immediate.

REMARK 1. The main emphasis in Theorem 1 is that (1.9) holds for all $\alpha \in (0,\infty]$. As such we restrict our attention to the range

$$\theta \in \left(0, 1-\frac{k}{m}\right).$$

Nevertheless, certain estimates of the form (1.9) remain valid for the endpoint values of θ. In particular, there exists a constant $C = C(p,r,m,d) > 0$ such that

$$\|f\|_{L^{p,\infty}(\mathbb{R}^d)} \leqslant C\|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)} \tag{1.10}$$

for all $f \in L^{r,\infty}(\mathbb{R}^d)$ with $\nabla^m f \in L^{r,\infty}(\mathbb{R}^d)$, where

$$1 < r < p < \infty, \quad m \in \mathbb{N}^* \quad \text{and} \quad m = d\left(\frac{1}{r}-\frac{1}{p}\right).$$

This means the case $\theta = 0$ is possible. To see this, recall from [3, Theorem 4.2(i)] that (1.10) holds for $m = 1$, i.e.,

$$\|f\|_{L^{p,\infty}(\mathbb{R}^d)} \leqslant C\|\nabla f\|_{L^{r,\infty}(\mathbb{R}^d)}. \tag{1.11}$$

Then (1.10) for a general $m \in \mathbb{N}^*$ follows by iterating (1.11) m times.

Next there exists a constant $C = C(p,q,r,k,m,d) > 0$ such that

$$\|\nabla^k f\|_{L^{p,\infty}(\mathbb{R}^d)} \leqslant C\|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{1-\frac{k}{m}}\|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)}^{\frac{k}{m}}$$

for all $f \in L^{q,\infty}(\mathbb{R}^d)$ with $\nabla^m f \in L^{r,\infty}(\mathbb{R}^d)$, where

$$1 < q < p = p_* < \infty, \quad k,m \in \mathbb{N}, \quad 0 \leqslant k < m$$

and p_* is given by (1.8). This implies the case $\theta = 1-\frac{k}{m}$ is also possible. See (3.7) and its related arguments below for the proof of this statement.

Our second theorem concerns the inequality of the form

$$\|\nabla^k f\|_{L^{p,\alpha_1}(\mathbb{R}^d)} \lesssim \|f\|_{L^{q,\alpha_2}(\mathbb{R}^d)}^{\theta}\|\nabla^m f\|_{BMO(\mathbb{R}^d)}^{1-\theta}.$$

The result extends [10, Theorem 1.3], in which $\alpha_1 = p$ and $\alpha_2 = q$ were considered.

THEOREM 2. *Let* $1<q<p<\infty$ *and* $1 \leqslant \alpha_1, \alpha_2 \leqslant \infty$. *Let* $k,m \in \mathbb{N}$ *be such that*

$$0<k<m, \quad q=\left(1-\frac{k}{m}\right)p \quad \text{and} \quad \alpha_2 \leqslant \left(1-\frac{k}{m}\right)\alpha_1.$$

Let $f \in L^{q,\alpha_2}(\mathbb{R}^d)$ *satisfy* $\nabla^m f \in BMO(\mathbb{R}^d)$. *Then* $\nabla^k f \in L^{p,\alpha_1}(\mathbb{R}^d)$. *Moreover, there exists a constant* $C = C(p,q,d,k,m,\alpha_1,\alpha_2) > 0$ *such that*

$$\|\nabla^k f\|_{L^{p,\alpha_1}(\mathbb{R}^d)} \leqslant C \|f\|_{L^{q,\alpha_2}(\mathbb{R}^d)}^{1-\frac{k}{m}} \|\nabla^m f\|_{BMO(\mathbb{R}^d)}^{\frac{k}{m}}. \tag{1.12}$$

As a concluding demonstration of Muramatu's integral formula, we prove an interpolation inequality involving the Lorentz space $L^{q,\infty}(\mathbb{R}^d)$ and the homogeneous Lipschitz space $\dot{\Lambda}_\eta(\mathbb{R}^d)$, which is Theorem 3. Here the space $\dot{\Lambda}_\eta(\mathbb{R}^d)$ is understood in the sense of [7, Definition 6.3.4]. The result of this type was lately updated in [4, Theorem 2.4] with indices $\eta \in (0,1)$ and $q>0$. Our result asserts the validity of the estimate for all $\eta>0$ and $q>1$. As such, Theorem 3 extends [4, Theorem 2.4] in certain aspects. That Theorem 3 holds only for $q>1$ is due to a technical reason. More specifically, its proof requires the sharpened version of Young's inequality in Proposition 3 below, which in turn dictates $q>1$.

THEOREM 3. *Let* $1<q<p<\infty$, $0<\alpha \leqslant \infty$ *and* $\eta>0$. *Let* $f \in L^{q,\infty}(\mathbb{R}^d) \cap \dot{\Lambda}_\eta(\mathbb{R}^d)$. *Then* $f \in L^{p,\alpha}(\mathbb{R}^d)$. *Moreover, there exists a constant* $C = C(p,q,\alpha,\eta) > 0$ *such that*

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^d)} \leqslant C \|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{\theta} \|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}^{1-\theta}, \tag{1.13}$$

where $\theta = \dfrac{\eta+\frac{d}{p}}{\eta+\frac{d}{q}} \in (0,1)$.

It is known that if $m \in \mathbb{N}$ and $r \in [1,\infty)$ satisfy $\eta = m - \frac{d}{r} > 0$, then the Sobolev space $W^{m,r}(\mathbb{R}^d)$ is continuously embedded in the Lipschitz space $\Lambda_\eta(\mathbb{R}^d)$. Hence Theorem 3 implies (1.5), which is also [4, Corollary 2.5].

The paper is structured as follows. In Section 2 we recall necessary definitions and results as well as briefly discuss Muramatu's integral formula. The proofs of Theorems 1, 2 and 3 are presented in Sections 3, 4 and 5 respectively.

2. Preliminaries

In this section, we collect the essential background on function spaces, embedding inequalities, Hardy-Littlewood maximal function and Muramatu's integral formula.

2.1. Function spaces

We provide the definitions of Lorentz, BMO and homogeneous Lipschitz spaces. We start with the Lorentz space.

DEFINITION 1. For $1 \leqslant p < \infty$ and $0 < \alpha \leqslant \infty$, the Lorentz space $L^{p,\alpha}(\mathbb{R}^d)$ consists of all measurable functions $g : \mathbb{R}^d \to \mathbb{R}$ such that $\|g\|_{L^{p,\alpha}(\mathbb{R}^d)} < \infty$, where

$$\|g\|_{L^{p,\alpha}(\mathbb{R}^d)} := \begin{cases} \left(p \int_0^\infty s^\alpha \left| \{x \in \mathbb{R}^d : |g(x)| > s\} \right|^{\frac{\alpha}{p}} \dfrac{ds}{s} \right)^{\frac{1}{\alpha}} & \text{if } 0 < \alpha < \infty, \\ \sup\limits_{s>0} s \left| \{x \in \mathbb{R}^d : |g(x)| > s\} \right|^{\frac{1}{p}} & \text{if } \alpha = \infty. \end{cases}$$

Lorentz spaces can be realized as interpolation spaces of the pair (L^1, L^∞) via K-method. A brief summary of this fact is as follows. More details can be found in [15] and [2]. Let X_0 and X_1 be Banach spaces which are continuously embedded in a Hausdorff topological vector space $\mathscr{X}$. For each $t > 0$ and $f \in X_0 + X_1$, define the K-functional by

$$K(t, f; X_0, X_1) := \inf\{\|u\|_{X_0} + t\|v\|_{X_1} : u \in X_0,\, v \in X_1,\, u + v = f\}.$$

Let $0 < \theta < 1$ and $1 \leqslant r \leqslant \infty$. We denote by $(X_0, X_1)_{\theta,r}$ the space consisting of all functions $g \in X_0 + X_1$ such that $\|g\|_{(X_0,X_1)_{\theta,r}} < \infty$, where

$$\|g\|_{(X_0,X_1)_{\theta,r}} := \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t, f; X_0, X_1) \right]^r \dfrac{dt}{t} \right)^{\frac{1}{r}} & \text{if } r < \infty, \\ \sup\limits_{t>0} t^{-\theta} K(t, f; X_0, X_1) & \text{if } r = \infty. \end{cases}$$

The space $(X_0, X_1)_{\theta,r}$ equipped with the norm $\|.\|_{(X_0,X_1)_{\theta,r}}$ is a Banach space. It turns out that for all $1 < p < \infty$ and $1 \leqslant \alpha \leqslant \infty$, we have the relation

$$(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))_{1-\frac{1}{p},\alpha} = L^{p,\alpha}(\mathbb{R}^d).$$

Moreover,

$$\|g\|_{L^{p,\alpha}(\mathbb{R}^d)} \leqslant \|g\|_{(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))_{1-\frac{1}{p},\alpha}} \leqslant \frac{p}{p-1} \|g\|_{L^{p,\alpha}(\mathbb{R}^d)} \tag{2.1}$$

for all $g \in L^{p,\alpha}(\mathbb{R}^d)$.

Next we define the space $BMO(\mathbb{R}^d)$.

DEFINITION 2. The space $BMO(\mathbb{R}^d)$ consists of all functions $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that

$$\|f\|_{BMO(\mathbb{R}^d)} := \sup \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < \infty,$$

where $f_B := \dfrac{1}{|B|} \int_B f(x)\, dx$ and the supremum is taken over all balls B in $\mathbb{R}^d$.

It is straightforward that $L^\infty(\mathbb{R}^d) \subset BMO(\mathbb{R}^d)$ and

$$\|f\|_{BMO(\mathbb{R}^d)} \leqslant 2\,\|f\|_{L^\infty(\mathbb{R}^d)}$$

for all $f \in L^\infty(\mathbb{R}^d)$.

Lastly, we define the homogeneous Lipschitz space. We follow [7, Definition 6.3.3].

DEFINITION 3. Let $h \in \mathbb{R}^d$. The *difference operator* Δ_h is defined by

$$\Delta_h f(x) := f(x+h) - f(x)$$

for all $f \in C(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

For each $k \in \mathbb{N}^*$, the operator Δ_h^k is defined recursively by

$$\begin{cases} \Delta_h^1 f = \Delta_h f, \\ \Delta_h^{k+1} f = \Delta_h(\Delta_h^k f). \end{cases}$$

Let $\eta > 0$. We denote by $\Lambda_\eta(\mathbb{R}^d)$ the *Lipschitz space* of order η, i.e., the space consisting of all $f \in C(\mathbb{R}^d)$ such that

$$\|f\|_{\Lambda_\eta(\mathbb{R}^d)} := \|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{x\in\mathbb{R}^d} \sup_{h\in\mathbb{R}^d\setminus\{0\}} \frac{|\Delta_h^{[\eta]+1} f(x)|}{|h|^\eta} < \infty.$$

Here $[\eta]$ denotes the largest integer which is smaller than η.

We denote by $\dot{\Lambda}_\eta(\mathbb{R}^d)$ the *homogeneous Lipschitz space* of order η, i.e., the space consisting of all $f \in C(\mathbb{R}^d)$ such that

$$\|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)} := \sup_{x\in\mathbb{R}^d} \sup_{h\in\mathbb{R}^d\setminus\{0\}} \frac{|\Delta_h^{[\eta]+1} f(x)|}{|h|^\eta} < \infty.$$

2.2. Necessary inequalities

We present some estimates to be used in the proofs of our main results. We start with an interpolation inequality.

PROPOSITION 1. ([4, Theorem 2.1]) Let $0 < q < p < r \leqslant \infty$ and $0 < \alpha \leqslant \infty$. If $f \in L^{q,\infty}(\mathbb{R}^d) \cap L^{r,\infty}(\mathbb{R}^d)$, then $f \in L^{p,\alpha}(\mathbb{R}^d)$ and there exists a constant $C = C(p,q,r,\alpha) > 0$ such that

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^d)} \leqslant C\,\|f\|^{\theta}_{L^{q,\infty}(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{L^{r,\infty}(\mathbb{R}^d)},$$

where $\theta \in (0,1)$ satisfies

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{r}.$$

Next we state Young's inequalities for L^p and weak L^p spaces.

PROPOSITION 2. ([8, Theorems 1.2.12 and 1.2.13]) Let $1 \leqslant p,q,r \leqslant \infty$ satisfy

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}.$$

Then for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^r(\mathbb{R}^d)$,

$$\|f \star g\|_{L^q(\mathbb{R}^d)} \leqslant \|g\|_{L^r(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.$$

Moreover, if $1<p,q<\infty$ and $1 \leqslant r<\infty$, then there exists a constant $C=C(p,q,r)>0$ such that

$$\|f \star g\|_{L^{q,\infty}(\mathbb{R}^d)} \leqslant C\|g\|_{L^r(\mathbb{R}^d)} \|f\|_{L^{p,\infty}(\mathbb{R}^d)}$$

for all $f \in L^{p,\infty}(\mathbb{R}^d)$ and $g \in L^r(\mathbb{R}^d)$.

Young's inequalities can be sharpened as follows.

PROPOSITION 3. ([8, Theorem 1.4.24]) Let $1<p,q,r<\infty$ satisfy

$$\frac{1}{q}+1=\frac{1}{p}+\frac{1}{r}.$$

Then there exists a constant $C=C(p,q,r)>0$ such that

$$\|f \star g\|_{L^q(\mathbb{R}^d)} \leqslant C\|g\|_{L^r(\mathbb{R}^d)} \|f\|_{L^{p,\infty}(\mathbb{R}^d)}$$

for all $f \in L^{p,\infty}(\mathbb{R}^d)$ and $g \in L^r(\mathbb{R}^d)$.

We also require a version of Hölder's inequality for weak spaces.

PROPOSITION 4. ([8, Exercise 1.1.15]) Let $k \in \mathbb{N}^*$ and $f_j \in L^{p_j,\infty}(\mathbb{R}^d)$, where $0<p_j<\infty$ and $1 \leqslant j \leqslant k$. Let

$$\frac{1}{p}=\frac{1}{p_1}+\ldots+\frac{1}{p_k}.$$

Then

$$\left\|\prod_{j=1}^k f_j\right\|_{L^{p,\infty}(\mathbb{R}^d)} \leqslant p^{-\frac{1}{p}} \prod_{j=1}^k p_j^{\frac{1}{p_j}} \prod_{j=1}^k \|f_j\|_{L^{p_j,\infty}(\mathbb{R}^d)}.$$

2.3. Hardy-Littlewood maximal operator

We recall the definition and some boundedness properties of the Hardy-Littlewood maximal function.

DEFINITION 4. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. The function

$$\mathscr{M}(f)(x) := \sup_{\delta>0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|\,dy$$

is called the (centered) Hardy-Littlewood maximal function of f, where $B(x,\delta) \subset \mathbb{R}^d$ denotes the open ball with center x and radius δ in $\mathbb{R}^d$.

It is well known that the Hardy-Littlewood maximal function acting boundedly between Lorentz spaces.

PROPOSITION 5. The following statements hold.

(i) $\|\mathscr{M}(f)\|_{L^\infty(\mathbb{R}^d)} \leqslant \|f\|_{L^\infty(\mathbb{R}^d)}$ for all $f \in L^\infty(\mathbb{R}^d)$.

(ii) There exists a constant $C > 0$ such that

$$\|\mathscr{M}(f)\|_{L^{1,\infty}(\mathbb{R}^d)} \leqslant C\|f\|_{L^1(\mathbb{R}^d)}$$

for all $f \in L^1(\mathbb{R}^d)$.

(iii) For all $1 < p < \infty$ and $1 < \alpha \leqslant \beta \leqslant \infty$ there exists a constant $C = C(p,\alpha,\beta) > 0$ such that

$$\|\mathscr{M}(f)\|_{L^{p,\beta}(\mathbb{R}^d)} \leqslant C\|f\|_{L^{p,\alpha}(\mathbb{R}^d)}$$

for all $f \in L^{p,\alpha}(\mathbb{R}^d)$.

See [8, Theorem 2.1.6] for (i) and (ii). Statement (iii) follows from (i) and (ii) as well as the Marcinkiewicz interpolation theorem and inclusion relation between Lorentz spaces.

2.4. Muramatu's integral formula

We quickly construct Muramatu's integral formula. For more details, see [9] and [10].

Define

$$K_t(x) = t^{-d} K\left(\frac{x}{t}\right)$$

for each function K on $\mathbb{R}^d$, $x \in \mathbb{R}^d$ and $t > 0$. Denote by $\mathscr{D}'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$, i.e., the dual space of $C_c^\infty(\mathbb{R}^d)$. For every $f \in \mathscr{D}'(\mathbb{R}^d)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$, the value of f at ϕ is written as $\langle f, \phi \rangle$ and the convolution $\phi \star f$ is given by

$$\phi \star f(x) = \langle f, \phi(x-\cdot) \rangle$$

for all $x \in \mathbb{R}^d$.

Let $\omega \in C_c^\infty(B(0,1))$ be such that $\int_{\mathbb{R}^d} \omega(x)\,dx = 1$. Let $m,N \in \mathbb{N}^*$. Set

$$\varphi(x) = \sum_{|\alpha|<N+m} \frac{1}{\alpha!}\partial^\alpha(x^\alpha \omega(x)) \quad \text{and} \quad K(x) = \sum_{|\alpha|=N+m} \frac{N+m}{\alpha!}\partial^\alpha(x^\alpha \omega(x)) \qquad (2.2)$$

for all $x \in \mathbb{R}^d$. Then K can be expressed as $K = \sum_{|\beta|=m}\partial^\beta K^{(\beta)}$, where $K^{(\beta)} \in C_c^\infty(B(0,1))$. Moreover, (2.2) implies that

$$\frac{\partial}{\partial t}\{\varphi_t(x)\} = -t^{-1}K_t(x) \qquad (2.3)$$

for all $x \in \mathbb{R}^d$.

Next let $f \in \mathscr{D}'(\mathbb{R}^d)$. Using equality (2.3), we can prove that

$$(\varphi_\lambda \star f)(x) - (\varphi_\varepsilon \star f)(x) = -\int_\varepsilon^\lambda (K_t \star f)(x)\,\frac{dt}{t} \qquad (2.4)$$

for all $x \in \mathbb{R}^d$ and for all $0 < \varepsilon < \lambda < \infty$. In addition,

$$\lim_{\varepsilon\to 0} \varphi_\varepsilon \star f = f \quad \text{in } \mathscr{D}'(\mathbb{R}^d). \qquad (2.5)$$

Combining (2.4) and (2.5), we obtain

$$f = \lim_{\varepsilon\to 0}\int_\varepsilon^\lambda K_t \star f\,\frac{dt}{t} + \varphi_\lambda \star f \quad \text{in } \mathscr{D}'(\mathbb{R}^d). \qquad (2.6)$$

We call (2.6) Muramatu's integral formula.

It is worth mentioning that when f belongs to a specific function space, the convergence in (2.6) can be interpreted in a different way. For instance, if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ then the limit in (2.6) holds pointwise in $\mathbb{R}^d$. Whereas, if $f \in L^p(\mathbb{R}^d)$ for some $1 \leqslant p < \infty$ then the limit in (2.6) converges in $L^p(\mathbb{R}^d)$. These follow from the fact that the convergence in (2.5) holds either pointwise in $\mathbb{R}^d$ or in $L^p(\mathbb{R}^d)$, depending on whether $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ or $f \in L^p(\mathbb{R}^d)$ respectively (cf. [1, Theorem 4.22] and [5, Theorem C.4.6]).

Now let $\gamma \in \mathbb{N}^d$ be such that $0 \leqslant k := |\gamma| < m$. We apply (2.6) with f replaced by $\partial^\gamma f \in \mathscr{D}'(\mathbb{R}^d)$ to obtain

$$\partial^\gamma f = \lim_{\varepsilon\to 0}\int_\varepsilon^\lambda K_t \star \partial^\gamma f\,\frac{dt}{t} + \varphi_\lambda \star \partial^\gamma f. \qquad (2.7)$$

Direct calculations confirm that

$$\partial^\gamma f = \lim_{\varepsilon\to 0}\int_\varepsilon^\lambda \sum_{|\beta|=m} t^{m-k}\Big((\partial^\gamma K^{(\beta)})_t \star \partial^\beta f\Big)\frac{dt}{t} + \lambda^{-k}\Big((\partial^\gamma\varphi)_\lambda \star f\Big). \qquad (2.8)$$

The first term on the right-hand side in (2.8) has the form

$$\lim_{\varepsilon\to 0}\int_\varepsilon^\lambda g(t,x)\,\frac{dt}{t} =: G(x).$$

We will use the fact that if $\int_0^\lambda ||g(t,\cdot)||_{L^p(\mathbb{R}^d)} \frac{dt}{t} < \infty$ with $1 \leqslant p \leqslant \infty$, then it follows that $G \in L^p(\mathbb{R}^d)$ and

$$||G||_{L^p(\mathbb{R}^d)} \leqslant \int_0^\lambda ||g(t,\cdot)||_{L^p(\mathbb{R}^d)} \frac{dt}{t}. \tag{2.9}$$

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We manage the task by dividing the proof into two cases, depending on whether $p > r$ or $q \leqslant p_* < p \leqslant r < \infty$. Hereafter, it is convenient to indicate the dependence of a constant $C > 0$ on certain parameters using sub-indices. For instance, we write

$$C_{p,q,r} := C(p,q,r) > 0.$$

Proof of Theorem 1. Let $f \in L^{q,\infty}(\mathbb{R}^d)$ satisfy $\nabla^m f \in L^{r,\infty}(\mathbb{R}^d)$. Fix $\gamma \in \mathbb{N}^d$ such that $|\gamma| = k$. We use Muramatu's integral formula (2.8).

Case 1: Suppose $p > r$. Let $\lambda > 0$ be arbitrary. Since $1 < q < p$ and $1 < r < p$, there exist indices $s, u \in (1,\infty)$ such that

$$\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{u} = \frac{1}{q} + \frac{1}{s}.$$

It follows from Proposition 3 that

$$\begin{aligned}
&\int_0^\lambda \left\| \sum_{|\beta|=m} t^{m-k-1} \left((\partial^\gamma K^{(\beta)})_t \star \partial^\beta f \right) \right\|_{L^p(\mathbb{R}^d)} dt + \lambda^{-k} \|(\partial^\gamma \varphi)_\lambda \star f\|_{L^p(\mathbb{R}^d)} \\
&\leqslant C_{p,q,r} \left(\int_0^\lambda t^{m-k-1} \sum_{|\beta|=m} \|(\partial^\gamma K^{(\beta)})_t\|_{L^u(\mathbb{R}^d)} \|\partial^\beta f\|_{L^{r,\infty}(\mathbb{R}^d)} dt \right. \\
&\qquad \left. + \lambda^{-k} \|(\partial^\gamma \varphi)_\lambda\|_{L^s(\mathbb{R}^d)} \|f\|_{L^{q,\infty}(\mathbb{R}^d)} \right) \\
&= C_{p,q,r} \left(\int_0^\lambda t^{m-k-1+\frac{d}{u}-d} \sum_{|\beta|=m} \|\partial^\gamma K^{(\beta)}\|_{L^u(\mathbb{R}^d)} \|\partial^\beta f\|_{L^{r,\infty}(\mathbb{R}^d)} dt \right. \\
&\qquad \left. + \lambda^{\frac{d}{s}-d-k} \|\partial^\gamma \varphi\|_{L^s(\mathbb{R}^d)} \|f\|_{L^{q,\infty}(\mathbb{R}^d)} \right) \\
&\leqslant C_{p,q,r,k,m} \left(\int_0^\lambda t^{m-k+d\left(\frac{1}{p}-\frac{1}{r}\right)-1} \|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)} dt + \lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)-k} \|f\|_{L^{q,\infty}(\mathbb{R}^d)} \right) \\
&\leqslant C_{p,q,r,k,m,d} \left(\lambda^{m-k-d\left(\frac{1}{r}-\frac{1}{p}\right)} \|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)} + \lambda^{-d\left(\frac{1}{q}-\frac{1}{p}\right)-k} \|f\|_{L^{q,\infty}(\mathbb{R}^d)} \right),
\end{aligned} \tag{3.1}$$

where we used $m - \dfrac{d}{r} > k - \dfrac{d}{p}$ in the last step.

Using (2.8), (2.9) and (3.1) yields

$$\|\partial^\gamma f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,q,r,k,m,d}\left(\lambda^{m-k-d\left(\frac{1}{r}-\frac{1}{p}\right)}\|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)} + \lambda^{-d\left(\frac{1}{q}-\frac{1}{p}\right)-k}\|f\|_{L^{q,\infty}(\mathbb{R}^d)}\right). \tag{3.2}$$

Since $\lambda > 0$ is arbitrary, we can optimize the right-hand side of (3.2) to obtain

$$\|\partial^\gamma f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,q,r,k,m,d}\|f\|^{\theta}_{L^{q,\infty}(\mathbb{R}^d)}\|\nabla^m f\|^{1-\theta}_{L^{r,\infty}(\mathbb{R}^d)}.$$

This estimate holds true for all multi-index γ of length k and therefore

$$\|\nabla^k f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,q,r,k,m,d}\|f\|^{\theta}_{L^{q,\infty}(\mathbb{R}^d)}\|\nabla^m f\|^{1-\theta}_{L^{r,\infty}(\mathbb{R}^d)}. \tag{3.3}$$

Choose

$$p_1 = \frac{p+\max\{q,r\}}{2} \quad \text{and} \quad p_2 = \begin{cases} \dfrac{p}{2} + \dfrac{d}{2\left(k+\frac{d}{r}-m\right)} & \text{if } k+\frac{d}{r}-m>0, \\ 2p & \text{otherwise.} \end{cases}.$$

Then

$$1<q<p_1<p<p_2, \quad m-\frac{d}{r} > k-\frac{d}{p_2} > k-\frac{d}{p_1} \quad \text{and} \quad p_1, p_2 > r.$$

We infer further from (3.3) that

$$\|\nabla^k f\|_{L^{p_i,\infty}(\mathbb{R}^d)} \leqslant C_{p_i}\|\nabla^k f\|_{L^{p_i}(\mathbb{R}^d)} \leqslant C_{k,m,p_i,q,r,d}\|f\|^{\theta_i}_{L^{q,\infty}(\mathbb{R}^d)}\|\nabla^m f\|^{1-\theta_i}_{L^{r,\infty}(\mathbb{R}^d)}, \tag{3.4}$$

where

$$\theta_i = \frac{m-k-d\left(\frac{1}{r}-\frac{1}{p_i}\right)}{m-d\left(\frac{1}{r}-\frac{1}{q}\right)}, \quad i\in\{1,2\}.$$

Next let $\eta \in (0,1)$ be such that

$$\frac{1}{p} = \frac{\eta}{p_1} + \frac{1-\eta}{p_2}.$$

Applying Proposition 1, we obtain

$$\begin{aligned} \|\nabla^k f\|_{L^{p,\alpha}(\mathbb{R}^d)} &\leqslant C_{p,\alpha,p_1,p_2}\|\nabla^k f\|^{\eta}_{L^{p_1,\infty}(\mathbb{R}^d)}\|\nabla^k f\|^{1-\eta}_{L^{p_2,\infty}(\mathbb{R}^d)} \\ &\leqslant C_{p,q,r,\alpha,k,m,d}\|f\|^{\theta_1\eta+\theta_2(1-\eta)}_{L^{q,\infty}(\mathbb{R}^d)}\|\nabla^m f\|^{1-\theta_1\eta-\theta_2(1-\eta)}_{L^{r,\infty}(\mathbb{R}^d)}. \end{aligned} \tag{3.5}$$

Since $\theta_1\eta+\theta_2(1-\eta)=\theta$, the desired inequality (1.9) follows.

Case 2: Suppose $q \leqslant p_* < p \leqslant r$, where

$$\frac{1}{p_*} = \frac{1-\frac{k}{m}}{q} + \frac{\frac{k}{m}}{r}.$$

In this case, we start with the inequality

$$|(\partial^\gamma f)(x)| \leqslant C_{k,m} \left(\mathscr{M}(f)(x)\right)^{1-\frac{k}{m}} \left(\mathscr{M}\left(|\nabla^m f|\right)(x)\right)^{\frac{k}{m}} \tag{3.6}$$

for a.e. $x \in \mathbb{R}^d$, whose proof can be found in [10] and [12]. The former uses the Muramatu representation formula, whereas the latter uses the Sobolev representation formula.

In view of (3.6) as well as Propositions 4 and 5, we deduce that

$$\begin{aligned}\|\partial^\gamma f\|_{L^{p_*,\infty}(\mathbb{R}^d)} &\leqslant C_{k,m,p,q,r} \|\mathscr{M}(f)\|_{L^{q,\infty}(\mathbb{R}^d)}^{1-\frac{k}{m}} \|\mathscr{M}\left(|\nabla^m f|\right)\|_{L^{r,\infty}(\mathbb{R}^d)}^{\frac{k}{m}} \\ &\leqslant C_{k,m,p,q,r} \|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{1-\frac{k}{m}} \|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)}^{\frac{k}{m}}.\end{aligned}$$

Therefore,

$$\|\nabla^k f\|_{L^{p_*,\infty}(\mathbb{R}^d)} \leqslant C_{k,m,p,q,r} \|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{1-\frac{k}{m}} \|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)}^{\frac{k}{m}}. \tag{3.7}$$

Choose

$$p_1 = p_* \quad \text{and} \quad p_2 = \begin{cases} \dfrac{1}{2}\left(1 + \dfrac{1}{1-\frac{r(m-k)}{d}}\right) r & \text{if } \frac{r(m-k)}{d} < 1, \\ 2r & \text{otherwise.} \end{cases}$$

Then

$$1 < q < p_1 < p < p_2, \quad m - \frac{d}{r} > k - \frac{d}{p_2} \quad \text{and} \quad p_2 > r.$$

Combining (3.3) and (3.7) together yields

$$\|\nabla^k f\|_{L^{p_1,\infty}(\mathbb{R}^d)} \leqslant C_{k,m,p_1,q,r,d} \|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{\theta_1} \|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)}^{1-\theta_1}$$

and

$$\|\nabla^k f\|_{L^{p_2,\infty}(\mathbb{R}^d)} \leqslant C_{p_2} \|\nabla^k f\|_{L^{p_2}(\mathbb{R}^d)} \leqslant C_{k,m,p_2,q,r,d} \|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{\theta_2} \|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)}^{1-\theta_2},$$

where

$$\theta_i = \frac{m-k-d\left(\frac{1}{r}-\frac{1}{p_i}\right)}{m-d\left(\frac{1}{r}-\frac{1}{q}\right)}, \qquad i \in \{1,2\}.$$

Note that

$$\theta_1 = 1 - \frac{k}{m}$$

by construction.

Next by repeating the procedure used to obtain (3.4) and (3.5) in Case 1, we arrive at

$$\|\nabla^k f\|_{L^{p,\alpha}(\mathbb{R}^d)} \leqslant C_{p,q,r,\alpha,k,m,d} \|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{\theta} \|\nabla^m f\|_{L^{r,\infty}(\mathbb{R}^d)}^{1-\theta}.$$

The proof is complete. $\square$

4. Proof of Theorem 2

Before proving Theorem 2, we comment on our method of proof.

REMARK 2. Theorem 2 extends [10, Theorem 1.3]. Despite the fact that our proof of Theorem 2 and [10, Proof of Theorem 1.3] are both based on Muramatu's integral formula, we present here a different viewpoint applying the K method in interpolation theory. In contrast, [10, Proof of Theorem 1.3] makes use of the Hardy-Littlewood maximal operator.

In the course of proof, we require the following observation.

LEMMA 1. *Let $\phi \in C_c^\infty(B(0,1))$ be such that $\int_{\mathbb{R}^d} \phi(x)\,dx = 0$. Then there exists a constant $C_{\phi,d} > 0$ such that*

$$\sup_{t>0} \|\phi_t \star g\|_{L^\infty(\mathbb{R}^d)} \leqslant C_{\phi,d}\, \|g\|_{BMO(\mathbb{R}^d)}$$

for all $g \in BMO(\mathbb{R}^d)$.

Proof. Let $t > 0$, $x \in \mathbb{R}^d$ and $g \in BMO(\mathbb{R}^d)$. Then

$$\begin{aligned}
|(\phi_t \star g)(x)| &= \left| t^{-d} \int_{B(0,t)} \left(g(x-y) - g_{B(x,t)}\right) \phi\left(\frac{y}{t}\right) dy \right| \\
&\leqslant t^{-d}\, \|\phi\|_{L^\infty(\mathbb{R}^d)} \int_{B(x,t)} |g(y) - g_{B(x,t)}|\, dy \\
&\leqslant \|\phi\|_{L^\infty(\mathbb{R}^d)}\, |B(0,1)|\, \|g\|_{BMO(\mathbb{R}^d)}.
\end{aligned}$$

This verifies the claim. $\square$

Now we prove Theorem 2.

Proof of Theorem 2. Fix $f \in L^{q,\infty}(\mathbb{R}^d)\backslash\{0\}$ with $\nabla^m f \in BMO(\mathbb{R}^d)$ and $\gamma \in \mathbb{N}^d$ such that $|\gamma| = k$. We use Muramatu's integral formula (2.8).

Now fix $s > 0$. By the definition of the K-functional, there exist functions $u_s \in L^1(\mathbb{R}^d)$ and $v_s \in L^\infty(\mathbb{R}^d)$ such that

$$u_s + v_s = f \quad \text{and} \quad \|u_s\|_{L^1(\mathbb{R}^d)} + s\|v_s\|_{L^\infty(\mathbb{R}^d)} \leqslant 2K(s, f; L^1, L^\infty). \tag{4.1}$$

Let $\lambda > 0$. By setting

$$U_{s,\lambda} = \lambda^{-k}\left((\partial^\gamma \varphi)_\lambda \star u_s\right)$$

and

$$V_{s,\lambda} = \lim_{\varepsilon \to 0} \int_\varepsilon^\lambda \left(\sum_{|\beta|=m} t^{m-k} \left((\partial^\gamma K^{(\beta)})_t \star \partial^\beta f \right) \right) \frac{dt}{t} + \lambda^{-k} \left((\partial^\gamma \varphi)_\lambda \star v_s \right),$$

we deduce from (2.8) that

$$\partial^\gamma f = U_{s,\lambda} + V_{s,\lambda} \qquad \text{a.e. on } \mathbb{R}^d. \tag{4.2}$$

Next estimate $U_{s,\lambda}$ and $V_{s,\lambda}$ separately. For $U_{s,\lambda}$ one has

$$\|U_{s,\lambda}\|_{L^1(\mathbb{R}^d)} \leqslant \lambda^{-k} \|(\partial^\gamma \varphi)_\lambda\|_{L^1(\mathbb{R}^d)} \|u_s\|_{L^1(\mathbb{R}^d)} = \|\partial^\gamma \varphi\|_{L^1(\mathbb{R}^d)} \lambda^{-k} \|u_s\|_{L^1(\mathbb{R}^d)} \tag{4.3}$$

by Proposition 2. To deal with $V_{s,\lambda}$, first observe that

$$\partial^\gamma K^{(\beta)} \in C_c^\infty(\mathbb{R}^d) \quad \text{and} \quad \int_{\mathbb{R}^d} \partial^\gamma K^{(\beta)}\, dx = 0.$$

Consequently, Lemma 1 and (2.9) assert that

$$\begin{aligned}
\|V_{s,\lambda}\|_{L^\infty(\mathbb{R}^d)} &\leqslant \int_0^\lambda \sum_{|\beta|=m} t^{m-k} \|(\partial^\gamma K^{(\beta)})_t \star \partial^\beta f\|_{L^\infty(\mathbb{R}^d)} \frac{dt}{t} + \lambda^{-k} \|(\partial^\gamma \varphi)_\lambda \star v_s\|_{L^\infty(\mathbb{R}^d)} \\
&\leqslant C_{m,k,d} \left(\int_0^\lambda \sum_{|\beta|=m} t^{m-k} \|\partial^\beta f\|_{BMO(\mathbb{R}^d)} \frac{dt}{t} + \lambda^{-k} \|v_s\|_{L^\infty(\mathbb{R}^d)} \right) \\
&= C_{m,k,d} \left(\lambda^{m-k} \|\nabla^m f\|_{BMO(\mathbb{R}^d)} + \lambda^{-k} \|v_s\|_{L^\infty(\mathbb{R}^d)} \right).
\end{aligned} \tag{4.4}$$

Combining (4.1), (4.2), (4.3) and (4.4) together, we may conclude that $\partial^\gamma f \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ and

$$\begin{aligned}
K(s, \partial^\gamma f; L^1, L^\infty) &\leqslant \|U_{s,\lambda}\|_{L^1(\mathbb{R}^d)} + s \|V_{s,\lambda}\|_{L^\infty(\mathbb{R}^d)} \\
&\leqslant C_{m,k,d} \left(\lambda^{-k} K(s, f; L^1, L^\infty) + s \lambda^{m-k} \|\nabla^m f\|_{BMO(\mathbb{R}^d)} \right).
\end{aligned} \tag{4.5}$$

If $\|\nabla^m f\|_{BMO(\mathbb{R}^d)} = 0$, then by letting $\lambda \longrightarrow \infty$ in (4.5) and recalling that $k > 0$ by hypothesis, we obtain $K(s, \partial^\gamma f; L^1, L^\infty) = 0$. This in turn implies $\|\nabla^k f\|_{L^{p,\alpha_1}(\mathbb{R}^d)} = 0$, whence the assertion of Theorem 1.2 is trivial.

Next suppose that $\|\nabla^m f\|_{BMO(\mathbb{R}^d)} > 0$. Since (4.5) holds for all $\lambda > 0$, we may choose λ such that

$$\lambda^{-k} K(s, f; L^1, L^\infty) = s \lambda^{m-k} \|\nabla^m f\|_{BMO(\mathbb{R}^d)}.$$

Then we obtain further from (4.5) that

$$K(s, \partial^\gamma f; L^1, L^\infty) \leqslant C_{m,k,d}\, s^{\frac{k}{m}} K^{1-\frac{k}{m}}(s, f; L^1, L^\infty) \|\nabla^m f\|_{BMO(\mathbb{R}^d)}^{\frac{k}{m}}. \tag{4.6}$$

Keeping in mind the inclusion relation between Lorentz spaces, it suffices to prove Theorem 2 when

$$\alpha_2 = \left(1 - \frac{k}{m}\right) \alpha_1.$$

There are two possibilities.

First we consider $\alpha_1, \alpha_2 < \infty$. Then by combining (4.6) and the assumption that

$$\frac{\alpha_2}{\alpha_1} = \frac{q}{p} = 1 - \frac{k}{m},$$

we derive

$$\|\partial^\gamma f\|_{(L^1,L^\infty)_{1-\frac{1}{p},\alpha_1}} \leqslant C_{m,k,d} \|\nabla^m f\|_{BMO(\mathbb{R}^d)}^{\frac{k}{m}} \left(\int_0^\infty \left[s^{\frac{1}{p}-1+\frac{k}{m}} K^{1-\frac{k}{m}}(s,f;L^1,L^\infty)\right]^{\alpha_1} \frac{ds}{s}\right)^{\frac{1}{\alpha_1}}$$

and hence

$$\|\partial^\gamma f\|_{(L^1,L^\infty)_{1-\frac{1}{p},\alpha_1}} \leqslant C_{m,k,d} \|\nabla^m f\|_{BMO(\mathbb{R}^d)}^{\frac{k}{m}} \|f\|_{(L^1,L^\infty)_{1-\frac{1}{q},\alpha_2}}^{1-\frac{k}{m}}. \tag{4.7}$$

Secondly, we consider $\alpha_1 = \alpha_2 = \infty$. Analogous arguments as in the first case lead to (4.7). By virtue of (2.1) and (4.7), we have

$$\|\partial^\gamma f\|_{L^{p,\alpha_1}(\mathbb{R}^d)} \leqslant \frac{q}{q-1} C_{m,k,d} \|\nabla^m f\|_{BMO(\mathbb{R}^d)}^{\frac{k}{m}} \|f\|_{L^{q,\alpha_2}(\mathbb{R}^d)}^{1-\frac{k}{m}}.$$

Thus we obtain Theorem 1.2. $\square$

It is worth pointing out that the proof of Theorem 2 also yields a specific case of Theorem 1 as follows.

PROPOSITION 6. Let $1 < p, q, r < \infty$, $1 < \alpha_1, \alpha_2 \leqslant \infty$ and $k, m \in \mathbb{N}^*$ such that

$$k < m - \frac{d}{r}, \quad q = \left(1 - \frac{k}{m - \frac{d}{r}}\right) p \quad \text{and} \quad \alpha_2 \leqslant \left(1 - \frac{k}{m - \frac{d}{r}}\right) \alpha_1.$$

Let $f \in L^{q,\alpha_2}(\mathbb{R}^d)$ satisfy $\nabla^m f \in L^r(\mathbb{R}^d)$. Then $\nabla^k f \in L^{p,\alpha_1}(\mathbb{R}^d)$. Moreover, there exists a constant $C = C(p,q,d,k,m,r,\alpha_1,\alpha_2) > 0$ such that

$$\|\nabla^k f\|_{L^{p,\alpha_1}(\mathbb{R}^d)} \leqslant C \|f\|_{L^{q,\alpha_2}(\mathbb{R}^d)}^{1-\frac{k}{m-\frac{d}{r}}} \|\nabla^m f\|_{L^r(\mathbb{R}^d)}^{\frac{k}{m-\frac{d}{r}}}.$$

5. Proof of Theorem 3

We start with a technical lemma.

LEMMA 2. ([9, Lemma 9.5]) *For each $u \in \mathbb{N}^*$, define the sequence $\{b_j\}_{j=0}^u$ so that the identity*

$$\prod_{v=1}^{u-1}\left(1 - \frac{u-v}{v} t\right) = \sum_{j=0}^{u-1} b_j t^j$$

holds as a polynomial in t. *If* $u=1$ *then the product on the left-hand side is regarded as* 1 *and hence* $b_0=1$. *Let* $K\in C_c^\infty(\mathbb{R}^d)$ *be written in the form*

$$K(x)=\sum_{|\alpha|=u-1}\partial^\alpha K^{(\alpha)}(x),$$

where each $K^{(\alpha)}$ *satisfies* $K^{(\alpha)}\in C_c^\infty(B(0,1))$ *and* $\int_{\mathbb{R}^d}K^{(\alpha)}(x)\,dx=0$.

Let $\omega\in C_c^\infty(B(0,1))$ *satisfy* $\int_{\mathbb{R}^d}\omega(x)\,dx=1$. *Set*

$$W(x,y)=(-1)^u u^d\sum_{|\alpha|=u-1}\sum_{j=0}^{u-1}\binom{u-1}{j}^{-1}b_j\sum_{\beta:\beta\leqslant\alpha,|\beta|=j}\binom{\alpha}{\beta}(\partial^{\alpha-\beta}K^{(\alpha)})(x)\,\partial^\beta\omega(y),$$

and

$$W_t(x,y)=t^{-2d}\,W\left(\frac{x}{t},\frac{y}{t}\right)$$

for each $t>0$. *Then*

$$K_t\star f(x)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}W_t(x-y,x-y-uz)\,\Delta_z^u f(y)\,dy\,dz$$

for all $x\in\mathbb{R}^d$.

Now we prove Theorem 3.

Proof of Theorem 3. Let $f\in L^{q,\infty}(\mathbb{R}^d)\cap\dot{\Lambda}_\eta(\mathbb{R}^d)$ and $\lambda>0$. Since

$$L^{q,\infty}(\mathbb{R}^d)\subset L^1(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)\subset L^1_{\mathrm{loc}}(\mathbb{R}^d),$$

it follows from (2.7) that

$$f(x)=\lim_{\varepsilon\to 0}\int_\varepsilon^\lambda(K_t\star f)(x)\frac{dt}{t}+(\varphi_\lambda\star f)(x) \tag{5.1}$$

for a.e. $x\in\mathbb{R}^d$.

Set $u=[\eta]+1$. In (2.2) we choose $N=u$ so that K can be written in the form of Lemma 2, from which it follows that

$$K_t\star f(x)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}W_t(x-y,x-y-uz)\,\Delta_z^u f(y)\,dy\,dz \tag{5.2}$$

for all $x\in\mathbb{R}^d$.

For each $x,y,z\in\mathbb{R}^d$, if $W_t(x-y,x-y-uz)\neq 0$ then

$$|x-y|<t \quad\text{and}\quad |x-y-uz|<t$$

since W vanishes outside $B(0,1)\times B(0,1)$, and hence

$$|uz|\leqslant|x-y-uz|+|x-y|<2t$$

and

$$|W_t(x-y,x-y-uz)| \leqslant t^{-2d}\,\|W\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\,\mathbb{1}_{[0,1)}\left(\frac{|x-y|}{t}\right)\mathbb{1}_{[0,1)}\left(\frac{u|z|}{2t}\right). \tag{5.3}$$

Applying (5.2), (5.3) and keeping in mind that

$$|\Delta_z^u f(y)| = |\Delta_z^{[\eta]+1} f(y)| \leqslant \|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}\,|z|^\eta$$

for all $y,z\in\mathbb{R}^d$, we obtain

$$\begin{aligned}
|K_t\star f(x)| &\leqslant C_\eta\, t^{-2d}\,\|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathbb{1}_{[0,1)}\left(\frac{|x-y|}{t}\right)\mathbb{1}_{[0,1)}\left(\frac{u|z|}{2t}\right)|z|^\eta\,dy\,dz\\
&\leqslant C_{\eta,d}\,t^{-d}\,\|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}\int_{\mathbb{R}^d}|z|^\eta\,\mathbb{1}_{[0,\frac{2t}{u})}(|z|)\,dz\\
&\leqslant C_{\eta,d}\,t^{-d}\,\|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}\int_0^{\frac{2t}{u}} r^{\eta+d-1}\,dr\\
&\leqslant C_{\eta,d}\,t^\eta\,\|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}
\end{aligned} \tag{5.4}$$

for all $x\in\mathbb{R}^d$. This leads to

$$\left\|\int_\varepsilon^\lambda K_t\star f\,\frac{dt}{t}\right\|_{L^\infty(\mathbb{R}^d)} \leqslant C_{\eta,d}\,\|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}\int_\varepsilon^\lambda t^{\eta-1}\,dt \leqslant C_{\eta,d}\,\lambda^\eta\,\|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)} \tag{5.5}$$

for all $\varepsilon>0$.

On the other hand, by Young's inequality,

$$\left\|\int_\varepsilon^\lambda K_t\star f\,\frac{dt}{t}\right\|_{L^{q,\infty}(\mathbb{R}^d)} = \|\varphi_\varepsilon\star f-\varphi_\lambda\star f\|_{L^{q,\infty}(\mathbb{R}^d)} \leqslant C_q\,\|f\|_{L^{q,\infty}(\mathbb{R}^d)} \tag{5.6}$$

for all $\varepsilon>0$.

For each $\varepsilon>0$, set

$$T_\varepsilon f = \int_\varepsilon^\lambda K_t\star f\,\frac{dt}{t}. \tag{5.7}$$

Then according to (5.5) and (5.6),

$$\begin{aligned}
\|T_\varepsilon f\|_{L^p(\mathbb{R}^d)} &= \left(p\int_0^{\|T_\varepsilon f\|_{L^\infty(\mathbb{R}^d)}} r^q\left|\{x\in\mathbb{R}^d : |T_\varepsilon f(x)|>r\}\right| r^{p-q-1}\,dr\right)^{\frac{1}{p}}\\
&\leqslant C_{p,q}\,\|T_\varepsilon f\|_{L^{q,\infty}(\mathbb{R}^d)}^{\frac{q}{p}}\,\|T_\varepsilon f\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{q}{p}}\\
&\leqslant C_{p,q,\eta,d}\,\|f\|_{L^{q,\infty}(\mathbb{R}^d)}^{\frac{q}{p}}\left(\lambda^\eta\,\|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)}\right)^{1-\frac{q}{p}}
\end{aligned} \tag{5.8}$$

for all $\varepsilon>0$.

Next, let $s \in (1,\infty)$ satisfy

$$\frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{s}.$$

Then Proposition 3 asserts that

$$\begin{aligned}\|\varphi_\lambda \star f\|_{L^p(\mathbb{R}^d)} &\leqslant C_{p,q} \|\varphi_\lambda\|_{L^s(\mathbb{R}^d)} \|f\|_{L^{q,\infty}(\mathbb{R}^d)} \\ &\leqslant C_{p,q} \lambda^{\frac{d}{s}-d} \|\varphi\|_{L^s(\mathbb{R}^d)} \|f\|_{L^{q,\infty}(\mathbb{R}^d)} \\ &\leqslant C_{p,q} \lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^{q,\infty}(\mathbb{R}^d)}. \end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9) together yields

$$\sup_{\varepsilon>0} \|T_\varepsilon f + \varphi_\lambda \star f\|_{L^p(\mathbb{R}^d)} < \infty. \tag{5.10}$$

Also recall from (5.1) that

$$\lim_{\varepsilon\to 0} (T_\varepsilon f + \varphi_\lambda f) = f \quad \text{a.e. on } \mathbb{R}^d. \tag{5.11}$$

As such, $f \in L^p(\mathbb{R}^d)$ and $\lim_{\varepsilon\to 0} \|T_\varepsilon f + \varphi_\lambda \star f - f\|_{L^p(\mathbb{R}^d)} = 0$. Indeed, the former follows since (5.10) and (5.11) together imply f belongs to the weak closure of $L^p(\mathbb{R}^d)$ which coincides with the norm closure of $L^p(\mathbb{R}^d)$. The latter is a direct consequence of Muramatu's integral formula.

Consequently,

$$\|f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,q,\eta,d} \left(\lambda^{\eta\left(1-\frac{q}{p}\right)} \|f\|^{1-\frac{q}{p}}_{\dot{\Lambda}_\eta(\mathbb{R}^d)} \|f\|^{\frac{q}{p}}_{L^{q,\infty}(\mathbb{R}^d)} + \lambda^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^{q,\infty}(\mathbb{R}^d)} \right). \tag{5.12}$$

Set

$$w(t) = \frac{\eta + \frac{d}{t}}{\eta + \frac{d}{q}}.$$

Since (5.12) holds for all $\lambda > 0$, we can optimize the right-hand side of (5.12) to arrive at

$$\|f\|_{L^p(\mathbb{R}^d)} \leqslant C_{p,q,\eta,d} \|f\|^{w(p)}_{L^{q,\infty}(\mathbb{R}^d)} \|f\|^{1-w(p)}_{\dot{\Lambda}_\eta(\mathbb{R}^d)}. \tag{5.13}$$

To finish the proof, we choose indices p_1 and p_2 such that

$$1 < q < p_1 < p < p_2 < \infty.$$

Then it follows from (5.13) that

$$\|f\|_{L^{p_i}(\mathbb{R}^d)} \leqslant C_{p_i,q,\eta,d} \|f\|^{w(p_i)}_{L^{q,\infty}(\mathbb{R}^d)} \|f\|^{1-w(p_i)}_{\dot{\Lambda}\eta(\mathbb{R}^d)}, \qquad i \in \{1,2\}.$$

Next choose $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

By virtue of Theorem 1, we obtain

$$\begin{aligned}
\|f\|_{L^{p,\alpha}(\mathbb{R}^d)} &\leqslant C_{p,q,\alpha} \|f\|^{1-\theta}_{L^{p_1,\infty}(\mathbb{R}^d)} \|f\|^{\theta}_{L^{p_2,\infty}(\mathbb{R}^d)} \\
&\leqslant C_{p,q,\alpha} \|f\|^{1-\theta}_{L^{p_1}(\mathbb{R}^d)} \|f\|^{\theta}_{L^{p_2}(\mathbb{R}^d)} \\
&\leqslant C_{p,q,\eta,d,\alpha} \|f\|^{(1-\theta)w(p_1)+\theta w(p_2)}_{L^{q,\infty}(\mathbb{R}^d)} \|f\|^{1-(1-\theta)w(p_1)-\theta w(p_2)}_{\dot{\Lambda}^{\eta}(\mathbb{R}^d)}.
\end{aligned}$$

Since $(1-\theta)w(p_1)+\theta w(p_2) = w(p)$, the proof is complete. $\square$

We conclude this section with two remarks concerning the above proof of Theorem 3.

REMARK 3. Lemma 2 is used to derive (5.4) which is valid for all $\eta > 0$. However, if $\eta \in (0,1)$ then (5.4) follows from a simple argument without using Lemma 2. Indeed, since

$$\int_{\mathbb{R}^d} K(y)\,dy = 0 \quad \text{and} \quad \operatorname{supp} K \subset B(0,1),$$

we have

$$\begin{aligned}
|K_t \star f(x)| &= \left| \int_{\mathbb{R}^d} t^{-d} K\left(\frac{x-y}{t}\right) (f(y)-f(x))\,dy \right| \\
&\leqslant t^{-d} \|K\|_{L^\infty(\mathbb{R}^d)} \|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)} \int_{\mathbb{R}^d} |x-y|^{\eta} \mathbb{1}_{[0,1)}\left(\frac{|x-y|}{t}\right) dy \\
&\leqslant \|K\|_{L^\infty(\mathbb{R}^d)} |S^{d-1}| t^{-d} \|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)} \int_0^t r^{\eta+d-1}\,dr \\
&\leqslant C_{\eta,d} \|f\|_{\dot{\Lambda}_\eta(\mathbb{R}^d)} t^{\eta}
\end{aligned}$$

for all $x \in \mathbb{R}^d$ and $t > 0$. Here $|S^{d-1}|$ denotes the surface measure of the unit sphere in $\mathbb{R}^d$.

REMARK 4. The operator T_ε given by (5.7) defines a bounded mapping from $L^{q,\infty}(\mathbb{R}^d)$ to $L^{q,\infty}(\mathbb{R}^d)$ whose norm is independent of ε and λ. It is interesting to note that we can prove this without using the identity

$$T_\varepsilon f = \varphi_\varepsilon \star f - \varphi_\lambda \star f.$$

To this end, we invoke the Calderón-Zygmund theory of singular integrals. The main ideas are as follows. See [9, Chapter 6] for a thorough treatment.

For each $\varepsilon > 0$ and $\lambda > 0$, set

$$\mathscr{K}(x) = \int_\varepsilon^\lambda K_t(x)\,\frac{dt}{t}.$$

Then $\mathscr{K}$ is a Schwartz function and the following properties hold:

$$\begin{cases} T_\varepsilon f = \mathscr{K} \star f, \\ M_1 = \|\mathscr{F}(\mathscr{K})\|_{L^\infty(\mathbb{R}^d)} < C_{K,d} < \infty, \\ M_2 = \sup\limits_{x \in \mathbb{R}^d} |x|^{d+1} |\nabla \mathscr{K}(x)| < C_{K,d} < \infty, \end{cases}$$

where $\mathscr{F}(\mathscr{K})$ denotes the Fourier transform of $\mathscr{K}$.

By Parseval's identity, T_ε maps boundedly from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. Using Calderón-Zygmund decomposition given in [8, Theorem 4.3.1], we can prove that T_ε maps boundedly from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. The Marcinkiewicz interpolation theorem then implies T_ε maps boundedly from $L^r(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for all $r \in (1,2)$. By duality, T_ε also maps boundedly from $L^r(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$ for all $r \in (1,\infty)$. Furthermore, the norms of all these mappings depend only on M_1, M_2, r and d. Another application of the Marcinkiewicz interpolation theorem verifies that $T_\varepsilon : L^{q,\infty}(\mathbb{R}^d) \to L^{q,\infty}(\mathbb{R}^d)$ is a bounded operator whose norm depends only on K, d, and q.

REFERENCES

[1] H. Brézis, *Analyse fonctionnelle, théorie et applications*, Collection Mathématiques appliquées pour la maîtrise, Masson, Paris etc., 1983.

[2] C. Bernett, and R. Sharpley, *Interpolation of Operators*, first edition, Academic Press, Inc., Boston etc., 1988.

[3] Ş. Costea, *Sobolev-Lorentz spaces in the Euclidean setting and counterexamples*, Nonlinear Anal. Theory Methods Appl. **152** (2017), 149–182.

[4] N. A. Dao, J. I. Diaz, and Q. H. Nguyen, *Generalized Gagliardo-Nirenberg inequalities using Lorentz spaces, BMO, Hölder spaces and fractional Sobolev spaces*, Nonlinear Anal. Theory Methods Appl. **173** (2018), 146–153.

[5] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, second edition, American Mathematical Society, USA, 2010.

[6] E. Gagliardo, *Ulteriori proprietà di alcune classi di funzioni in più variabilia*, Ricerche Mat. **8** (1959), 24–51.

[7] L. Grafakos, *Modern Fourier Analysis*, vol. 250 of Graduate Texts in Mathematics, 2nd edition, Springer, New York, 2009.

[8] L. Grafakos, *Classical Fourier Analysis*, vol. 249 of Graduate Texts in Mathematics, 3rd edition, Springer, New York, 2014.

[9] Y. Miyazaki, *Introduction to L_p Sobolev spaces via Muramatu's integral formula*, Milan J. Math. **85** (2017), 103–148.

[10] Y. Miyazaki, *A short proof of the Gagliardo-Nirenberg inequality with BMO term*, Proc. Amer. Math. Soc. **148** (2020), 4257–4261.

[11] D. S. McCormick, J. C. Robinson, and J. L. Rodrigo, *Generalised Gagliardo-Nirenberg Inequalities Using Weak Lebesgue Spaces and BMO*, Milan J. Math. **81** (2013), 265–289.

[12] V. G. Mazya, and T. O. Shaposhnikova, *On pointwise interpolation inequalities for derivatives*, Math. Bohem. **124** (1999), 131–148.

[13] T. Muramatu, *On Besov spaces and Sobolev spaces of generalized functions defined on a general region*, Publ. Res. Inst. Math. Sci. **9** (1974), 325–396.

[14] L. NIRENBERG, *On elliptic partial differential equations*, Ann. Sc. Norm. Super Pisa Cl. Sci. **13** (1959), 115–162.

[15] H. TRIEBEL, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.

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