

MONOTONICITY PROPERTIES OF GAUSSIAN HYPERGEOMETRIC FUNCTIONS WITH RESPECT TO THE PARAMETER

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Abstract. The authors establish the necessary and sufficient conditions under which certain combinations of Gaussian hypergeometric function and elementary function are monotone in the parameter, which generalize the recent results of generalized elliptic integrals of the first and second kinds obtained by Qiu et al. Moreover, the authors also prove two monotonicity theorems of generalized elliptic integrals from another point of view.

1. Introduction

Throughout this paper, we always let $r' = \sqrt{1-r^2}$ for $r \in [0, 1]$, denote by \mathbb{N} (resp. \mathbb{R}) the set of positive integers (resp. real numbers), and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For complex number x with $\operatorname{Re} x > 0$, let

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad (1.1)$$

be the classical Euler gamma, beta and psi (digamma) functions, respectively (cf. [1, 5, 7, 8]). For complex numbers a, b and c with $c \neq 0, -1, -2, \dots$, the Gaussian hypergeometric function is defined by

$$F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{x^n}{n!} \quad (|x| < 1), \quad (1.2)$$

where (a, n) is the Pochhammer symbol or shifted factorial defined as $(a, 0) = 1$, and $(a, n) = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(n+a)/\Gamma(a)$ for $n \in \mathbb{N}$ (cf. [1, 7, 8, 10, 17]). $F(a, b; c; x)$ is said to be zero-balanced if $c = a + b$, and it converges absolutely for all $|x| < 1$ (cf. [7, Theorem 2.1.1]). It is well known that $F(a, b; c; x)$ has many important applications in several branches of mathematics, physics and engineering, and many other special functions in mathematical physics and even some elementary functions are particular or limiting cases of this function.

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In 1769, Euler gave an important integral representation of Gaussian hypergeometric function ${}_2F_1$ (cf. [1, 15.3.1]). If $\operatorname{Re} c > \operatorname{Re} b > 0$ and $x \in \mathbb{C} \setminus [1, +\infty)$, then

$${}_2F_1(a, b; c; x) = \frac{2\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^{\pi/2} (\sin u)^{2b-1} (\cos u)^{2c-2b-1} (1-x\sin^2 u)^{-a} du. \tag{1.3}$$

For $a \in (0, 1)$ and $r \in (0, 1)$, the generalized elliptic integrals of the first and second kinds are defined as

$$\begin{cases} \mathcal{K}_a = \mathcal{K}_a(r) = \pi F(a, 1-a; 1; r^2) / 2, \\ \mathcal{K}'_a = \mathcal{K}'_a(r) = \mathcal{K}_a(r'), \\ \mathcal{K}_a(0) = \pi/2, \mathcal{K}_a(1) = \infty, \end{cases} \tag{1.4}$$

and

$$\begin{cases} \mathcal{E}_a = \mathcal{E}_a(r) = \pi F(a-1, 1-a; 1; r^2) / 2, \\ \mathcal{E}'_a = \mathcal{E}'_a(r) = \mathcal{E}_a(r'), \\ \mathcal{E}_a(0) = \pi/2, \mathcal{E}_a(1) = [\sin(\pi a)] / [2(1-a)], \end{cases} \tag{1.5}$$

respectively (cf. [6, 4, 11]). Taking $a = 1/2$, $\mathcal{K}_{1/2} \equiv \mathcal{K}$ and $\mathcal{E}_{1/2} \equiv \mathcal{E}$ are the complete elliptic integrals of the first and second kinds, respectively (cf. [1, 17.3.9–17.3.10]).

During the past few years, both complete elliptic integrals (\mathcal{K} and \mathcal{E}) and generalized elliptic integrals (\mathcal{K}_a and \mathcal{E}_a) have been widely studied and applied in the theories of conformal invariants, quasiconformal mappings and Ramanujan’s modular equations [2, 3, 4, 9, 12, 13, 15, 16, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]. One of the meaningful tasks is to investigate the dependence on the parameter a in \mathcal{K}_a and \mathcal{E}_a , and thus show some stabilities of \mathcal{K}_a and \mathcal{E}_a with respect to a and establish several sharp bounds for \mathcal{K}_a and \mathcal{E}_a in terms of \mathcal{K} and \mathcal{E} .

In 2000, Anderson, Qiu, Vamanamurthy and Vuorinen [4, Theorem 7.2] proved the following Theorem 1.1.

THEOREM 1.1. *For each $r \in (0, 1)$, let μ, ν be defined on $[0, 1]$ by $\mu(a) = F(a-1, 1-a; 1; r^2)$ and $\nu(a) = F(a, 1-a; 1; r^2)$.*

(1) If $1/2 \leq a < b \leq 1$, then all coefficients are positive in the Taylor series for $\mu(b) - \mu(a)$ in powers of r^2 .

(2) If $0 \leq a < b \leq 1 - 1/\sqrt{2}$, then all coefficients are negative in the Taylor series for $\mu(b) - \mu(a) - (b-a)(2-a-b)r^2$ in powers of r^2 .

(3) If $0 \leq a < b \leq 1/2$ (resp. $1/2 \leq a < b \leq 1$), then all coefficients are positive (resp. negative) in the Taylor series for $\nu(b) - \nu(a)$ in powers of r^2 .

It is apparent from Theorem 1.1 that \mathcal{K}_a and \mathcal{E}_a are both strictly increasing with respect to the parameter $a \in (0, 1/2]$. Recently, Qiu, Ma and Bao [19, Theorems 1.1 and 1.2] presented the necessary and sufficient conditions under which certain familiar combinations, defined in terms of $\mathcal{K}_a, \mathcal{E}_a$ and elementary functions, are monotone in $a \in (0, 1/2]$, so that some known related results were proved substantially. For example, they proved the following monotonicity theorem.

THEOREM 1.2. *Let $1.118763390276 < \lambda_0 < 1.118763390286$ be as in [19, Lemma 2.1 (3)], for each $r \in (0, 1)$ and $\lambda \in \mathbb{R}$, define the functions $\varphi_{1,\lambda}$, $\varphi_{2,\lambda}$, $\varphi_{3,\lambda}$ and $\varphi_{4,\lambda}$ on $(0, 1/2]$ by*

$$\begin{aligned} \varphi_{1,\lambda}(a) &= \frac{\mathcal{K}_a(r) - \pi/2}{a^\lambda}, & \varphi_{2,\lambda}(a) &= \frac{\pi/2 - \mathcal{E}_a(r)}{a^\lambda}, \\ \varphi_{3,\lambda}(a) &= \frac{\mathcal{K}_a(r) - \mathcal{E}_a(r)}{a^\lambda} & \text{and} & \quad \varphi_{4,\lambda}(a) = \frac{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r)}{a^\lambda}, \end{aligned}$$

respectively. Then the following conclusions hold:

- (1) $\varphi_{1,\lambda}$ is strictly increasing (resp. decreasing) on $(0, 1/2]$ if and only if $\lambda \leq 0$ (resp. $\lambda \geq 1$).
- (2) $\varphi_{2,\lambda}$ is strictly increasing (resp. decreasing) on $(0, 1/2]$ if and only if $\lambda \leq -2$ (resp. $\lambda \geq 0$).
- (3) $\varphi_{3,\lambda}$ is strictly increasing (resp. decreasing) on $(0, 1/2]$ if and only if $\lambda \leq -1$ (resp. $\lambda \geq 1$).
- (4) $\varphi_{4,\lambda}$ is strictly increasing (resp. decreasing) on $(0, 1/2]$ if and only if $\lambda \leq 1$ (resp. $\lambda \geq \lambda_0$).

Besides, the authors [19, Theorem 6.2] further proved

THEOREM 1.3. *For each $\lambda \in \mathbb{R}$, $n \in \mathbb{N}_0$ and for each $r \in (0, 1)$, define the functions $\varphi_{5,\lambda}$ and $\varphi_{6,\lambda}$ on $(0, 1/2]$ by*

$$\varphi_{5,\lambda}(a) = \frac{\mathcal{K}_a(r) - \frac{\pi}{2} \sum_{k=0}^n \frac{(a,k)(1-a,k)}{(k!)^2} r^{2k}}{a^\lambda}$$

and

$$\varphi_{6,\lambda}(a) = \frac{\frac{\pi}{2} \sum_{k=0}^n \frac{(a-1,k)(1-a,k)}{(k!)^2} r^{2k} - \mathcal{E}_a(r)}{a^\lambda},$$

respectively. Then the following conclusions hold:

- (1) $\varphi_{5,\lambda}$ is strictly increasing (resp. decreasing) on $(0, 1/2]$ if and only if $\lambda \leq 0$ (resp. $\lambda \geq 1$).
- (2) $\varphi_{6,\lambda}$ is strictly increasing (resp. decreasing) on $(0, 1/2]$ if and only if $\lambda \leq -1 - \frac{1}{2n+1}$ (resp. $\lambda \geq 0$ if $n = 0$ or $\lambda \geq 1$ if $n \in \mathbb{N}$).

Substituting $n = 0$ in Theorem 1.3, Theorem 1.3 (1) and (2) reduce to Theorem 1.2 (1) and (2), respectively. In this paper, we shall consider more general situations. For fixed $c \in (0, \infty)$, let $a \in (0, c/2]$, $\lambda \in \mathbb{R}$ and $x \in (0, 1)$, then we define

$$f_{1,\lambda}(a) = \frac{F(a, c-a; c; x) - 1}{a^\lambda}, \quad f_{2,\lambda}(a) = \frac{1 - F(a-1, c-a; c; x)}{a^\lambda}, \tag{1.6}$$

$$f_{3,\lambda}(a) = \frac{F(a, c-a; c; x) - F(a-1, c-a; c; x)}{a^\lambda} \tag{1.7}$$

and

$$f_{4,\lambda}(a) = \frac{F(a-1, c-a; c; x) - (1-x)F(a, c-a; c; x)}{a^\lambda}. \tag{1.8}$$

Obviously, when $c = 1$ and $x = r^2$, $f_{1,\lambda}$, $f_{2,\lambda}$, $f_{3,\lambda}$ and $f_{4,\lambda}$ become $\varphi_{1,\lambda}$, $\varphi_{2,\lambda}$, $\varphi_{3,\lambda}$ and $\varphi_{4,\lambda}$ in Theorem 1.2, respectively. Naturally, the following Question 1.4 is proposed.

QUESTION 1.4. *For what values of $\lambda \in \mathbb{R}$, $f_{i,\lambda}$ ($i = 1, 2, 3, 4$) are increasing (or decreasing) on $(0, c/2]$?*

Motivated by the Theorems 1.2 and 1.3, and Question 1.4, we firstly shall give the complete answer to Question 1.4 in Section 2 (See Theorem 2.1 and Theorem 2.2). Besides, In Section 3, we shall also generalize parts (3) and (4) in Theorem 1.2 in the similar way as the extension from parts (1) and (2) in Theorem 1.2 to Theorem 1.3, and thus derive several sharp lower and upper bounds for $\mathcal{K}_a - \mathcal{E}_a$ and $\mathcal{E}_a - r^{1/2}\mathcal{K}_a$ in terms of \mathcal{K} , \mathcal{E} and elementary functions.

Let us recall some well-known formulas, which can be found in [1, 4].

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \quad (0 < x < 1), \tag{1.9}$$

$$\psi(1-x) - \psi(x) = \pi \cot(\pi x) \quad (0 < x < 1), \tag{1.10}$$

$$\Gamma(x+1) = x\Gamma(x), \quad \psi(x+1) = \psi(x) + \frac{1}{x} \quad (x > 0), \tag{1.11}$$

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (a+b < c), \tag{1.12}$$

$$F(a, b; a+b; x) \sim -\frac{1}{B(a, b)} \log(1-x), \quad x \rightarrow 1. \tag{1.13}$$

Let $\gamma = \lim_{n \rightarrow \infty} (\sum_{k=1}^n \frac{1}{k} - \log n) = 0.577 \dots$ be the Euler-Mascheroni constant. Then the psi function has the following representation (cf. [1, 6.3.16])

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}. \tag{1.14}$$

The following two technical lemmas are useful for proving the monotonicity of functions.

LEMMA 1.5. ([4, Lemma 5.1]) *For $-\infty < a < b < \infty$, let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$, and be differentiable on (a, b) . Let $g'(x) \neq 0$ on (a, b) . If $f'(x)/g'(x)$ is increasing (or decreasing) on (a, b) , then so are*

$$[f(x) - f(a)]/[g(x) - g(a)] \quad \text{and} \quad [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 1.6. ([18, Lemma 2.1]) For $n \in \mathbb{N}_0$, let r_n and s_n be real numbers, and let the power series $R(x) = \sum_{n=0}^{\infty} r_n x^n$ and $S(x) = \sum_{n=0}^{\infty} s_n x^n$ be convergent for $|x| < 1$. If $s_n \geq 0$ and not all vanish for $n \in \mathbb{N}_0$, and if r_n/s_n is strictly increasing (resp. decreasing) in $n \in \mathbb{N}_0$, then the function $x \mapsto R(x)/S(x)$ is strictly increasing (resp. decreasing) on $(0, 1)$.

2. The answer to question 1.4

In this section, we always assume that c is a fixed constant in $(0, \infty)$, and $a \in (0, c/2]$. Let

$$\lambda_1 = \lambda_1(a) = \frac{a}{a-c}, \quad \lambda_2 = \lambda_2(a) = \lambda_1 + 1 = \frac{c-2a}{c-a}, \tag{2.15}$$

$$\lambda_3 = \lambda_3(a) = a[\psi(c-a) - \psi(a)], \quad \lambda_4 = \lambda_4(a) = \frac{a(2a-c-1)}{(c-a)(1-a)}, \tag{2.16}$$

$$\lambda_5 = \lambda_5(a) = \frac{a\Gamma(c)[\psi(c+1-a) - \psi(a)]}{\Gamma(c) - \Gamma(a)\Gamma(c+1-a)}, \tag{2.17}$$

$$\lambda_6 = \lambda_6(a) = \lambda_3 + \frac{a}{c-a} = a[\psi(c+1-a) - \psi(a)]. \tag{2.18}$$

For $\lambda \in \mathbb{R}$ and $|x| < 1$, let

$$P_1(\lambda, c, x) = \left(\frac{2}{c}\right)^\lambda \sum_{n=1}^{\infty} \frac{(\frac{c}{2}, n)^2}{(c, n)n!} x^n, \quad \bar{P}_1(c, x) = P_1(0, c, x), \tag{2.19}$$

$$P_2(\lambda, c, x) = \left(\frac{2}{c}\right)^\lambda \sum_{n=1}^{\infty} \frac{(1-\frac{c}{2})(\frac{c}{2}, n-1)(\frac{c}{2}, n)}{(c, n)n!} x^n, \quad \bar{P}_2(c, x) = P_2(0, c, x), \tag{2.20}$$

$$P_3(\lambda, c, x) = \left(\frac{2}{c}\right)^\lambda \sum_{n=1}^{\infty} \frac{(\frac{c}{2}, n-1)(\frac{c}{2}, n)}{(c, n)(n-1)!} x^n, \quad \bar{P}_3(c, x) = P_3(0, c, x), \tag{2.21}$$

$$P_4(\lambda, c, x) = \left(\frac{2}{c}\right)^{\lambda-1} \sum_{n=0}^{\infty} \frac{(\frac{c}{2}, n)^2}{(n+c)(c, n)n!} x^{n+1}, \quad \bar{P}_4(c, x) = P_4(0, c, x). \tag{2.22}$$

Now we state our main results below.

THEOREM 2.1. Let λ_3^* be as in Lemma 2.3 (3) and let $P_1(\lambda, c, x)$, $P_2(\lambda, c, x)$, $\bar{P}_1(c, x)$ and $\bar{P}_2(c, x)$ be as in (2.19)–(2.20). For each fixed $c \in (0, \infty)$, $a \in (0, c/2]$, $x \in (0, 1)$ and $\lambda \in \mathbb{R}$, the functions $f_{1,\lambda}$ and $f_{2,\lambda}$ are given in (1.6). Then we have the following conclusions:

(1) $f_{1,\lambda}$ is strictly increasing (resp. decreasing) on $(0, c/2]$ if and only if $\lambda \leq 0$ (resp. $\lambda \geq 1$ if $c \in (0, 1]$ or $\lambda \geq \lambda_3^*$ if $c \in (1, \infty)$), with

$$f_{1,\lambda}(0^+) = \begin{cases} 0, & \text{if } \lambda < 1, \\ \log \frac{1}{1-x}, & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda > 1, \end{cases} \quad f_{1,\lambda}\left(\frac{c}{2}\right) = P_1(\lambda, c, x).$$

In particular, for $c \in (0, 1]$ (resp. $c \in (1, \infty)$), $a \in (0, c/2]$ and $x \in (0, 1)$,

$$1 + \frac{2a}{c} \bar{P}_1(c, x) \leq F(a, c - a; c; x) \leq 1 + \min \left\{ \bar{P}_1(c, x), a \log \frac{1}{1-x} \right\}, \tag{2.23}$$

$$\left(\text{resp. } 1 + \left(\frac{2a}{c} \right)^{\lambda_3^*} \bar{P}_1(c, x) \leq F(a, c - a; c; x) \leq 1 + \min \left\{ \bar{P}_1(c, x), a \log \frac{1}{1-x} \right\} \right).$$

The first (resp. second) equality holds if and only if $a = c/2$ (resp. $a = c/2$ or $a \rightarrow 0$).

(2) If $c \in (0, 2)$, then $f_{2,\lambda}$ is strictly increasing (resp. decreasing) on $(0, c/2]$ if and only if $\lambda \leq 2/(c - 2)$ (resp. $\lambda \geq 0$), with

$$f_{2,\lambda}(0^+) = \begin{cases} 0, & \text{if } \lambda < 0, \\ x, & \text{if } \lambda = 0, \\ \infty, & \text{if } \lambda > 0, \end{cases} \quad f_{2,\lambda} \left(\frac{c}{2} \right) = P_2(\lambda, c, x).$$

If $c \in [2, \infty)$, then there does not exist $\lambda \in \mathbb{R}$ for which $f_{2,\lambda}$ is strictly increasing (or decreasing) on $(0, c/2]$. In particular, for $c \in (0, 2)$, $a \in (0, c/2]$ and $x \in (0, 1)$,

$$1 - \min \left\{ x, \left(\frac{2a}{c} \right)^{\frac{2}{c-2}} \bar{P}_2(c, x) \right\} \leq F(a - 1, c - a; c; x) \leq 1 - \bar{P}_2(c, x). \tag{2.24}$$

The first (resp. second) equality holds if and only if $a = c/2$ or $a \rightarrow 0$ (resp. $a = c/2$).

THEOREM 2.2. Let λ_3^* and $\bar{\lambda}_6$ be as in Lemma 2.3 (3) and (6), and let $P_3(\lambda, c, x)$, $P_4(\lambda, c, x)$, $\bar{P}_3(c, x)$ and $\bar{P}_4(c, x)$ as in (2.21)–(2.22). For each fixed $c \in (0, \infty)$ and $a \in (0, c/2]$, $x \in (0, 1)$ and $\lambda \in \mathbb{R}$, the functions $f_{3,\lambda}$ and $f_{4,\lambda}$ are given in (1.7) and (1.8), respectively. Then we have the following conclusions:

(1) $f_{3,\lambda}$ is strictly increasing (resp. decreasing) on $(0, c/2]$ if and only if $\lambda \leq -1$ (resp. $\lambda \geq 1$ if $c \in (0, 1]$ or $\lambda \geq \lambda_3^*$ if $c \in (1, \infty)$), with

$$f_{3,\lambda}(0^+) = \begin{cases} 0, & \text{if } \lambda < 0, \\ x, & \text{if } \lambda = 0, \\ \infty, & \text{if } \lambda > 0, \end{cases} \quad f_{3,\lambda} \left(\frac{c}{2} \right) = P_3(\lambda, c, x).$$

In particular, for $c \in (0, 1]$ (resp. $c \in (1, \infty)$), $a \in (0, c/2]$ and $x \in (0, 1)$,

$$\frac{2a}{c} \bar{P}_3(c, x) \leq F(a, c - a; c; x) - F(a - 1, c - a; c; x) \leq \frac{c}{2a} \bar{P}_3(c, x), \tag{2.25}$$

$$\left(\text{resp. } \left(\frac{2a}{c} \right)^{\lambda_3^*} \bar{P}_3(c, x) \leq F(a, c - a; c; x) - F(a - 1, c - a; c; x) \leq \frac{c}{2a} \bar{P}_3(c, x) \right)$$

with equality in each instance if and only if $a = c/2$.

(2) $f_{4,\lambda}$ is strictly increasing (resp. decreasing) on $(0, c/2]$ if and only if $\lambda \leq 1$ (resp. $\lambda \geq \bar{\lambda}_6$), with

$$f_{4,\lambda}(0^+) = \begin{cases} 0, & \text{if } \lambda < 1, \\ \frac{x}{c}, & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda > 1, \end{cases} \quad f_{4,\lambda}\left(\frac{c}{2}\right) = P_4(\lambda, c, x).$$

In particular, for $c \in (0, \infty)$, $a \in (0, c/2]$ and $x \in (0, 1)$,

$$\begin{aligned} \max \left\{ \frac{ax}{c}, \left(\frac{2a}{c}\right)^{\bar{\lambda}_6} \bar{P}_4(c, x) \right\} &\leq F(a-1, c-a; c; x) - (1-x)F(a, c-a; c; x) \\ &\leq \frac{2a}{c} \bar{P}_4(c, x). \end{aligned} \tag{2.26}$$

The first (resp. second) equality holds if and only if $a = c/2$ or $a \rightarrow 0$ (resp. $a = c/2$).

The following three lemmas are required in the proofs of Theorems 2.1–2.2.

LEMMA 2.3. For each fixed $c \in (0, \infty)$, let λ_i ($0 \leq i \leq 6$) be as in (2.1)–(2.4). Then the following statements hold true:

- (1) $\tilde{\lambda}_1 \equiv \inf_{a \in (0, c/2]} \lambda_1(a) = -1$;
- (2) $\tilde{\lambda}_2 \equiv \inf_{a \in (0, c/2]} \lambda_2(a) = 0$;
- (3) $\bar{\lambda}_3 \equiv \sup_{a \in (0, c/2]} \lambda_3(a) = \begin{cases} 1, & \text{if } c \in (0, 1], \\ \lambda_3^*, & \text{if } c \in (1, \infty), \end{cases}$ where $\lambda_3^* \geq 1$;
- (4) $\tilde{\lambda}_4 \equiv \inf_{a \in (0, c/2]} \lambda_4(a) = \begin{cases} c/(c-2), & \text{if } c \in (0, 2), \\ -\infty, & \text{if } c \in [2, \infty); \end{cases}$
- (5) $\bar{\lambda}_5 \equiv \sup_{a \in (0, c/2]} \lambda_5(a) = \begin{cases} 0, & \text{if } c \in (0, 2), \\ \infty, & \text{if } c \in [2, \infty); \end{cases}$
- (6) $\bar{\lambda}_6 \equiv \sup_{a \in (0, c/2]} \lambda_6(a) \geq 1$.

Proof. Since the function $a \mapsto \lambda_1$ and $a \mapsto \lambda_2$ are both strictly decreasing on $(0, c/2]$ with ranges $[-1, 0)$ and $[0, 1)$ respectively, then parts (1) and (2) follows.

For part (3), due to the fact that $a \mapsto \lambda_3(a)$ is a continuous function on $(0, c/2]$ with $\lambda_3(c/2) = 0$ and

$$\lambda_3(0^+) = \lim_{a \rightarrow 0^+} a\psi(c-a) - \lim_{a \rightarrow 0^+} a\psi(a) = 1,$$

the supremum $\bar{\lambda}_3$ of the set $\{\lambda_3(a) : a \in (0, c/2]\}$ exists and $\bar{\lambda}_3 \geq 1$. If $c \in (0, 1]$, then from the monotonicity property of ψ on $(0, \infty)$ and (1.10) we get

$$\bar{\lambda}_3 \leq \sup_{a \in (0, 1/2]} a[\psi(1-a) - \psi(a)] = \sup_{a \in (0, 1/2]} \frac{\pi a}{\tan(\pi a)} = 1.$$

Consequently, it follows that $\bar{\lambda}_3 = 1$ if $c \in (0, 1]$. If $c \in (1, \infty)$, then we denote $\bar{\lambda}_3$ by λ_3^* , and thereby $\lambda_3^* \geq 1$.

For part (4), if $c \in (0, 2)$, then $\lambda_4(c/2) = 2/(c - 2)$, so that $\tilde{\lambda}_4 \leq 2/(c - 2)$. On the other hand,

$$\lambda_4(a) = \frac{a(2a - c - 1)}{(c - a)(1 - a)} \geq \frac{2}{c - 2}$$

for $a \in (c/2]$ because it is equivalent to the simple inequality $2 - a(3 - c) \geq 0$ for $a \in (0, c/2]$. This yields $\tilde{\lambda}_4 = 2/(c - 2)$ if $c \in (0, 2)$. If $c \in [2, \infty)$, then from the representation of λ_4 one has $\tilde{\lambda}_4 = -\infty$ immediately.

For part (5), if $c \in [2, \infty)$, then

$$\lim_{a \rightarrow 1^+} \lambda_5(a) = \Gamma(c) [\psi(c) - \psi(1)] \lim_{a \rightarrow 1^+} \frac{1}{\Gamma(c) - \Gamma(a)\Gamma(c + 1 - a)} = \infty,$$

which shows that $\bar{\lambda}_5 = \infty$.

If $c \in (0, 2)$, then by (1.14),

$$\begin{aligned} \bar{\lambda}_5 \geq \lambda_5(0^+) &= \lim_{a \rightarrow 0^+} \frac{a\Gamma(c)[\psi(c - a + 1) - \psi(a)]}{\Gamma(c) - \Gamma(a)\Gamma(c - a + 1)} \\ &= \lim_{a \rightarrow 0^+} \frac{a\psi(a)}{\Gamma(a)\Gamma(c - a + 1)/\Gamma(c) - 1} \\ &= \lim_{a \rightarrow 0^+} a\psi(a) \cdot \lim_{a \rightarrow 0^+} \frac{1}{\Gamma(a)\Gamma(c - a + 1)/\Gamma(c) - 1} = 0. \end{aligned} \tag{2.27}$$

On the other hand, set $\lambda_5(a) = p_1(a)/[1 - p_2(a)]$, where $p_1(a) = a[\psi(c - a + 1) - \psi(a)]$ and $p_2(a) = \Gamma(a)\Gamma(c - a + 1)/\Gamma(c)$. Since $p_2(a) > 0$ for $a \in (0, c/2]$, logarithmic differentiation leads to

$$\frac{p_2'(a)}{p_2(a)} = \psi(a) - \psi(c - a + 1) < 0$$

for $a \in (0, c/2]$, which shows that p_2 is strictly decreasing on $a \in (0, c/2]$. Thus $1 - p_2(a) \leq 1 - p_2(c/2) = 0$ for $a \in (0, c/2]$ by [14, Theorem], and therefore $\lambda_5(a) \leq 0$ for $a \in (0, c/2]$. This, in conjunction with (2.27), implies that $\bar{\lambda}_5 = 0$.

For part (6), by (2.15), (2.16) and (2.18), $\lambda_6(a) = \lambda_3(a) - \lambda_1(a)$. Obviously, $\lambda_6(0^+) = \lambda_6(c/2) = 1$. This, together with the continuous property of $a \mapsto \lambda_6(a)$ on $(0, c/2]$, implies that the supremum $\bar{\lambda}_6$ of the set $\{\lambda_6(a) : a \in (0, c/2]\}$ exists. By parts (1) and (3), we know that $\lambda_6 \geq 1$. \square

LEMMA 2.4. For each fixed $c \in (0, \infty)$, let $a \in (0, c/2]$ and $n \in \mathbb{N}_0$, $a_n = \psi(n + a) - \psi(n + c - a)$ and $b_n = \psi(n + a) - \psi(n + 1 + c - a)$. Then the sequences $\{a_n\}$ and $\{b_n\}$ are both strictly increasing in $n \in \mathbb{N}_0$ with $a_\infty = \lim_{n \rightarrow \infty} a_n = b_\infty = \lim_{n \rightarrow \infty} b_n = 0$.

Proof. By the asymptotic formula for the psi function (cf. [1, 6.3.18]), $a_\infty = b_\infty = 0$. By (1.11), we have

$$a_{n+1} - a_n = \frac{c - 2a}{(a + n)(c - a + n)} \quad \text{and} \quad b_n = a_n - \frac{c}{c - a + n},$$

from which the monotonicity properties of $\{a_n\}$ and $\{b_n\}$ follow. \square

LEMMA 2.5. For each fixed $c \in (0, \infty)$, let $a \in (0, c/2]$ and $n \in \mathbb{N}_0$. Define

$$\begin{aligned} g_{1,n}(a) &= (a, n)(c - a, n + 1) \\ g_{2,n}(a) &= (a, n)(c - a, n) \\ g_{3,n}(a) &= (1 - a)(a, n)(c - a, n + 1) \end{aligned}$$

and

$$g_{4,n}(a) = a(a, n)(c - a, n).$$

Then the sequence $\{g'_{i,n}(a)/g_{i,n}(a)\} (i = 1, 2, 3, 4)$ is strictly increasing in $n \in \mathbb{N}_0$.

Proof. Let a_n and b_n be as in Lemma 2.4. By logarithmic differentiations, we obtain

$$\frac{g'_{1,n}(a)}{g_{1,n}(a)} = a_{n+1} - a_0 - \frac{1}{n + a}, \tag{2.28}$$

$$\frac{g'_{2,n}(a)}{g_{2,n}(a)} = a_n - a_0, \tag{2.29}$$

$$\frac{g'_{3,n}(a)}{g_{3,n}(a)} = a_{n+1} - a_0 - \frac{1}{n + a} + \frac{1}{a - 1}, \tag{2.30}$$

$$\frac{g'_{4,n}(a)}{g_{4,n}(a)} = a_n - a_0 + \frac{1}{a}. \tag{2.31}$$

Therefore, Lemma 2.5 follows from (2.28)–(2.31) and Lemma 2.4. \square

Proof of Theorem 2.1. For part (1), by (1.2) and (1.11) we obtain

$$f_{1,\lambda} \left(\frac{c}{2} \right) = \left(\frac{2}{c} \right)^\lambda \sum_{n=1}^\infty \frac{(c/2, n)^2}{(c, n)n!} x^n = P_1(\lambda, c, x) \tag{2.32}$$

and

$$f_{1,\lambda}(a) = \frac{a^{1-\lambda} \Gamma(c)}{\Gamma(a + 1) \Gamma(c - a)} \sum_{n=1}^\infty \frac{\Gamma(n + a) \Gamma(n + c - a)}{\Gamma(n + c) n!} x^n,$$

from which it follows that

$$f_{1,\lambda}(0^+) = \lim_{a \rightarrow 0^+} a^{1-\lambda} \sum_{n=1}^\infty \frac{x^n}{n} = \begin{cases} 0, & \text{if } \lambda < 1, \\ \log \frac{1}{1-x}, & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda > 1. \end{cases} \tag{2.33}$$

Logarithmic differentiation yields

$$a \frac{f'_{1,\lambda}(a)}{f_{1,\lambda}(a)} = F_{1,a}(x) - \lambda, \tag{2.34}$$

where

$$F_{1,a}(x) = a \frac{\frac{\partial}{\partial a} [F(a, c - a; c; x) - 1]}{F(a, c - a; c; x) - 1} = a \frac{\sum_{n=0}^{\infty} \frac{g'_{2,n+1}(a)}{(c,n+1)(n+1)!} x^n}{\sum_{n=0}^{\infty} \frac{g_{2,n+1}(a)}{(c,n+1)(n+1)!} x^n}, \tag{2.35}$$

and $g_{2,n}$ is defined in Lemma 2.5. Noting that $g_{2,n+1} > 0$ for $n \in \mathbb{N}_0$, and $g'_{2,n+1}/g_{2,n+1}$ is strictly increasing in $n \in \mathbb{N}_0$ by Lemma 2.5. Then by application of Lemma 1.6, we derive that $F_{1,a}$ is strictly increasing in $x \in (0, 1)$. Furthermore, by (2.29) and (2.35), $F_{1,a}(0^+) = ag'_{2,1}(a)/g_{2,1}(a) = (c - 2a)/(c - a) = \lambda_2$. For the limiting value of $F_{1,a}(x)$ at 1, firstly, written $F_{1,a}(x)$ as

$$F_{1,a}(x) = \frac{1}{1 - 1/F(a, c - a; c; x)} F_{2,a}(x), \tag{2.36}$$

where

$$F_{2,a}(x) = a \frac{\frac{\partial}{\partial a} F(a, c - a; c; x)}{F(a, c - a; c; x)}. \tag{2.37}$$

Next, for $a \in (0, c/2]$ and $x \in (0, 1)$, set

$$F_3(a, x) = \int_0^{\pi/2} (\sin u)^{2c-2a-1} (\cos u)^{2a-1} (1 - x \sin^2 u)^{-a} du,$$

$$F_4(a, u) = (\sin u)^{2c-2a-1} (\cos u)^{-1} \log(\sin u)$$

and

$$F_5(a, u) = (\sin u)^{2c-2a-1} (\cos u)^{2a-1} (1 - x \sin^2 u)^{-a} [2 \log(\tan u) + \log(1 - x \sin^2 u)].$$

Then $F(a, c - a; c; x) = [2\Gamma(c)/\Gamma(a)\Gamma(c - a)]F_3(a, x)$ by (1.3), so that

$$F_{2,a}(x) = \frac{2a\Gamma(c)}{\Gamma(a)\Gamma(c - a)F(a, c - a; c; x)} \left\{ \frac{\partial F_3}{\partial a} + [\psi(c - a) - \psi(a)]F_3 \right\}$$

$$= a[\psi(c - a) - \psi(a)] + \frac{2a\Gamma(c)}{\Gamma(a)\Gamma(c - a)F(a, c - a; c; x)} \frac{\partial F_3}{\partial a}, \tag{2.38}$$

$$\frac{\partial F_3}{\partial a} = - \int_0^{\pi/2} F_5(a, u) du, \quad \lim_{x \rightarrow 1} \frac{\partial F_3}{\partial a} = -2 \int_0^{\pi/2} F_4(a, u) du. \tag{2.39}$$

It is well known that for $n \in \mathbb{N}$ (cf. [1, 6.4.1]),

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^{\infty} \frac{t^n e^{-xt}}{1 - e^{-t}} dt = - \int_0^1 \frac{t^{x-1}}{1 - t} (\log t)^n dt.$$

Employing (2.39) and substituting $t = \sin^2 u$, we have

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{\partial F_3}{\partial a} &= -2 \int_0^{\pi/2} \frac{(\sin u)^{2c-2a-1} \log(\sin u)}{1 - \sin^2 u} d(\sin u) \\ &= -\frac{1}{2} \int_0^{\pi/2} \frac{(\sin u)^{2(c-a-1)} \log(\sin^2 u)}{1 - \sin^2 u} d(\sin^2 u) \\ &= -\frac{1}{2} \int_0^1 \frac{t^{(c-a)-1} \log t}{1-t} dt = \frac{1}{2} \psi'(c-a). \end{aligned}$$

Finally, according to (1.13), (2.36) and (2.38) we get

$$F_{1,a}(1^-) = F_{2,a}(1^-) = a[\psi(c-a) - \psi(a)] = \lambda_3. \tag{2.40}$$

In conclusion, $F_{1,a}$ is strictly increasing from $(0, 1)$ onto (λ_2, λ_3) . Combining (2.34) with Lemma 2.3(1) and (3), we obtain that, for all $c \in (0, \infty)$, $a \in (0, c/2]$ and $x \in (0, 1)$,

$$f'_{1,\lambda}(a) \geq 0 \iff \lambda \leq \inf_{\substack{a \in (0, c/2] \\ x \in (0, 1)}} \{F_{1,a}(x)\} = \inf_{a \in (0, c/2]} \{\lambda_2(a)\} = \tilde{\lambda}_2 = 0,$$

$$f'_{1,\lambda}(a) \leq 0 \iff \lambda \geq \sup_{\substack{a \in (0, c/2] \\ x \in (0, 1)}} \{F_{1,a}(x)\} = \sup_{a \in (0, c/2]} \{\lambda_3(a)\} = \bar{\lambda}_3 = \begin{cases} 1, & \text{if } 0 < c \leq 1, \\ \lambda_3^*, & \text{if } c > 1. \end{cases}$$

This, together with (2.32) and (2.33), yields the first assertion in part (1). Employing the monotonicity properties and ranges of $f_{1,0}$, $f_{1,1}$ and f_{1,λ_3^*} , we get (2.23) immediately.

For part (2), by (1.2) and (1.11), one can easily obtain

$$f_{2,\lambda} \left(\frac{c}{2} \right) = \left(\frac{2}{c} \right)^\lambda \sum_{n=1}^\infty \frac{(1-c/2)(c/2, n-1)(c/2, n)}{(c, n)n!} x^n = P_2(\lambda, c, x) \tag{2.41}$$

and

$$\begin{aligned} f_{2,\lambda}(a) &= -a^{-\lambda} \sum_{n=1}^\infty \frac{(a-1, n)(c-a, n)}{(c, n)n!} x^n \\ &= -a^{-\lambda} \sum_{n=1}^\infty \frac{\Gamma(n+a-1)\Gamma(n+c-a)}{\Gamma(a-1)\Gamma(c-a)(c, n)n!} x^n \\ &= (1-a)a^{1-\lambda} \sum_{n=1}^\infty \frac{\Gamma(n+a-1)\Gamma(n+c-a)}{\Gamma(a+1)\Gamma(c-a)(c, n)n!} x^n \\ &= (1-a)a^{-\lambda} \left[\frac{c-a}{c} x + a \sum_{n=2}^\infty \frac{\Gamma(n+a-1)\Gamma(n+c-a)}{\Gamma(a+1)\Gamma(c-a)(c, n)n!} x^n \right], \end{aligned}$$

from which it follows that

$$f_{2,\lambda}(0^+) = \begin{cases} 0, & \text{if } \lambda < 0, \\ x, & \text{if } \lambda = 0, \\ \infty, & \text{if } \lambda > 0. \end{cases} \tag{2.42}$$

Logarithmic differentiation gives

$$a \frac{f'_{2,\lambda}(a)}{f_{2,\lambda}(a)} = F_{6,a}(x) - \lambda, \tag{2.43}$$

where

$$F_{6,a}(x) = a \frac{\frac{\partial}{\partial a} \sum_{n=1}^{\infty} \frac{(1-a)(a,n-1)(c-a,n)}{(c,n)n!} x^n}{\sum_{n=1}^{\infty} \frac{(1-a)(a,n-1)(c-a,n)}{(c,n)n!} x^n} = a \frac{\sum_{n=0}^{\infty} \frac{g'_{3,n}(a)}{(c,n+1)(n+1)!} x^n}{\sum_{n=0}^{\infty} \frac{g_{3,n}(a)}{(c,n+1)(n+1)!} x^n}, \tag{2.44}$$

and $g_{3,n}$ is defined in Lemma 2.5. Since $g_{3,n} > 0$ for $n \in \mathbb{N}_0$ and $g'_{3,n}/g_{3,n}$ is strictly increasing in $n \in \mathbb{N}_0$ by Lemma 2.5, then applying Lemma 1.6 we derive that $F_{6,a}$ is strictly increasing in $x \in (0, 1)$. Furthermore, by (2.30) and (2.44), we obtain

$$F_{6,a}(0^+) = a \frac{g'_{3,0}(a)}{g_{3,0}(a)} = \frac{a(2a - c - 1)}{(c - a)(1 - a)} = \lambda_4. \tag{2.45}$$

For the limiting value of $F_{6,a}(x)$ at 1, firstly, write $F_{6,a}(x)$ as

$$F_{6,a}(x) = a \frac{\frac{\partial}{\partial a} [1 - F(a - 1, c - a; c; x)]}{1 - F(a - 1, c - a; c; x)} = \frac{1}{1 - 1/F(a - 1, c - a; c; x)} F_{7,a}(x), \tag{2.46}$$

where

$$F_{7,a}(x) = a \frac{\frac{\partial}{\partial a} F(a - 1, c - a; c; x)}{F(a - 1, c - a; c; x)}. \tag{2.47}$$

Next, for $a \in (0, c/2]$ and $x \in (0, 1)$, let

$$F_8(a, x) = \int_0^{\pi/2} (\sin u)^{2c-2a-1} (\cos u)^{2a-1} (1 - x \sin^2 u)^{1-a} du,$$

$$F_9(a) = \int_0^{\pi/2} (\sin u)^{2c-2a-1} (\cos u) \log(\sin u) du.$$

Then $F(a - 1, c - a; c; x) = [2\Gamma(c)/(\Gamma(a)\Gamma(c - a))] F_8(a, x)$ by (1.3), we obtain

$$\frac{\partial F_8}{\partial a} = - \int_0^{\pi/2} (\sin u)^{2c-2a-1} (\cos u)^{2a-1} (1 - x \sin^2 u)^{1-a} [2 \log(\tan u) + \log(1 - x \sin^2 u)] du$$

and

$$F_{7,a}(x) = \frac{2a\Gamma(c)}{\Gamma(a)\Gamma(c - a)F(a - 1, c - a; c; x)} \left\{ \frac{\partial F_8}{\partial a} + [\psi(c - a) - \psi(a)] F_8 \right\}$$

$$= a[\psi(c - a) - \psi(a)] + \frac{2a\Gamma(c)}{\Gamma(a)\Gamma(c - a)F(a - 1, c - a; c; x)} \frac{\partial F_8}{\partial a}. \tag{2.48}$$

It follows from (1.12) and (2.48) that

$$\begin{aligned}
 F_{7,a}(1^-) &= a[\psi(c-a) - \psi(a)] + \lim_{x \rightarrow 1^-} \frac{2a\Gamma(c)}{\Gamma(a)\Gamma(c-a)F(a-1, c-a; c; 1)} \frac{\partial F_8}{\partial a} \\
 &= a[\psi(c-a) - \psi(a)] - 4a(c-a)F_9(a).
 \end{aligned}
 \tag{2.49}$$

Using the substitution $t = \sin u$, and integrating by parts, we obtain

$$F_9(a) = \int_0^1 t^{2c-2a-1} \log t \, dt = \frac{1}{2(c-a)} \int_0^1 \log t \, dt^{2(c-a)} = -\frac{1}{4(c-a)^2}.
 \tag{2.50}$$

Hence by (1.12), (2.46), (2.49) and (2.50), we obtain

$$\begin{aligned}
 F_{6,a}(1^-) &= \frac{1}{1 - 1/F(a-1, c-a; c; 1)} \cdot F_{7,a}(1^-) \\
 &= \frac{a\Gamma(c)[\psi(c-a+1) - \psi(a)]}{\Gamma(c) - \Gamma(a)\Gamma(c-a+1)} = \lambda_5.
 \end{aligned}
 \tag{2.51}$$

In conclusion, $F_{6,a}$ is strictly increasing from $(0, 1)$ onto (λ_4, λ_5) . By (2.43) and Lemma 2.3 (3) and (4), we obtain that, for all $c \in (0, \infty)$, $a \in (0, c/2]$ and $x \in (0, 1)$,

$$f'_{2,\lambda}(a) \geq 0 \iff \lambda \leq \inf_{\substack{a \in (0, c/2] \\ x \in (0, 1)}} \{F_{6,a}(x)\} = \inf_{a \in (0, c/2]} \{\lambda_4(a)\} = \tilde{\lambda}_4 = \begin{cases} \frac{2}{c-2} & \text{if } 0 < c < 2, \\ \infty & \text{if } c \geq 2, \end{cases}$$

$$f'_{2,\lambda}(a) \leq 0 \iff \lambda \geq \sup_{\substack{a \in (0, c/2] \\ x \in (0, 1)}} \{F_{6,a}(x)\} = \sup_{a \in (0, c/2]} \{\lambda_5(a)\} = \bar{\lambda}_5 = \begin{cases} 0, & \text{if } 0 < c < 2, \\ \infty, & \text{if } c \geq 2. \end{cases}$$

This, together with (2.41)–(2.42), yields the the first assertion in part (2).

Inequality (2.24) follows from the monotonicity properties and ranges of the particular cases $f_{2,0}$ and $f_{2,2/(c-2)}$. The condition of each equality in (2.24) is clear. This completes the proof. \square

Proof of Theorem 2.2. For part (1), by (1.2) and (1.11), we have

$$\begin{aligned}
 f_{3,\lambda}(a) &= a^{-\lambda} \sum_{n=0}^{\infty} \frac{(a,n)(c-a,n)}{(c,n)n!} x^n - a^{-\lambda} \sum_{n=0}^{\infty} \frac{(a-1,n)(c-a,n)}{(c,n)n!} x^n \\
 &= a^{-\lambda} \sum_{n=1}^{\infty} \frac{(a,n-1)(c-a,n)}{(c,n)(n-1)!} x^n,
 \end{aligned}
 \tag{2.52}$$

from which one can easily obtain

$$f_{3,\lambda}\left(\frac{c}{2}\right) = \left(\frac{2}{c}\right)^\lambda \sum_{n=1}^{\infty} \frac{(c/2, n-1)(c/2, n)}{(c, n)(n-1)!} x^n = P_3(\lambda, c, x),
 \tag{2.53}$$

$$\begin{aligned}
 f_{3,\lambda}(a) &= a^{1-\lambda} \sum_{n=1}^{\infty} \frac{\Gamma(n+a-1)\Gamma(n+c-a)}{\Gamma(1+a)\Gamma(c-a)(c,n)n!} x^n \\
 &= a^{-\lambda} \left[\frac{c-a}{c} x + a \sum_{n=2}^{\infty} \frac{\Gamma(n+a-1)\Gamma(n+c-a)}{\Gamma(1+a)\Gamma(c-a)(c,n)n!} x^n \right],
 \end{aligned}$$

and therefore

$$f_{3,\lambda}(0^+) = \begin{cases} 0, & \text{if } \lambda < 0, \\ x, & \text{if } \lambda = 0, \\ \infty, & \text{if } \lambda > 0. \end{cases} \tag{2.54}$$

Logarithmic differentiation of $f_{3,\lambda}$ gives

$$a \frac{f'_{3,\lambda}(a)}{f_{3,\lambda}(a)} = F_{10,a}(x) - \lambda, \tag{2.55}$$

where

$$F_{10,a}(x) = a \frac{\frac{\partial}{\partial a} [F(a, c-a; c; x) - F(a-1, c-a; c; x)]}{F(a, c-a; c; x) - F(a-1, c-a; c; x)} = a \frac{\sum_{n=0}^{\infty} \frac{g'_{1,n}(a)}{(c,n+1)n!} x^n}{\sum_{n=0}^{\infty} \frac{g_{1,n}(a)}{(c,n+1)n!} x^n}, \tag{2.56}$$

and $g_{1,n}$ is defined in Lemma 2.5.

Since $g_{1,n} > 0$ for $n \in \mathbb{N}_0$, and $g'_{1,n}/g_{1,n}$ is strictly increasing in $n \in \mathbb{N}_0$ by Lemma 2.5, then $F_{10,a}$ is strictly increasing in $x \in (0, 1)$ by application of Lemma 1.6. Moreover, $F_{10,a}(0^+) = ag'_{1,0}(a)/g_{1,0}(a) = a/(a-c) = \lambda_1$ by (2.56), and from (1.12), (1.13), (2.37), (2.40), (2.47) and (2.51) we obtain

$$\begin{aligned}
 F_{10,a}(1^-) &= a \lim_{x \rightarrow 1^-} \left\{ \frac{1}{1 - \frac{F(a-1, c-a; c; x)}{F(a, c-a; c; x)}} \left[\frac{\frac{\partial}{\partial a} F(a, c-a; c; x)}{F(a, c-a; c; x)} - \frac{\frac{\partial}{\partial a} F(a-1, c-a; c; x)}{F(a, c-a; c; x)} \right] \right\} \\
 &= \lim_{x \rightarrow 1^-} F_{2,a}(x) - \lim_{x \rightarrow 1^-} F_{7,a}(x) \lim_{x \rightarrow 1^-} \frac{F(a-1, c-a; c; x)}{F(a, c-a; c; x)} \\
 &= \lim_{x \rightarrow 1^-} F_{2,a}(x) = \lambda_3.
 \end{aligned}$$

In conclusion, $F_{10,a}$ is strictly increasing from $(0, 1)$ onto (λ_1, λ_3) . Incorporated with Lemma 2.3 (1) and (3), equation (2.55) gives that, for all $c \in (0, \infty)$, $a \in (0, c/2]$ and $x \in (0, 1)$,

$$f'_{3,\lambda}(a) \geq 0 \iff \lambda \leq \inf_{\substack{a \in (0, c/2] \\ x \in (0, 1)}} \{F_{10,a}(x)\} = \inf_{a \in (0, c/2]} \{\lambda_1(a)\} = \tilde{\lambda}_1 = -1,$$

$$f'_{3,\lambda}(a) \leq 0 \iff \lambda \geq \sup_{\substack{a \in (0, c/2] \\ x \in (0, 1)}} \{F_{10,a}(x)\} = \sup_{a \in (0, c/2]} \{\lambda_3(a)\} = \bar{\lambda}_3 = \begin{cases} 1, & \text{if } 0 < c \leq 1, \\ \lambda_3^*, & \text{if } c > 1. \end{cases}$$

Therefore, the first assertion in part (1) holds true. Applying the monotonicity properties of $f_{4,-1}$, $f_{4,1}$ and f_{4,λ_3^*} , inequality (2.25) and its equality cases follow immediately.

For part (2), by (1.2), we obtain

$$\begin{aligned} f_{4,\lambda}(a) &= a^{-\lambda} \left[\sum_{n=0}^{\infty} \frac{(a-1,n)(c-a,n)}{(c,n)n!} x^n - (1-x) \sum_{n=0}^{\infty} \frac{(a,n)(c-a,n)}{(c,n)n!} x^n \right] \\ &= a^{1-\lambda} \sum_{n=0}^{\infty} \frac{1}{n+c} \frac{(a,n)(c-a,n)}{(c,n)n!} x^{n+1}, \end{aligned} \tag{2.57}$$

from which it follows that

$$f_{4,\lambda} \left(\frac{c}{2} \right) = \left(\frac{c}{2} \right)^{1-\lambda} \sum_{n=0}^{\infty} \frac{(c/2,n)^2}{(n+c)(c,n)n!} x^{n+1} = P_4(\lambda, c, x),$$

$$\begin{aligned} f_{4,\lambda}(a) &= a^{1-\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+a)\Gamma(n+c-a)}{(n+c)\Gamma(a)\Gamma(c-a)(c,n)n!} x^{n+1} \\ &= a^{1-\lambda} x \left[\frac{1}{c} + \frac{a}{\Gamma(a+1)\Gamma(c-a)} \sum_{n=1}^{\infty} \frac{\Gamma(n+a)\Gamma(n+c-a)}{(n+c)(c,n)n!} x^n \right], \end{aligned} \tag{2.58}$$

and therefore

$$f_{4,\lambda}(0^+) = \lim_{a \rightarrow 0^+} f_{4,\lambda}(a) = \begin{cases} 0, & \text{if } \lambda < 1, \\ \frac{x}{c}, & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda > 1. \end{cases}$$

By logarithmic differentiation, we have

$$a \frac{f'_{4,\lambda}(a)}{f_{4,\lambda}(a)} = F_{11,a}(x) - \lambda, \tag{2.59}$$

where

$$\begin{aligned} F_{11,a}(x) &= a \frac{\frac{\partial}{\partial a} [F(a-1, c-a; c; x) - (1-x)F(a, c-a; c; x)]}{F(a-1, c-a; c; x) - (1-x)F(a, c-a; c; x)} \\ &= a \frac{\sum_{n=0}^{\infty} \frac{g'_{4,n}(a)}{(n+c)(c,n)n!} x^n}{\sum_{n=0}^{\infty} \frac{g_{4,n}(a)}{(n+c)(c,n)n!} x^n}, \end{aligned} \tag{2.60}$$

and $g_{4,n}$ is defined in Lemma 2.5 (4).

Since $g_{4,n} > 0$ for $n \in \mathbb{N}_0$, and $g'_{4,n}/g_{4,n}$ is strictly increasing in $n \in \mathbb{N}_0$ by Lemma 2.5, then $F_{11,a}$ is strictly increasing in $x \in (0, 1)$ by application of Lemma 1.6. Furthermore, by (2.60) $F_{11,a}(0^+) = a(g'_{4,0}(a))/g_{4,0}(a) = 1$, and from (1.12), (1.13), (2.37),

(2.40), (2.47) and (2.51) we have

$$\begin{aligned}
 & F_{11,a}(1^-) \\
 &= a \lim_{x \rightarrow 1^-} \left\{ \frac{1}{1 - (1-x) \frac{F(a,c-a;c;x)}{F(a-1,c-a;c;x)}} \right. \\
 &\quad \left. \times \left[\frac{\frac{\partial}{\partial a} F(a-1,c-a;c;x)}{F(a-1,c-a;c;x)} - (1-x) \frac{\frac{\partial}{\partial a} F(a,c-a;c;x)}{F(a-1,c-a;c;x)} \right] \right\} \\
 &= a \lim_{x \rightarrow 1^-} \frac{\frac{\partial}{\partial a} F(a-1,c-a;c;x)}{F(a-1,c-a;c;x)} - a \lim_{x \rightarrow 1^-} \left\{ \frac{(1-x)F(a,c-a;c;x)}{F(a-1,c-a;c;x)} \frac{\frac{\partial}{\partial a} F(a,c-a;c;x)}{F(a,c-a;c;x)} \right\} \\
 &= \lim_{x \rightarrow 1^-} F_{7,a}(x) - \lim_{x \rightarrow 1^-} \frac{(1-x)F(a,c-a;c;x)}{F(a-1,c-a;c;x)} \lim_{x \rightarrow 1^-} F_{2,a}(x) = \lambda_6.
 \end{aligned}$$

Therefore, $F_{11,a}$ is strictly increasing from $(0, 1)$ onto $(1, \lambda_6)$ by Lemma 1.6. Using (2.59) and Lemma 2.3 (5), we obtain that, for all $c \in (0, \infty)$, $a \in (0, c/2]$ and $x \in (0, 1)$,

$$\begin{aligned}
 f'_{4,\lambda}(a) \geq 0 &\iff \lambda \leq \inf_{\substack{a \in (0, c/2] \\ x \in (0, 1)}} \{F_{11,a}(x)\} = F_{11,a}(0^+) = 1, \\
 f'_{4,\lambda}(a) \leq 0 &\iff \lambda \geq \sup_{\substack{a \in (0, c/2] \\ x \in (0, 1)}} \{F_{11,a}(x)\} = \sup_{a \in (0, c/2]} \{\lambda_6(a)\} = \bar{\lambda}_6,
 \end{aligned}$$

which yields the monotonicity properties of $f_{4,\lambda}$. The remaining conclusions are obvious. \square

3. Monotonicity of generalized elliptic integrals with respect to a

The purpose of this section is to generalize parts (3) and (4) in Theorem 1.2, thus find the analogous extension from parts (1) and (2) in Theorem 1.2 to Theorem 1.3. For $a \in (0, 1/2]$, let

$$g_{5,n}(a) \equiv g_{1,n}(a)|_{c=1} = (a,n)(1-a,n+1), \tag{3.61}$$

$$g_{6,n}(a) \equiv g_{4,n}(a)|_{c=1} = a(a,n)(1-a,n), \tag{3.62}$$

$$\lambda_7 = \lambda_7(a) = \pi a / \tan(\pi a), \tag{3.63}$$

$$\lambda_8 = \lambda_8(a) = a[\psi(n+a+1) - \psi(n+3-a) + \psi(1-a) - \psi(a)], \tag{3.64}$$

$$\lambda_9 = \lambda_9(a) = 1 + a[\psi(n+1+a) - \psi(n+2-a) + \psi(1-a) - \psi(a)], \tag{3.65}$$

$$\lambda_{10} = \lambda_{10}(a) = \frac{a}{1-a} \frac{\sin(\pi a) + \pi(1-a)\cos(\pi a) - \pi(1-a)^2 \sum_{k=0}^n (k+1)^{-1} (k!)^{-2} g'_{6,k}(a)}{\sin(\pi a) - \pi(1-a) \sum_{k=0}^n (k+1)^{-1} (k!)^{-2} g_{6,k}(a)}. \tag{3.66}$$

For $r \in (0, 1)$ and $a \in (0, 1/2]$, let

$$P_{5,n}(a,r) = \frac{\pi}{2} \sum_{k=0}^n \frac{g_{5,k}(a)}{k!(k+1)!} r^{2(k+1)}, \quad \bar{P}_{5,n}(r) = P_{5,n}\left(\frac{1}{2}, r\right), \tag{3.67}$$

$$P_{6,n}(a,r) = \frac{\pi}{2} \sum_{k=0}^n \frac{g_{6,k}(a)}{k!(k+1)!} r^{2(k+1)}, \quad \bar{P}_{6,n}(r) = P_{6,n}\left(\frac{1}{2}, r\right), \tag{3.68}$$

$$P_{7,n}(r) = \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{1}{k} r^{2(k+1)} = \begin{cases} -\pi r^2 \log r', & \text{if } n = 0, \\ -\pi r^2 \log r' - \frac{\pi}{2} \sum_{k=1}^n \frac{1}{k} r^{2(k+1)}, & \text{if } n \geq 1, \end{cases} \tag{3.69}$$

$$P_{8,n}(r) = \begin{cases} \frac{\pi}{2} (r^2 + 2r'^2 \log r'), & \text{if } n = 0, \\ \frac{\pi}{2} (r^2 + 2r'^2 \log r') - \frac{\pi}{2} \sum_{k=1}^n \frac{1}{k(k+1)} r^{2(k+1)}, & \text{if } n \geq 1. \end{cases} \tag{3.70}$$

Now we state our main result of this section.

THEOREM 3.1. *For each $\lambda \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $r \in (0, 1)$, let $\bar{P}_{5,n}$, $\bar{P}_{6,n}$ and $P_{i,n}$ ($5 \leq i \leq 8$) be as in (3.67)–(3.70), let $\tilde{\lambda}_8$ and $\bar{\lambda}_{10}$ be as in Lemma 3.2(2) and (4), respectively. Define the functions $f_{5,\lambda}$ and $f_{6,\lambda}$ on $(0, 1/2]$ by*

$$f_{5,\lambda}(a) = \frac{\mathcal{K}_a(r) - \mathcal{E}_a(r) - P_{5,n}(a,r)}{a^\lambda} \quad \text{and} \quad f_{6,\lambda}(a) = \frac{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r) - P_{6,n}(a,r)}{a^\lambda},$$

respectively. Then we have the following conclusions:

(1) $f_{5,\lambda}$ is strictly increasing (resp. decreasing) on $(0, 1/2]$ if and only if $\lambda \leq \tilde{\lambda}_8 = -1/(2n+3)$ (resp. $\lambda \geq 1$), with

$$f_{5,\lambda}(0^+) = \begin{cases} 0, & \lambda < 1, \\ P_{7,n}(r), & \lambda = 1, \\ \infty, & \lambda > 1, \end{cases} \quad f_{5,\lambda}\left(\frac{1}{2}\right) = 2^\lambda [\mathcal{K}(r) - \mathcal{E}(r) - \bar{P}_{5,n}(r)].$$

In particular, for $n \in \mathbb{N}_0$, $a \in (0, 1/2]$ and $r \in (0, 1)$,

$$2a [\mathcal{K}(r) - \mathcal{E}(r) - \bar{P}_{5,n}(r)] \leq \mathcal{K}_a(r) - \mathcal{E}_a(r) - P_{5,n}(a,r) \leq \min \left\{ aP_{7,n}(r), (2a)^{-\frac{1}{2n+3}} [\mathcal{K}(r) - \mathcal{E}(r) - \bar{P}_{5,n}(r)] \right\}, \tag{3.71}$$

with equality in each instance if and only if $a = 1/2$ or $a \rightarrow 0$.

(2) $f_{6,\lambda}$ is strictly increasing (resp. decreasing) on $(0, 1/2]$ if and only if $\lambda \leq 1$ (resp. $\lambda \geq \bar{\lambda}_{10}$), with

$$f_{6,\lambda}(0^+) = \begin{cases} 0, & \lambda < 2, \\ P_{8,n}(r), & \lambda = 2, \\ \infty, & \lambda > 2, \end{cases} \quad f_{6,\lambda}\left(\frac{1}{2}\right) = 2^\lambda [\mathcal{E}(r) - r'^2 \mathcal{K}(r) - \bar{P}_{6,n}(r)].$$

In particular, for $n \in \mathbb{N}_0$, $a \in (0, 1/2]$ and $r \in (0, 1)$,

$$(2a)^{\bar{\lambda}_{10}} [\mathcal{E}(r) - r'^2 \mathcal{K}(r) - \bar{P}_{6,n}(r)] \leq \mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r) - P_{6,n}(a,r) \leq 2a [\mathcal{E}(r) - r'^2 \mathcal{K}(r) - \bar{P}_{6,n}(r)], \tag{3.72}$$

with equality in each case if and only if $a = 1/2$.

The proof of Theorem 3.1 requires some properties of $\lambda_i(a)$ ($7 \leq i \leq 10$), which are given in the following Lemma 3.2.

LEMMA 3.2. For $n \in \mathbb{N}_0$ and $a \in (0, 1/2]$, let λ_i ($7 \leq i \leq 10$) be as in (3.63)–(3.66). Then we have the following conclusions:

$$(1) \bar{\lambda}_7 = \sup_{a \in (0, 1/2]} \{\lambda_7(a)\} = 1;$$

$$(2) \tilde{\lambda}_8 = \inf_{a \in (0, 1/2]} \{\lambda_8(a)\} = -\frac{1}{2n+3};$$

$$(3) \tilde{\lambda}_9 = \inf_{a \in (0, 1/2]} \{\lambda_9(a)\} = 1;$$

(4) Let $\bar{\lambda}_{10} = \sup_{a \in (0, 1/2]} \{\lambda_{10}(a)\}$ for $n \in \mathbb{N}_0$. Then $\bar{\lambda}_{10} = 2$ if $n = 0$. That is, for each $a \in (0, 1/2]$, if we let

$$z(a) \equiv \lambda_{10}(a)|_{n=0} = \frac{a}{1-a} \frac{\sin(\pi a) + \pi(1-a)\cos(\pi a) - \pi(1-a)^2}{\sin(\pi a) - \pi a(1-a)},$$

then $\sup_{a \in (0, 1/2]} \{z(a)\} = 2$.

Proof. Since the function $x \mapsto x/\tan x$ is strictly decreasing from $(0, \pi/2)$ onto $(0, 1)$, then part (1) follows.

For part (2), by (1.11), $\lambda_8(1/2) = -1/(2n+3)$, so that

$$\tilde{\lambda}_8 = \inf_{a \in (0, 1/2]} \{\lambda_8(a)\} \leq -\frac{1}{2n+3}. \tag{3.73}$$

On the other hand, it is easy to verify that the function $a \mapsto a/(n+2-a)$ is strictly increasing from $(0, 1/2]$ onto $(0, 1/(2n+3)]$ for each fixed $n \in \mathbb{N}_0$. Combining (1.11) and Lemma 2.4 gives

$$\begin{aligned} \lambda_8(a) &= a[\psi(n+a+1) - \psi(n+2-a) + \psi(1-a) - \psi(a)] - \frac{a}{n+2-a} \\ &\geq \frac{1-2a}{1-a} - \frac{a}{n+2-a} \geq -\frac{1}{2n+3}. \end{aligned} \tag{3.74}$$

Consequently, it follows from (3.73) and (3.74) that $\tilde{\lambda}_8 = -1/(2n+3)$.

For part (3), similarly, one can easily obtain $\lambda_9(1/2) = 1$, so that

$$\tilde{\lambda}_9 = \inf_{a \in (0, 1/2]} \{\lambda_9(a)\} \leq 1. \tag{3.75}$$

On the other hand, since the function $a \mapsto a/(1-a)$ is strictly increasing from $(0, 1/2]$ onto $(0, 1]$, combining (1.11) and Lemma 2.4 we obtain

$$\begin{aligned} \lambda_9(a) &= 1 + a[\psi(n+1+a) - \psi(n+2-a) + \psi(1-a) - \psi(a)] \\ &\geq 1 + a[\psi(1+a) - \psi(2-a) + \psi(1-a) - \psi(a)] \\ &= 2 - \frac{a}{1-a} \geq 1. \end{aligned} \tag{3.76}$$

Consequently, it follows from (3.75) and (3.76) that $\tilde{\lambda}_9 = 1$ for $n \in \mathbb{N}_0$.

For part (4), let $z_1(a) = a [\sin(\pi a) + \pi(1-a)\cos(\pi a) - \pi(1-a)^2]$ and $z_2(a) = (1-a)[\sin(\pi a) - \pi a(1-a)]$. Then $z(a) = z_1(a)/z_2(a)$. Utilizing the following series expansions

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (x \in \mathbb{R}),$$

we derive that $\sin(\pi a) > \pi a - \pi^3 a^3/6$ and $\cos(\pi a) < 1 - \pi^2 a^2/2 + \pi^4 a^4/24$ for $a \in (0, 1/2]$,

$$\lim_{a \rightarrow 0^+} z(a) = \lim_{a \rightarrow 0^+} \frac{a}{1-a} \left[\frac{\pi a + \pi(1-a)(1 - \pi^2 a^2/2) - \pi(1-a)^2 + o(a^3)}{\pi a - \pi a + \pi a^2 + o(a^2)} \right] = 2$$

and therefore

$$\frac{z_2(a)}{1-a} = \sin(\pi a) - \pi a(1-a) > \pi a - \frac{\pi^3}{6} a^3 - \pi a(1-a) = \pi a^2(1 - \pi^2 a/6) > 0$$

for $a \in (0, 1/2]$.

Following it suffices to show that $z_1(a) < 2z_2(a)$ for each $a \in (0, 1/2]$, which is equivalent to $z_3(a) \equiv \pi a(1-a)\cos(\pi a) - (2-3a)\sin(\pi a) + \pi a(1-a)^2 < 0$ for $a \in (0, 1/2]$. Noting that

$$\begin{aligned} z_3(a) &< \pi a(1-a)(1 - \pi^2 a^2/2 + \pi^4 a^4/24) - (2-3a)(\pi a - \pi^3 a^3/6) + \pi a(1-a)^2 \\ &= -\frac{\pi a^3}{24} [4\pi^2 - 24 - \pi^4 a^2(1-a)] < 0 \end{aligned}$$

for $a \in (0, 1/2]$. This yields the assertion of part (4). \square

Proof of Theorem 3.1. (1) If $n = 0$, then Theorem 3.1 (1) has been proved in [19, Theorem 1.2 (3)]. Now we suppose that $n \in \mathbb{N}_0$, let $g_{5,n}(a)$ and $P_{5,n}(a, r)$ be as in (3.61) and (3.67), respectively. Let $h_1(a) = \mathcal{H}_a(r) - \mathcal{E}_a(r) - P_{5,n}(a, r)$, then by (1.4) and (1.5),

$$h_1(a) = \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{g_{5,k}(a)}{k!(k+1)!} r^{2k+2} = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{g_{5,k+n+1}(a)}{(k+n+1)!(k+n+2)!} r^{2(k+n+2)}, \quad (3.77)$$

from which it follows that

$$\begin{aligned} f_{5,\lambda}(a) &= a^{-\lambda} h_1(a) = \frac{\pi}{2} a^{-\lambda} \sum_{k=0}^{\infty} \frac{g_{5,k+n+1}(a)}{(k+n+1)!(k+n+2)!} r^{2(k+n+2)} \\ &= \frac{\pi}{2} a^{1-\lambda} \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1+a)\Gamma(k+n+3-a)}{\Gamma(a+1)\Gamma(1-a)(k+n+1)!(k+n+2)!} r^{2(k+n+2)}, \end{aligned}$$

and therefore

$$f_{5,\lambda}(0^+) = \begin{cases} 0 & \text{if } \lambda < 1, \\ P_{7,n}(r), & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda > 1, \end{cases} \quad f_{5,\lambda} \left(\frac{1}{2} \right) = 2^\lambda [\mathcal{H}(r) - \mathcal{E}(r) - \bar{P}_{5,n}(r)]. \quad (3.78)$$

Logarithmic differentiation of $f_{5,\lambda}$ leads to

$$a \frac{f'_{5,\lambda}(a)}{f_{5,\lambda}(a)} = a \frac{h'_1(a)}{h_1(a)} - \lambda = G_{1,a}(r) - \lambda, \tag{3.79}$$

where

$$\begin{aligned} G_{1,a}(r) &= a \left\{ \sum_{k=0}^{\infty} \frac{g'_{5,k+n+1}(a)}{(k+n+1)!(k+n+2)!} r^{2k} \right\} \left\{ \sum_{k=0}^{\infty} \frac{g_{5,k+n+1}(a)}{(k+n+1)!(k+n+2)!} r^{2k} \right\}^{-1} \\ &= a \left\{ \sum_{k=0}^{\infty} C_k r^{2k} \right\} \left\{ \sum_{k=0}^{\infty} D_k r^{2k} \right\}^{-1}, \end{aligned} \tag{3.80}$$

$$C_k = \frac{g'_{5,k+n+1}(a)}{(k+n+1)!(k+n+2)!}, \quad D_k = \frac{g_{5,k+n+1}(a)}{(k+n+1)!(k+n+2)!}.$$

Clearly, $D_k > 0$ for $k \in \mathbb{N}_0$, and C_k/D_k is strictly increasing in $k \in \mathbb{N}_0$ by Lemma 2.5. Applying Lemma 1.6, $G_{1,a}$ is strictly increasing in $r \in (0, 1)$. Moreover, by (3.61) and (3.80),

$$G_{1,a}(0^+) = a \frac{C_0}{D_0} = a \frac{g'_{5,n+1}(a)}{g_{5,n+1}(a)} = \lambda_8,$$

and from (1.4), (1.5) and the proof of [19, Theorem 1.1] one has

$$\begin{aligned} G_{1,a}(1^-) &= \lim_{r \rightarrow 1^-} \frac{a}{1 - \mathcal{E}_a/\mathcal{K}_a - P_{5,n}(a,r)/\mathcal{K}_a} \left[\frac{1}{\mathcal{K}_a} \frac{\partial \mathcal{K}_a}{\partial a} - \frac{1}{\mathcal{K}_a} \frac{\partial \mathcal{E}_a}{\partial a} - \frac{1}{\mathcal{K}_a} \frac{\partial P_{5,n}(a,r)}{\partial a} \right] \\ &= \lim_{r \rightarrow 1^-} \frac{a}{\mathcal{K}_a} \frac{\partial \mathcal{K}_a}{\partial a} = \lambda_7. \end{aligned}$$

Combining with (3.79), and Lemma 3.2 (1) and (2), we obtain that, for all $a \in (0, 1/2]$ and $r \in (0, 1)$,

$$\begin{aligned} f'_{5,\lambda}(a) \geq 0 &\iff \lambda \leq \inf_{a \in (0, 1/2], r \in (0, 1)} \{G_{1,a}(r)\} = \inf_{a \in (0, 1/2]} \{\lambda_8(a)\} = \tilde{\lambda}_8 = -\frac{1}{2n+3}, \\ f'_{5,\lambda}(a) \leq 0 &\iff \lambda \geq \sup_{a \in (0, 1/2], r \in (0, 1)} \{G_{1,a}(r)\} = \sup_{a \in (0, 1/2]} \{\lambda_7(a)\} = \bar{\lambda}_7 = 1. \end{aligned}$$

This, together with (3.78), yields the first assertion of part (1). The double inequality (3.71) and its equality case are clear.

(2) For $n \in \mathbb{N}_0$, let $g_{6,n}(a)$ and $P_{6,n}(a,r)$ be as in (3.62) and (3.68), respectively. Define $h_2(a) = \mathcal{E}_a(r) - r^2 \mathcal{K}_a(r) - P_{6,n}(a,r)$, then by (1.4) and (1.5) one has

$$h_2(a) = \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{g_{6,k}(a)}{k!(k+1)!} r^{2(k+1)} = \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{g_{6,k+n+1}(a)}{(k+n+1)!(k+n+2)!} r^{2(k+n+2)}, \tag{3.81}$$

from which it follows that

$$\begin{aligned}
 f_{6,\lambda}(a) &= a^{-\lambda} h_2(a) = \frac{\pi}{2} a^{-\lambda} \sum_{k=0}^{\infty} \frac{g_{6,k+n+1}(a)}{(k+n+1)!(k+n+2)!} r^{2(k+n+2)} \\
 &= \frac{\pi}{2} a^{2-\lambda} \sum_{k=0}^{\infty} \frac{\Gamma(k+n+1+a)\Gamma(k+n+2-a)}{\Gamma(a+1)\Gamma(1-a)(k+n+1)!(k+n+2)!} r^{2(k+n+2)}, \tag{3.82}
 \end{aligned}$$

and therefore

$$f_{6,\lambda}(0^+) = \begin{cases} 0 & \text{if } \lambda < 2, \\ \frac{\pi}{2} \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} r^{2k+2} = P_{8,n}(r), & \text{if } \lambda = 2, \\ \infty, & \text{if } \lambda > 2, \end{cases} \tag{3.83}$$

$$f_{6,\lambda}\left(\frac{1}{2}\right) = 2^\lambda [\mathcal{E}(r) - r'^2 \mathcal{K}(r) - \bar{P}_{6,n}(r)]. \tag{3.84}$$

It is easy to verify that for $x \in (0, 1)$,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} x^{k+1} = x - (1-x) \log \frac{1}{1-x},$$

and hence $P_{8,n}(r) = \frac{\pi}{2} (r^2 + 2r'^2 \log r')$ if $n = 0$, and

$$P_{8,n}(r) = \frac{\pi}{2} \left[r^2 + 2r'^2 \log r' - \sum_{k=1}^n \frac{1}{k(k+1)} r^{2(k+1)} \right] \text{ if } n \geq 1.$$

Logarithmic differentiation of $f_{6,\lambda}$ leads to

$$a \frac{f'_{6,\lambda}}{f_{6,\lambda}} = a \frac{h'_2(a)}{h_2(a)} - \lambda = G_{2,a}(r) - \lambda, \tag{3.85}$$

where

$$\begin{aligned}
 G_{2,a}(r) &= a \left\{ \sum_{k=0}^{\infty} \frac{g'_{6,k+n+1}(a)}{(k+n+1)!(k+n+2)!} r^{2k} \right\} \left\{ \sum_{k=0}^{\infty} \frac{g_{6,k+n+1}(a)}{(k+n+1)!(k+n+2)!} r^{2k} \right\}^{-1} \\
 &= a \left\{ \sum_{k=0}^{\infty} E_k r^{2k} \right\} \left\{ \sum_{k=0}^{\infty} F_k r^{2k} \right\}^{-1}, \tag{3.86}
 \end{aligned}$$

$$E_k = \frac{g'_{6,k+n+1}(a)}{(k+n+1)!(k+n+2)!}, \quad F_k = \frac{g_{6,k+n+1}(a)}{(k+n+1)!(k+n+2)!}.$$

Clearly, $F_k > 0$ for $k \in \mathbb{N}_0$, and the sequence $\{E_k/F_k\}$ is strictly increasing in $k \in \mathbb{N}_0$ by Lemma 2.5, so that the function $r \mapsto G_{2,a}$ is strictly increasing in $(0, 1)$ by application of Lemma 1.6. Moreover, by (3.62) and (3.86),

$$G_{2,a}(0^+) = a \frac{E_0}{F_0} = a \frac{g'_{6,n+1}(a)}{g_{6,n+1}(a)} = \lambda_9(a). \tag{3.87}$$

and from (1.5) and [19, equation (4.19)] we obtain

$$\begin{aligned}
 G_{2,a}(1^-) &= \lim_{r \rightarrow 1^-} \left\{ \frac{a[\mathcal{E}_a(r) - \pi/2]}{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r) - P_{6,n}(a,r)} \right. \\
 &\quad \times \left. \left[\frac{\frac{\partial}{\partial a} [\pi/2 - \mathcal{E}_a(r)]}{\pi/2 - \mathcal{E}_a(r)} + \frac{\frac{\partial}{\partial a} [r'^2 \mathcal{K}_a(r)]}{\pi/2 - \mathcal{E}_a(r)} + \frac{\frac{\partial}{\partial a} P_{6,n}(a,r)}{\pi/2 - \mathcal{E}_a(r)} \right] \right\} \\
 &= \frac{a[\sin(\pi a) - \pi(1-a)]}{\sin(\pi a) - \pi(1-a) \sum_{k=0}^n (k+1)^{-1} (k!)^{-2} g_{6,k}(a)} \\
 &\quad \times \left\{ -\frac{\sin(\pi a) + \pi(1-a) \cos(\pi a)}{(1-a)[\pi(1-a) - \sin(\pi a)]} + \frac{\pi(1-a) \sum_{k=0}^n (k+1)^{-1} (k!)^{-2} g'_{6,k}(a)}{\pi(1-a) - \sin(\pi a)} \right\} \\
 &= \frac{a}{\sin(\pi a) - \pi(1-a) \sum_{k=0}^n (k+1)^{-1} (k!)^{-2} g_{6,k}(a)} \\
 &\quad \times \left\{ \frac{\sin(\pi a) + \pi(1-a) \cos(\pi a)}{1-a} - \pi(1-a) \sum_{k=0}^n (k+1)^{-1} (k!)^{-2} g'_{6,k}(a) \right\} \\
 &= \frac{a}{1-a} \frac{\sin(\pi a) + \pi(1-a) \cos(\pi a) - \pi(1-a)^2 \sum_{k=0}^n (k+1)^{-1} (k!)^{-2} g'_{6,k}(a)}{\sin(\pi a) - \pi(1-a) \sum_{k=0}^n (k+1)^{-1} (k!)^{-2} g_{6,k}(a)} \\
 &= \lambda_{10}. \tag{3.88}
 \end{aligned}$$

Combining with (3.85), and Lemma 3.2 (3) and (4), we conclude that for all $a \in (0, 1/2]$ and $r \in (0, 1)$,

$$f'_{6,\lambda}(a) \geq 0 \iff \lambda \leq \inf_{a \in (0,1/2], r \in (0,1)} \{G_{2,a}(r)\} = \inf_{a \in (0,1/2]} \{\lambda_9(a)\} = \tilde{\lambda}_9 = 1,$$

$$f'_{6,\lambda}(a) \leq 0 \iff \lambda \geq \sup_{a \in (0,1/2], r \in (0,1)} \{G_{2,a}(r)\} = \sup_{a \in (0,1/2]} \{\lambda_{10}(a)\} = \bar{\lambda}_{10}.$$

This, together with (3.83) and (3.84), yields the first assertion of part (2). Inequality (3.72) and its equality case are clear. This completes the proof. \square

Taking $n = 0$ in Theorem 3.1 (2) and using Lemma 3.2 (4), the following Corollary 3.3 can be obtained immediately.

COROLLARY 3.3. *For each $\lambda \in \mathbb{R}$ and $r \in (0, 1)$, define the function $f_{7,\lambda}$ on $(0, 1/2]$ by*

$$f_{7,\lambda}(a) = \frac{\mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r) - \pi ar^2/2}{a^\lambda}.$$

Then $f_{7,\lambda}$ is strictly decreasing on $(0, 1/2]$ if and only if $\lambda \geq 2$. In particular, for all $a \in (0, 1/2]$ and $r \in (0, 1)$,

$$\begin{aligned}
 a^2 [4(\mathcal{E} - r'^2 \mathcal{K}) - \pi r^2] &\leq \mathcal{E}_a(r) - r'^2 \mathcal{K}_a(r) - \frac{\pi}{2} ar^2 \\
 &\leq \frac{\pi a^2}{2} (r^2 + 2r'^2 \log r').
 \end{aligned}$$

The first (resp. second) equality holds if and only if $a = 1/2$ (resp. $a \rightarrow 0$).

REMARK 3.4. It is not easy to calculate the exact values of λ_3^* , $\bar{\lambda}_6$ and $\bar{\lambda}_{10}$ ($n \in \mathbb{N}$) appeared in Lemma 2.3 (3), (6) and Lemma 3.2 (4), respectively. Here it is left as an open problem for the readers.

OPEN PROBLEM 3.5. (1) What are the values of λ_3^* and $\bar{\lambda}_6$ in Lemma 2.3 (3) and (6)?

(2) What is the value of $\bar{\lambda}_{10}$ in Lemma 3.2 (4) for $n \in \mathbb{N}$?

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