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HARDY–HILBERT TYPE INEQUALITIES FOR MATRICES

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Abstract. By making use of the weighted geometric mean of positive definite matrices, we extend Hardy-Hilbert type inequalities to matrix case. Our results complement those of Hansen [Internat. J. Math. 21 (2010), no. 10, 1283–1295] and Kian [Ann. Funct. Anal. 3 (2012), no. 2, 128–134].

1. Introduction

Hardy-Hilbert's inequality has been studied by many great mathematicians since the beginning of the last century. It has been refined and generalized in many different ways. We refer the reader to [5, 9] and references therein for more details. The inequality reads as follows:

If $\sum\limits_{m=1}^{\infty} a_m^p$ and $\sum\limits_{n=1}^{\infty} b_n^q$ are convergent for nonnegative a_m and b_n, and $p,q > 1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1.1}$$

The integral form of the Hardy-Hilbert's inequality is also well-known. Its operator version was studied by Hansen et al. in [4] and Kian in [6].

In this paper, we consider Hardy-Hilbert's inequality for positive definite matrices and sector matrices. Since the product of positive definite matrices does not preserve its positivity, the geometric mean is used to replace the ordinary product.

The main results are Theorem 4.1 and its sharper form Theorem 4.3 for positive semidefinite matrices, and Theorem 4.7 for sector matrices. More generalization is discussed in the last section.

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2. Notation and definitions

Let $\mathbb{M}_r$ be the set of $r \times r$ complex matrices and $\mathbb{M}_r^+$ the subset of $\mathbb{M}_r$ consisting of positive definite matrices. If $A \in \mathbb{M}_r$ is positive semidefinite (definite), then we write $A \geqslant 0$ $(A > 0)$. For two Hermitian matrices A and B of the same size, we write $A \geqslant B$ $(A > B)$ to mean that $A - B$ is positive semidefinite (positive definite). This $\geqslant$ is a partial ordering, called Löwner partial ordering, on the set of Hermitian matrices.

The geometric mean of two $r \times r$ positive definite matrices A and B, denoted by $A\#B$, is the positive definite solution of the Riccati equation $XB^{-1}X = A$ and it has the explicit expression

$$A\#B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

If $A, B \in \mathbb{M}_r^+$ and $\lambda \in (0,1)$, then the λ-weighted geometric mean of A and B is defined by

$$A\#_\lambda B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\lambda}A^{\frac{1}{2}}. \tag{2.1}$$

Consider the Cartesian decomposition of A, $A = \Re A + i\Im A$, where $\Re A = \frac{1}{2}(A + A^*)$ is the real part of A and $\Im A = \frac{1}{2i}(A - A^*)$ is the imaginary part of A. A more general class of matrices than that of positive definite ones is the so called accretive matrices. A matrix $A \in \mathbb{M}_r$ is said to be accretive if $\Re A > 0$. In [7], Raïssouli et al. extended the definition of the λ-weighted geometric mean from positive definite matrices to accretive matrices. The definition is

$$A\#_\lambda B = \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty t^{\lambda-1}(A^{-1} + tB^{-1})^{-1}\mathrm{d}t, \tag{2.2}$$

where $\lambda \in (0,1)$. If A and B are positive definite, then (2.2) coincides with (2.1).

A special class of accretive matrices is sector matrices. The numerical range of a square matrix A is defined to be

$$\mathbb{W}(A) = \{x^*Ax \mid x \in \mathbb{C}^r, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, let S_α be the sector in the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} \mid \Re z > 0, |\Im z| \leqslant (\Re z)\tan\alpha\}.$$

If $\mathbb{W}(A) \subset S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then A is called a sector matrix. It is clear that every sector matrix is nonsingular and its real part is positive definite.

We recall the notions of convergence of matrix sequence and matrix series. A sequence of matrices is called convergent if it is convergent entrywise. Let $\{A_k\} \subset \mathbb{M}_r$ be an infinite sequence of matrices. The series $\sum\limits_{k=1}^{\infty} A_k$ is convergent if the sequence of its partial sums is convergent. For positive semidefinite matrices A_k, the series $\sum\limits_{k=1}^{\infty} A_k$ is convergent if and only if its partial sums $\sum\limits_{k=1}^{m} A_k$ are bounded from above in the Löwner partial ordering sense.

3. Preliminaries

Before presenting and proving our results, we need the following lemmas.

LEMMA 3.1. ([7, Proposition 4.1]) *For any accretive $A,B \in \mathbb{M}_r$ and $\lambda \in (0,1)$ the following equality*

$$(aA)\#_\lambda(bB) = (a^{1-\lambda}b^\lambda)(A\#_\lambda B)$$

holds for every real numbers $a,b>0$.

Directly induced from (2.1), we get

LEMMA 3.2. *Let $A,B \in \mathbb{M}_r^+$ and $\lambda \in (0,1)$. If $AB = BA$, then $A\#_\lambda B = A^{1-\lambda}B^\lambda$.*

The operator monotonicity of the map $A \mapsto A^\lambda$ on $\mathbb{M}_r^+$ for $0<\lambda<1$ yields the operator monotonicity of the weighted geometric mean.

LEMMA 3.3. *Let $A,B,C,D \in \mathbb{M}_r^+$ and $\lambda \in (0,1)$. If $A \leqslant C$ and $B \leqslant D$, then $A\#_\lambda B \leqslant C\#_\lambda D$.*

Recall that a map $f : \mathbb{M}_r^+ \times \mathbb{M}_r^+ \to \mathbb{M}_r^+$ is called *jointly concave*, if

$$f(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda)B_2) \geqslant \lambda f(A_1,B_1) + (1-\lambda)f(A_2,B_2)$$

for all $A_1,A_2,B_1,B_2 \in \mathbb{M}_r^+$ and $0<\lambda<1$. The geometric mean is jointly concave on pairs of positive definite matrices because of its maximal characterization, that is,

$$A\#B = \max\left\{ X>0 \,\middle|\, \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geqslant 0 \right\}.$$

One can see [1] for more detailed discussion.

The concavity of the weighted geometric mean is pointed out by Bourin et al. [2] without explicit proof. Since the weighted geometric mean does not preserve the maximal characterization, the proof of its concavity is a little different. We supplement a proof here for the convenience of readers.

PROPOSITION 3.4. *The λ-weighted geometric mean defined on $\mathbb{M}_r^+ \times \mathbb{M}_r^+$ is jointly concave.*

Proof. From (2.2), it is sufficient to show that the map $(A,B) \mapsto (A^{-1}+tB^{-1})^{-1}$ is jointly concave for any $t>0$. Note that

$$(A^{-1}+tB^{-1})^{-1} = A : t^{-1}B$$

is the parallel sum of A and $t^{-1}B$. And the parallel sum is jointly concave on $\mathbb{M}_r^+ \times \mathbb{M}_r^+$ (see [1, Theorem 4.1.1]). Thus the integrand in (2.2) is jointly concave, so is $A\#_\lambda B$. $\square$

The matrix version of the Hölder inequality [2, Corollary 2] can be generalized to the convergent series of positive definite matrices.

LEMMA 3.5. *Let $A_k, B_k \in \mathbb{M}_r^+$ for all $k \geqslant 1$. If the series $\sum_{k=1}^{\infty} A_k$ and $\sum_{k=1}^{\infty} B_k$ are convergent, then*

$$\sum_{k=1}^{\infty} A_k \#_\lambda B_k \leqslant \left(\sum_{k=1}^{\infty} A_k\right) \#_\lambda \left(\sum_{k=1}^{\infty} B_k\right).$$

Proof. Since the weighted geometric mean is monotone increasing, using [2, Corollary 2], we see that the partial sums satisfy

$$\begin{aligned}\sum_{k=1}^{m} A_k \#_\lambda B_k &\leqslant \left(\sum_{k=1}^{m} A_k\right) \#_\lambda \left(\sum_{k=1}^{m} B_k\right)\\ &\leqslant \left(\sum_{k=1}^{\infty} A_k\right) \#_\lambda \left(\sum_{k=1}^{\infty} B_k\right).\end{aligned}$$

So the series $\sum_{k=1}^{\infty} A_k \#_\lambda B_k$ is convergent and satisfies the asserted inequality. □

4. Main results

The main result of this paper is the following theorem which can be regarded as a matrix version of Hardy-Hilbert's inequality (1.1).

THEOREM 4.1. *Let $A_m, B_n \in \mathbb{M}_r^+$ for all $m,n \geqslant 1$, and $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\sum_{m=1}^{\infty} A_m^p$ and $\sum_{n=1}^{\infty} B_n^q$ are convergent, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_m^p \#_{\frac{1}{q}} B_n^q}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left[\left(\sum_{m=1}^{\infty} A_m^p\right) \#_{\frac{1}{q}} \left(\sum_{n=1}^{\infty} B_n^q\right)\right]. \tag{4.1}$$

Proof. For $\frac{1}{p}+\frac{1}{q}=1$, Lemma 3.1 yields

$$\frac{A_m^p \#_{\frac{1}{q}} B_n^q}{m+n} = \left[\frac{A_m^p}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right] \#_{\frac{1}{q}} \left[\frac{B_n^q}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right].$$

Then using Lemma 3.5 twice, we get

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{A_m^p \#_{\frac{1}{q}} B_n^q}{m+n}\right)\\ &\leqslant \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{A_m^p}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right] \#_{\frac{1}{q}} \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{B_n^q}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right]\\ &= \left[\sum_{m=1}^{\infty} A_m^p \sum_{n=1}^{\infty} \frac{1}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right] \#_{\frac{1}{q}} \left[\sum_{n=1}^{\infty} B_n^q \sum_{m=1}^{\infty} \frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right].\end{aligned}$$

Note that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}} &= \sum_{n=1}^{\infty}\frac{1}{1+\frac{n}{m}}\left(\frac{n}{m}\right)^{-\frac{1}{q}}\frac{1}{m}\\ &< \int_0^{\infty}\frac{1}{1+\frac{t}{m}}\left(\frac{t}{m}\right)^{-\frac{1}{q}}\frac{1}{m}\mathrm{d}t\\ &= \int_0^{\infty}\frac{1}{1+x}x^{-\frac{1}{q}}\mathrm{d}x = \frac{\pi}{\sin(\frac{\pi}{p})},\end{aligned}$$

where the first inequality follows from the fact that $\frac{1}{1+x}x^{-\frac{1}{p}}$ is a decreasing function on x, and the last equality follows from the formula (see e.g., [3, p. 135])

$$\int_0^{\infty}\frac{x^{\lambda-1}}{x+1}\mathrm{d}x = \frac{\pi}{\sin(\lambda\pi)} \quad \text{for } 0<\lambda<1. \tag{4.2}$$

So

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{A_m^p \#_{\frac{1}{q}} B_n^q}{m+n}\right) < \left(\frac{\pi}{\sin(\frac{\pi}{p})}\sum_{m=1}^{\infty}A_m^p\right)\#_{\frac{1}{q}}\left(\frac{\pi}{\sin(\frac{\pi}{q})}\sum_{n=1}^{\infty}B_n^q\right).$$

As $\sin(\frac{\pi}{q}) = \sin(\frac{\pi}{p})$, applying Lemma 3.1 again yields the conclusion. □

For commuting pairs of positive definite matrices, we have the following simple consequence of Lemma 3.2 and Theorem 4.1.

COROLLARY 4.2. *Let $A_m, B_n \in \mathbb{M}_r^+$ for all $m,n \geqslant 1$, and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\sum_{m=1}^{\infty} A_m^p$ and $\sum_{n=1}^{\infty} B_n^q$ are convergent, and $A_mB_n = B_nA_m$ for each m,n, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_mB_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}B_n^q\right)^{\frac{1}{q}}.$$

Similar to the idea in Section 1.1.2 [9, p. 2], we may consider the inequality when the subscripts m and n of the double series begin with 0.

THEOREM 4.3. *Let $A_m, B_n \in \mathbb{M}_r^+$ for all $m,n \geqslant 0$, and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\sum_{m=0}^{\infty} A_m^p$ and $\sum_{n=0}^{\infty} B_n^q$ are convergent, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_m^p \#_{\frac{1}{q}} B_n^q}{m+n+a} < \frac{\pi}{\sin(\frac{\pi}{p})}\left[\left(\sum_{m=0}^{\infty}A_m^p\right)\#_{\frac{1}{q}}\left(\sum_{n=0}^{\infty}B_n^q\right)\right] \tag{4.3}$$

holds for any $a \geqslant 1$.

Proof. The proof is similar to the one for Theorem 4.1. We have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_m^p\#_{\frac{1}{q}}B_n^q}{m+n+a}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\frac{A_m^p}{m+n+a}\left(\frac{m+\frac{a}{2}}{n+\frac{a}{2}}\right)^{\frac{1}{q}}\right]\#_{\frac{1}{q}}\left[\frac{B_n^q}{m+n+a}\left(\frac{n+\frac{a}{2}}{m+\frac{a}{2}}\right)^{\frac{1}{p}}\right]\\
&\leqslant\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{A_m^p}{m+n+a}\left(\frac{m+\frac{a}{2}}{n+\frac{a}{2}}\right)^{\frac{1}{q}}\right]\#_{\frac{1}{q}}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{B_n^q}{m+n+a}\left(\frac{n+\frac{a}{2}}{m+\frac{a}{2}}\right)^{\frac{1}{p}}\right]\\
&=\left[\sum_{m=0}^{\infty}A_m^p\sum_{n=0}^{\infty}\frac{1}{m+n+a}\left(\frac{m+\frac{a}{2}}{n+\frac{a}{2}}\right)^{\frac{1}{q}}\right]\#_{\frac{1}{q}}\left[\sum_{n=0}^{\infty}B_n^q\sum_{m=0}^{\infty}\frac{1}{m+n+a}\left(\frac{n+\frac{a}{2}}{m+\frac{a}{2}}\right)^{\frac{1}{p}}\right]\\
&<\frac{\pi}{\sin(\frac{\pi}{p})}\left[\left(\sum_{m=0}^{\infty}A_m^p\right)\#_{\frac{1}{q}}\left(\sum_{n=0}^{\infty}B_n^q\right)\right].
\end{aligned}$$

For completeness, we give a proof of the last inequality here. Set

$$f(x)=\frac{1}{m+\frac{a}{2}+x}\left(\frac{m+\frac{a}{2}}{x}\right)^{\frac{1}{q}}.$$

It is easy to check that $f(x)$ is decreasing and convex for $x>0$. So

$$2f\left(\frac{x+y}{2}\right)\leqslant f(x)+f(y),\quad \text{for } x,y>0.$$

Then, for $a\geqslant 1$, we have

$$\begin{aligned}
f\left(n+\frac{a}{2}\right)&\leqslant f\left(n+\frac{1}{2}\right)=\int_n^{n+\frac{1}{2}}2f\left(\frac{2n+1}{2}\right)\mathrm{d}x\\
&\leqslant\int_n^{n+\frac{1}{2}}\big(f(x)+f(2n+1-x)\big)\mathrm{d}x=\int_n^{n+1}f(x)\mathrm{d}x.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\left[\frac{1}{m+n+a}\left(\frac{m+\frac{a}{2}}{n+\frac{a}{2}}\right)^{\frac{1}{q}}\right]&=\sum_{n=0}^{\infty}f\left(n+\frac{a}{2}\right)\leqslant\sum_{n=0}^{\infty}\int_n^{n+1}f(x)\mathrm{d}x\\
&=\int_0^{\infty}f(x)\mathrm{d}x=\int_0^{\infty}\frac{1}{m+\frac{a}{2}+x}\left(\frac{m+\frac{a}{2}}{x}\right)^{\frac{1}{q}}\mathrm{d}x\\
&=\int_0^{\infty}\frac{1}{1+t}\left(\frac{1}{t}\right)^{\frac{1}{q}}\mathrm{d}t=\frac{\pi}{\sin(\frac{\pi}{p})}.
\end{aligned}$$

Then the desired inequality follows from Lemma 3.1. $\square$

In fact, Theorem 4.3 is a slightly sharper form of Theorem 4.1, since (4.1) is a special case of (4.3) in which $a=2$. Next, in order to extend Theorem 4.3 to sector matrices, we need some lemmas.

LEMMA 4.4. ([7, Theorem 2.4]) *Let $A,B\in\mathbb{M}_r$ be accretive and $\lambda\in(0,1)$. Then*

$$\Re(A)\#_\lambda\Re(B)\leqslant\Re(A\#_\lambda B).$$

LEMMA 4.5. ([8, Lemma 5]) *Let $A,B\in\mathbb{M}_r$ be sector matrices with $\mathbb{W}(A),\mathbb{W}(B)\subset S_\alpha$, and $\lambda\in(0,1)$. Then*

$$\Re(A\#_\lambda B)\leqslant\sec^2\alpha(\Re A\#_\lambda\Re B).$$

LEMMA 4.6. *Let $\{A_k\}\subset\mathbb{M}_r$ be an infinite sequence of matrices. If the series $\sum_{k=1}^{\infty}A_k$ converges to A, then the series $\sum_{k=1}^{\infty}\Re A_k$ and $\sum_{k=1}^{\infty}\Im A_k$ converge to $\Re A$ and $\Im A$, respectively.*

Proof. It is clear that $\sum_{k=1}^{\infty}A_k=A$ implies $\sum_{k=1}^{\infty}A_k^*=A^*$. So

$$\sum_{k=1}^{\infty}\Re A_k=\sum_{k=1}^{\infty}\frac{A_k+A_k^*}{2}=\frac{A+A^*}{2}=\Re A,$$

and

$$\sum_{k=1}^{\infty}\Im A_k=\sum_{k=1}^{\infty}\frac{A_k-A_k^*}{2i}=\frac{A-A^*}{2i}=\Im A.\quad\square$$

THEOREM 4.7. *Let $A_m^p,B_n^q\in\mathbb{M}_r$ be sector matrices with $\mathbb{W}(A_m^p),\mathbb{W}(B_n^q)\subset S_\alpha$ for all $m,n\geqslant 0$, and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\sum_{m=0}^{\infty}A_m^p$ and $\sum_{n=0}^{\infty}B_n^q$ are convergent, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Re\left(\frac{A_m^p\#_{\frac{1}{q}}B_n^q}{m+n+a}\right)<\frac{\pi\sec^2\alpha}{\sin(\frac{\pi}{p})}\Re\left(\sum_{m=0}^{\infty}A_m^p\#_{\frac{1}{q}}\sum_{n=0}^{\infty}B_n^q\right)$$

holds for any $a\geqslant 1$.

Proof. Applying Theorem 4.3 and Lemmas 4.4-4.6, we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Re\left(\frac{A_m^p\#_{\frac{1}{q}}B_n^q}{m+n+a}\right)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Re(A_m^p\#_{\frac{1}{q}}B_n^q)}{m+n+a}$$

$$\leqslant\sec^2\alpha\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Re(A_m^p)\#_{\frac{1}{q}}\Re(B_n^q)}{m+n+a}\qquad\text{(by Lemma 4.5)}$$

$$< \frac{\pi \sec^2 \alpha}{\sin(\frac{\pi}{p})} \left[\left(\sum_{m=0}^{\infty} \Re(A_m^p) \right) \#_{\frac{1}{q}} \left(\sum_{n=0}^{\infty} \Re(B_n^q) \right) \right] \qquad \text{(by Theorem 4.3)}$$

$$= \frac{\pi \sec^2 \alpha}{\sin(\frac{\pi}{p})} \left[\Re \left(\sum_{m=0}^{\infty} A_m^p \right) \#_{\frac{1}{q}} \Re \left(\sum_{n=0}^{\infty} B_n^q \right) \right] \qquad \text{(by Lemma 4.6)}$$

$$\leqslant \frac{\pi \sec^2 \alpha}{\sin(\frac{\pi}{p})} \Re \left(\sum_{m=0}^{\infty} A_m^p \#_{\frac{1}{q}} \sum_{n=0}^{\infty} B_n^q \right). \qquad \text{(by Lemma 4.4)} \quad \square$$

5. More generalization

Now, we consider a general class. The following theorem can be regarded as a matrix version of Theorem 318 in [5].

THEOREM 5.1. *Let $A_m, B_n \in \mathbb{M}_r^+$ for all $m,n \geqslant 1$, and $p,q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $K(x,y)$ for $x,y > 0$ has the following properties:*

(i) $K(x,y)$ is non-negative and homogeneous of degree -1;

(ii) $\int_0^\infty K(x,1) x^{-\frac{1}{p}} \mathrm{d}x = \int_0^\infty K(1,y) y^{-\frac{1}{q}} \mathrm{d}y = k$;

(iii) $K(x,1)x^{-\frac{1}{p}}$ is a strictly decreasing function of x, and $K(1,y)y^{-\frac{1}{q}}$ of y.

If $\sum\limits_{m=1}^{\infty} A_m^p$ and $\sum\limits_{n=1}^{\infty} B_n^q$ are convergent, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} K(m,n) \left(A_m^p \#_{\frac{1}{q}} B_n^q \right) < k \left[\left(\sum_{m=1}^{\infty} A_m^p \right) \#_{\frac{1}{q}} \left(\sum_{n=1}^{\infty} B_n^q \right) \right].$$

Recall that a function $K(x,y)$ is said to be homogeneous of degree -1 if $K(tx,ty) = t^{-1}K(x,y)$ for any $t \neq 0$. Then $\int_0^\infty K(x,1)x^{-\frac{1}{p}} \mathrm{d}x$ is always equal to $\int_0^\infty K(1,y) y^{-\frac{1}{q}} \mathrm{d}y$ when it is convergent. The inequalities

$$\sum_{n=1}^{\infty} K(m,n) \left(\frac{m}{n} \right)^{\frac{1}{q}} = \sum_{n=1}^{\infty} \frac{1}{m} K\left(1, \frac{n}{m}\right) \left(\frac{n}{m} \right)^{-\frac{1}{q}} < \int_0^\infty K(1,y) y^{-\frac{1}{q}} \mathrm{d}y,$$
$$\sum_{m=1}^{\infty} K(m,n) \left(\frac{n}{m} \right)^{\frac{1}{p}} = \sum_{m=1}^{\infty} \frac{1}{n} K\left(\frac{m}{n}, 1\right) \left(\frac{m}{n} \right)^{-\frac{1}{p}} < \int_0^\infty K(x,1) x^{-\frac{1}{p}} \mathrm{d}x$$

follow from the condition (iii). Note that Theorem 4.1 is the special case of Theorem 5.1 in which $K(x,y) = \frac{1}{x+y}$. In fact, Theorem 5.1 can be verified by replacing $\frac{1}{m+n}$ with $K(m,n)$ in the proof of Theorem 4.1 and using the property (ii).

As discussed in [9, p. 4], we can apply Theorem 5.1 to some explicit cases. It yields some new matrix inequalities. We list our results here without proof.

PROPOSITION 5.2. *Let $A_m, B_n \in \mathbb{M}_r^+$ for all $m,n \geqslant 1$, and $p,q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. If $\sum\limits_{m=1}^{\infty} A_m^p$ and $\sum\limits_{n=1}^{\infty} B_n^q$ are convergent, then*

(I) taking $K(x,y)=\frac{1}{a\cdot\min\{x,y\}+b\cdot\max\{x,y\}}$ with $a,b>0$ gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m^p \#_{\frac{1}{q}} B_n^q}{a\cdot\min\{m,n\}+b\cdot\max\{m,n\}} < k\left[\left(\sum_{m=1}^{\infty}A_m^p\right)\#_{\frac{1}{q}}\left(\sum_{n=1}^{\infty}B_n^q\right)\right],$$

where $k=\int_0^{\infty}K(x,1)x^{-\frac{1}{p}}\mathrm{d}x=\int_0^{\infty}K(1,y)y^{-\frac{1}{q}}\mathrm{d}y$;

(II) taking $K(x,y)=\frac{1}{\max\{x,y\}}$ gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m^p \#_{\frac{1}{q}} B_n^q}{\max\{m,n\}} < pq\left[\left(\sum_{m=1}^{\infty}A_m^p\right)\#_{\frac{1}{q}}\left(\sum_{n=1}^{\infty}B_n^q\right)\right];$$

(III) taking $K(x,y)=\frac{\log(x)-\log(y)}{x-y}$ gives

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\log(m)-\log(n)}{m-n}(A_m^p \#_{\frac{1}{q}} B_n^q) < \left(\frac{\pi}{\sin(\frac{\pi}{q})}\right)^2\left[\left(\sum_{m=1}^{\infty}A_m^p\right)\#_{\frac{1}{q}}\left(\sum_{n=1}^{\infty}B_n^q\right)\right].$$

Similarly to Theorem 4.7, using Theorem 5.1 and Lemmas 4.4-4.6, we obtain an inequality for sector matrices.

THEOREM 5.3. *Let $A_m^p, B_n^q \in \mathbb{M}_r$ be sector matrices with $\mathbb{W}(A_m^p), \mathbb{W}(B_n^q) \subset S_\alpha$ for all $m,n \geqslant 1$, and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Suppose that $K(x,y)$ satisfies the conditions of Theorem 5.1. If $\sum\limits_{m=1}^{\infty} A_m^p$ and $\sum\limits_{n=1}^{\infty} B_n^q$ are convergent, then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(m,n)\Re\left(A_m^p \#_{\frac{1}{q}} B_n^q\right) < k(\sec^2\alpha)\Re\left[\left(\sum_{m=1}^{\infty}A_m^p\right)\#_{\frac{1}{q}}\left(\sum_{n=1}^{\infty}B_n^q\right)\right].$$

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