RUBIO DE FRANCIA EXTRAPOLATION RESULTS
FOR GRAND LEBESGUE SPACES DEFINED ON
SETS HAVING POSSIBLY INFINITE MEASURE

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Abstract. In this paper, we present Rubio de Francia type extrapolation results for certain generalized grand Lebesgue spaces defined on sets $\Omega \subseteq \mathbb{R}^n$ with $|\Omega| \leqslant \infty$. Both diagonal and off-diagonal cases have been considered. As applications to these results, boundedness of certain integral operators has been studied and also a vector valued inequality has been established.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be open and w be a weight, i.e., positive finite almost everywhere (a.e.) measurable locally integrable function defined on Ω. For $1 \leqslant p < \infty$, the weighted Lebesgue space, denoted by $L^p(\Omega, w)$, consists of all measurable functions f defined on Ω for which

$$\|f\|_{L^p(\Omega, w)} := \left(\int_\Omega |f(t)|^p w(t) dt \right)^{1/p} < \infty.$$

It is known that for $1 \leqslant p < \infty$, $L^p(\Omega, w)$ is a Banach space and for $1 < p < \infty$, the space is reflexive too.

After the celebrity extrapolation result of J. L. Rubio de Francia [24], a lot of development has taken place in this direction. A generalized form of this result can be stated as follows: If (f, g) is a pair of non-negative measurable functions such that for some $1 \leqslant p_0 < \infty$, the inequality

$$\int_{\mathbb{R}^n} f^{p_0}(x) w(x) dx \leqslant C \int_{\mathbb{R}^n} g^{p_0}(x) w(x) dx$$

holds for every $w \in A_{p_0}$ (the Muckenhoupt class of weights, see [21]), then for every $1 < p < \infty$, the inequality

$$\int_{\mathbb{R}^n} f^p(x) w(x) dx \leqslant C \int_{\mathbb{R}^n} g^p(x) w(x) dx \tag{1}$$

holds for every $w \in A_p$, where C depends on $[w]_{A_p}$ (defined in Section 3). A justification for $1 < p < \infty$ in (1) has been given in [3].

In view of its usefulness, this theory has drawn the attention of many researchers. It has been generalized to the case of A_∞ weights also (see [5]). One may refer to [4], [25], [26] for the relevant literature. In [2], a parallel theory has been established for B_p weights. We may mention a very recent paper [31] for extrapolation results in the framework of Lebesgue spaces with variable exponents.

In the above discussion, the integrals in the inequality represent L^p-norms. In [17], Kokilashvili and Meskhi have developed the Rubio de Francia extrapolation results in the frame work of grand Lebesgue spaces defined on $\Omega \subseteq \mathbb{R}^n$ with $|\Omega| < \infty$: Let $I = (0, 1)$ and $1 < p < \infty$. The grand Lebesgue space, denoted by $L^{p)}(I)$ is the space of all measurable finite a.e. functions f defined on I for which

$$\|f\|_{L^{p)}(I)} := \sup_{0 < \varepsilon < p-1} \left(\varepsilon \int_I |f(x)|^{p-\varepsilon} dx \right)^{1/(p-\varepsilon)} < \infty.$$

These spaces were defined by Iwaniec and Sbordone [11], later studied and developed, e.g., in [7], [9], [13] and the references therein. During the recent past, grand Lebesgue spaces have caught the attention of many researchers. A lot of significant work has been done in this direction. Unlike L^p-spaces, grand Lebesgue spaces are not reflexive for any $p > 1$. Moreover, the duality in these spaces does not behave the same way as in L^p-spaces. For such information and many more including the mapping properties of various integral operators in these spaces, one may refer to [12], [16], [18], [20], [27], [28] [29] and [32]. Similar results for B_p weights have also been obtained in the framework of grand Lebesgue spaces, see [13], [19].

Let us point it out that the grand Lebesgue spaces discussed above consist of functions on sets having finite measure. The case of sets having infinite measure was first studied in [27] and later generalized in [28], [32]: the weighted grand Lebesgue space $L_a^{q)}(\Omega, w)$ with suitable weight functions w and a on $\Omega \subseteq \mathbb{R}^n, |\Omega| \leqslant \infty$ is defined to be the space of all measurable functions f for which

$$\|f\|_{L_a^{q)}(\Omega, w)} := \sup_{0 < \varepsilon < q-1} \left(\varepsilon \int_\Omega |f(x)|^{q-\varepsilon} w(x) a^\varepsilon(x) dx \right)^{1/(q-\varepsilon)} < \infty. \tag{2}$$

In the present paper, we deal with a variant of the space $L_a^{q)}(\Omega, w)$ replacing the multiple ε of the integral in (2) by a more general function $\phi(\varepsilon)$ and denote the space by $L_a^{q), \phi}(\Omega, w)$. We study certain embedding properties of the space $L_a^{q), \phi}(\Omega, w)$. The main aim of the paper is to establish Rubio de Francia extrapolation results in the framework of $L_a^{q), \phi}(\Omega, w)$ spaces. We discuss both diagonal and off diagonal cases. As applications, we deduce boundedness of various integral operators.

In various inequalities we may have used the same symbol to denote the constant, however its value may be different at different places. Also, for an index τ (say), we shall denote its conjugate by τ', i.e., $\frac{1}{\tau} + \frac{1}{\tau'} = 1$.

2. Generalized grand Lebesgue spaces

Let $\Omega \subseteq \mathbb{R}^n$, $|\Omega| \leqslant \infty$, and w, a be weights defined on Ω. Let $\mathscr{M}^0(\Omega)$ denote the set of finite a.e. measurable functions defined on Ω.

DEFINITION 1. Let $1 < q < \infty$ and ϕ be a positive non-decreasing function on $(0, q-1)$ satisfying the condition $\phi(0^+) = 0$. Then, the *generalized grand Lebesgue space* denoted by $L_a^{q),\phi}(\Omega,w)$, is defined to be the space of all those $f \in \mathscr{M}^0(\Omega)$ for which

$$\|f\|_{L_a^{q),\phi}(\Omega,w)} := \sup_{0<\varepsilon<q-1} \left(\phi(\varepsilon) \int_{\Omega} |f(x)|^{q-\varepsilon} w(x) a^{\varepsilon}(x) dx \right)^{1/(q-\varepsilon)} < \infty,$$

where $wa^{\varepsilon} \in L_{loc}^1(\Omega)$ for all $\varepsilon \in (0, q-1)$. If $\phi(t) = t^{\theta}$, $\theta > 0$ then $L_a^{q),\phi}(\Omega,w)$ shall be denoted by $L_a^{q),\theta}(\Omega,w)$.

We begin with the following result, which establishes the embedding $L^q(\Omega,w) \hookrightarrow L_a^{q),\phi}(\Omega,w)$. This result was proved in [32] and [33] for ε instead of $\phi(\varepsilon)$. For completeness and convenience of the readers, we give proof with more clear explanation.

PROPOSITION 1. *Let $q > 1$. The inequality*

$$\|f\|_{L_a^{q),\phi}(\Omega,w)} \leqslant C\|f\|_{L^q(\Omega,w)} \tag{3}$$

holds for all $f \in L^q(\Omega,w)$ if and only if $a \in L^q(\Omega,w)$, where $C > 0$ is a positive constant.

Proof. Let $a \in L^q(\Omega,w)$. Then $0 < \|a\|_{L^q(\Omega,w)} < \infty$. For $0 < \varepsilon < q-1$ using Hölder's inequality with exponents $\frac{q}{q-\varepsilon}$, $\frac{q}{\varepsilon}$, we get

$$\|f\|_{L_a^{q),\phi}(\Omega,w)} \leqslant \sup_{0<\varepsilon<q-1} \phi(\varepsilon)^{1/(q-\varepsilon)} \left(\int_{\Omega} |f(x)|^q w(x) dx \right)^{\frac{1}{q}} \left(\int_{\Omega} a^q(x) w(x) dx \right)^{\frac{\varepsilon}{q(q-\varepsilon)}}$$

$$= \|f\|_{L^q(\Omega,w)} \sup_{0<\varepsilon<q-1} \left(\phi(\varepsilon)\|a\|_{L^q(\Omega,w)}^q \right)^{\frac{1}{q-\varepsilon}} \|a\|_{L^q(\Omega,w)}^{-1} \tag{4}$$

$$\leqslant C(a,\phi,q)\|f\|_{L^q(\Omega,w)},$$

so that (3) holds with $C(a,\phi,q) := \|a\|_{L^q(\Omega,w)}^{-1}(\phi(q-1)+1)\left(\|a\|_{L^q(\Omega,w)}+1\right)^q$.

Conversely, suppose the inequality (3) holds. Taking $g := |f|^q w$ in (3), we get

$$\int_{\Omega} g(x)^{\frac{q-\varepsilon}{q}} w^{\frac{\varepsilon}{q}}(x) a^{\varepsilon}(x) dx < \infty, \quad \text{for all} \quad 0 < \varepsilon < q-1, \tag{5}$$

for every $g^{\frac{q-\varepsilon}{q}} \in L^{\frac{q}{q-\varepsilon}}(\Omega)$. Therefore, by Riesz Lemma (see, e.g., [23], p. 120) and (5), it follows that $w^{\frac{\varepsilon}{q}} a^{\varepsilon} \in L^{\frac{q}{\varepsilon}}$, i.e., $a \in L^q(\Omega,w)$. $\square$

REMARK 1. In view of Proposition 1, the following embeddings hold for all $a \in L^q(\Omega, w)$ and for all $0 < \varepsilon < q - 1$:

$$L^q(\Omega, w) \hookrightarrow L_a^{q),\phi}(\Omega, w) \hookrightarrow L^{q-\varepsilon}(\Omega, wa^\varepsilon).$$

For a particular case of ϕ, namely, $\phi(t) = t^\theta$ in Proposition 1, the sharp value of the constant C can be calculated. In order to prove that, we need the following lemma:

LEMMA A [32]. *Let* $q > 1$, $s > 0$ *and* $t \in [0, q-1)$. *Then*

$$\sup_{t < x < q-1} (xs)^{\frac{1}{q-x}} = \begin{cases} (ts)^{\frac{1}{q-t}}, & \text{if } s \leqslant \frac{1}{t} e^{1 - \frac{1}{t}}; \\ e^{W(-1, -qse^{-1})/q}, & \text{if } \frac{1}{t} e^{1 - \frac{1}{t}} < s < \frac{1}{q-1} e^{1 - \frac{1}{q-1}}; \\ (q-1)s, & \text{if } s \geqslant \frac{1}{q-1} e^{1 - \frac{1}{q-1}}, \end{cases}$$

where $W(\cdot)$ *is the Lambert's function (see* [32] *for definition).*

COROLLARY 1. *Let* $q > 1$. *The inequality*

$$\|f\|_{L_a^{q),\theta}(\Omega, w)} \leqslant C\|f\|_{L^q(\Omega, w)} \tag{6}$$

holds for all $f \in L^q(\Omega, w)$ *if and only if* $a \in L^q(\Omega, w)$. *Moreover, the constant* C *involved in inequality* (6) *is sharp and is given by*

$$C \equiv C(a, \theta, q) = \begin{cases} (q-1)^\theta \|a\|_{L^q(\Omega, w)}^{q-1}, & \text{if } \|a\|_{L^q(\Omega, w)}^q \leqslant \frac{1}{q-1} e^{1 - \frac{1}{q-1}}; \\ \|a\|_{L^q(\Omega, w)}^{-1} e^{W(-1, -q\|a\|_{L^q(\Omega, w)}^q e^{-1})/q}, & \text{if } \|a\|_{L^q(\Omega, w)}^q > \frac{1}{q-1} e^{1 - \frac{1}{q-1}}. \end{cases}$$

Proof. On taking $\phi(\varepsilon) = \varepsilon^\theta$ in Proposition 1, we get the first part of the corollary. To get the sharp value of the constant, take $x = \varepsilon^\theta$ and $s = \|a\|_{L^q(\Omega, w)}^q$ in (4) and use Lemma A with $t \to 0^+$. $\square$

The following theorem indicates some of the important properties of the space $L_a^{q),\phi}(\Omega, w)$:

THEOREM 1. *Let* $1 < q < \infty$ *and* $a \in L^q(\Omega, w)$. *The following hold:*

(a) *(Completeness) The generalized grand Lebesgue space* $L_a^{q),\phi}(\Omega, w)$ *is a Banach space.*

(b) *(Lattice property) If* $|f| \leqslant |g|$ *a.e. on* Ω, *then* $\|f\|_{L_a^{q),\phi}(\Omega, w)} \leqslant \|g\|_{L_a^{q),\phi}(\Omega, w)}$.

(c) *(Fatou property) If* $0 \leqslant f_n \uparrow f$ *a.e. in* Ω, *then* $\|f_n\|_{L_a^{q),\phi}(\Omega, w)} \uparrow \|f\|_{L_a^{q),\phi}(\Omega, w)}$.

(d) *If* $E \subset \Omega, |E| < \infty$ *then*

$$\|\chi_E\|_{L_a^{q),\phi}(\Omega, w)} \leqslant \|\chi_E\|_{L_q(\Omega, w)} \|a\|_{L_q(\Omega, w)}^{-1} \|a\|_{L_a^{q),\phi}(\Omega, w)}. \tag{7}$$

(e) *For all $f \in L_a^{q),\phi}(\Omega,w)$ and $0 < \sigma < q-1$, the following estimate holds:*

$$\|f\|_{L_a^{q),\phi}(\Omega,w)} \leqslant C(a,\sigma,\phi,q) \sup_{0<\varepsilon\leqslant\sigma} \left(\phi(\varepsilon) \int_\Omega |f(x)|^{q-\varepsilon} w(x) a^\varepsilon(x) dx \right)^{1/(q-\varepsilon)}.$$

(8)

Proof. We shall only prove (d) and (e). The remaining are easy to prove.

(d) For $0 < \varepsilon < q-1$, using Hölder's inequality with the exponents $\frac{q}{q-\varepsilon}$ and $\frac{q}{\varepsilon}$, we get

$$\|\chi_E\|_{L_a^{q),\phi}(\Omega,w)} = \sup_{0<\varepsilon<q-1} \left(\phi(\varepsilon) \int_\Omega |\chi_E(x)|^{q-\varepsilon} w(x) a^\varepsilon(x) dx \right)^{\frac{1}{q-\varepsilon}}$$

$$\leqslant \sup_{0<\varepsilon<q-1} \phi(\varepsilon)^{\frac{1}{q-\varepsilon}} \left(\int_\Omega |\chi_E(x)|^q w(x) dx \right)^{\frac{1}{q}} \left(\int_\Omega a^q(x) w(x) dx \right)^{\frac{\varepsilon}{q(q-\varepsilon)}}$$

(9)

$$= \|\chi_E\|_{L^q(\Omega,w)} \sup_{0<\varepsilon<q-1} \phi(\varepsilon)^{\frac{1}{q-\varepsilon}} \left(\int_\Omega a^{q-\varepsilon}(x) w(x) a^\varepsilon(x) dx \right)^{\frac{1}{q-\varepsilon}} \|a\|_{L^q(\Omega,w)}^{-1}$$

$$= \|\chi_E\|_{L^q(\Omega,w)} \|a\|_{L^q(\Omega,w)}^{-1} \|a\|_{L_a^{q),\phi}(\Omega,w)}.$$

(e) Let $f \in L_a^{q),\phi}(\Omega,w)$ and $0 < \sigma < q-1$. For $\sigma < \varepsilon < q-1$, on applying Hölder's inequality with conjugate indices $\lambda := \frac{q-\sigma}{q-\varepsilon}$, $\lambda' := \frac{q-\sigma}{\varepsilon-\sigma}$, we have

$$\|f\|_{L^{q-\varepsilon}(\Omega,wa^\varepsilon)} \leqslant \left(\int_\Omega |f|^{q-\sigma}(x) w(x) a^\sigma(x) dx \right)^{\frac{1}{q-\sigma}} \left(\int_\Omega a^{(\varepsilon-\frac{\sigma}{\lambda})\lambda'}(x) w(x) dx \right)^{\frac{1}{(q-\varepsilon)\lambda'}}$$

$$= \|f\|_{L^{q-\sigma}(\Omega,wa^\sigma)} \|a\|_{L^q(\Omega,w)}^{\frac{q(\varepsilon-\sigma)}{(q-\sigma)(q-\varepsilon)}}.$$

Also, for $\sigma < \varepsilon < q-1$

$$0 < \frac{\varepsilon-\sigma}{(q-\sigma)(q-\varepsilon)} < \frac{q-1-\sigma}{q-\sigma}.$$

Using the above estimates, we have

$$\|f\|_{L_a^{q),\phi}(\Omega,w)} =$$

$$= \max \left\{ \sup_{0<\varepsilon\leqslant\sigma} \phi(\varepsilon)^{1/(q-\varepsilon)} \|f\|_{L^{q-\varepsilon}(\Omega,wa^\varepsilon)}, \quad \sup_{\sigma<\varepsilon<q-1} \phi(\varepsilon)^{1/(q-\varepsilon)} \|f\|_{L^{q-\varepsilon}(\Omega,wa^\varepsilon)} \right\}$$

$$\leqslant \max \left\{ 1, \sup_{\sigma<\varepsilon<q-1} \phi(\varepsilon)^{1/(q-\varepsilon)} \phi(\sigma)^{-\frac{1}{q-\sigma}} \|a\|_{L^q(\Omega,w)}^{\frac{q(\varepsilon-\sigma)}{(q-\sigma)(q-\varepsilon)}} \right\} \times$$

$$\times \sup_{0<\varepsilon\leqslant\sigma} \phi(\varepsilon)^{1/(q-\varepsilon)} \|f\|_{L^{q-\varepsilon}(\Omega,wa^\varepsilon)}$$

$$\leqslant \max\left\{1,(\phi(q-1)+1)\phi(\sigma)^{-\frac{1}{q-\sigma}}\left(\|a\|_{L^q(\Omega,w)}^q+1\right)^{\frac{q-1-\sigma}{q-\sigma}}\right\}\times$$

$$\times \sup_{0<\varepsilon\leqslant\sigma}\phi(\varepsilon)^{1/(q-\varepsilon)}\|f\|_{L^{q-\varepsilon}(\Omega,wa^\varepsilon)}.$$

from which it follows that the estimate (8) holds with

$$C(a,\sigma,\phi,q) = \max\left\{1,(\phi(q-1)+1)\phi(\sigma)^{-\frac{1}{q-\sigma}}\left(\|a\|_{L^q(\Omega,w)}^q+1\right)^{\frac{q-1-\sigma}{q-\sigma}}\right\} \quad (10)$$

and we are done. $\square$

REMARK 2. In view of Remark 1 and (7), it follows that $\chi_E \in L_a^{q),\phi}(\Omega,w)$.

REMARK 3. For $\phi(t)=t^\theta$, the constant $C(a,\sigma,\phi,q)$ can be obtained in a more precise form. Indeed, on using Lemma A for $x=\varepsilon^\theta$, $s=\|a\|_{L^q(\Omega,w)}^q$ and letting $t\to 0^+$, we have

$$\sup_{\sigma<\varepsilon<q-1}\varepsilon^{\frac{\theta}{q-\varepsilon}}\|f\|_{L^{q-\varepsilon}(\Omega,wa^\varepsilon)}$$

$$\leqslant \|f\|_{L^{q-\sigma}(\Omega,wa^\sigma)}\|a\|_{L^q(\Omega,w)}^{\frac{-q}{q-\sigma}}\sup_{\sigma<\varepsilon<q-1}(\varepsilon^\theta\|a\|_{L^q(\Omega,w)}^q)^{1/(q-\varepsilon)}$$

$$= C(a,\sigma,\theta,q)\sigma^{\frac{\theta}{q-\sigma}}\|f\|_{L^{q-\sigma}(\Omega,wa^\sigma)},$$

where the constant $C(a,\sigma,\theta,q)$ is

$$\begin{cases}\|a\|_{L^q(\Omega,w)}^{\frac{-q}{q-\sigma}}\left(\sigma^\theta\|a\|_{L^q(\Omega,w)}^q\right)^{\frac{1}{q-\sigma}}\sigma^{\frac{-\theta}{q-\sigma}}, & \text{if } \|a\|_{L^q(\Omega,w)}^q\leqslant\frac{1}{\sigma}e^{1-\frac{1}{\sigma}};\\[2mm]\|a\|_{L^q(\Omega,w)}^{\frac{-q}{q-\sigma}}e^{W(-1,-q\|a\|_{L^q(\Omega,w)}^qe^{-1})/q}\sigma^{\frac{-\theta}{q-\sigma}}, & \text{if } \frac{1}{\sigma}e^{1-\frac{1}{\sigma}}<\|a\|_{L^q(\Omega,w)}^q<\frac{1}{q-1}e^{1-\frac{1}{q-1}};\\[2mm]\|a\|_{L^q(\Omega,w)}^{\frac{-q}{q-\sigma}}(q-1)\sigma^{\frac{-\theta}{q-\sigma}}, & \text{if } \|a\|_{L^q(\Omega,w)}^q\geqslant\frac{1}{q-1}e^{1-\frac{1}{q-1}},\end{cases}$$

where W is the Lambert's function. Consequently, (9) gives that

$$\|f\|_{L_a^{q),\theta}(\Omega,w)}\leqslant C_1(a,\sigma,\theta,q)\sup_{0<\varepsilon\leqslant\sigma}\varepsilon^{\theta/(q-\varepsilon)}\left(\int_\Omega|f(x)|^{q-\varepsilon}w(x)a^\varepsilon(x)dx\right)^{1/(q-\varepsilon)},$$

and

$$C_1(a,\sigma,\theta,q)=\max\{1,C(a,\sigma,\theta,q)\}.$$

3. Extrapolation theorem for the diagonal case

In this section, our aim is to prove extrapolation theorem for the diagonal case in the framework of generalized grand Lebesgue space $L_a^{q),\phi}(\Omega, w)$, when $|\Omega| \leqslant \infty$.

Let us recall A_r, the Muckenhoupt class of weights. For $1 < r < \infty$, we say that a weight $w \in A_r$ if

$$[w]_{A_r} := \sup_{B \subseteq \Omega} \left(\frac{1}{|B|} \int_B w(t)dt \right) \left(\frac{1}{|B|} \int_B w^{1-r'}(t)dt \right)^{r-1} < \infty,$$

where $|B|$ being the Lebesgue measure of the ball B. The term $[w]_{A_r}$ is usually known as the A_r constant of the weight w. In the sequel, we shall assume that the weight class A_r satisfies the following properties when considered on the general domain Ω. This seems reasonable to assume these properties since they are known to hold when $\Omega = \mathbb{R}^n$.

(i) If $1 \leqslant s \leqslant r < \infty$, then $A_s \subseteq A_r$ and $[w]_{A_r} \leqslant [w]_{A_s}$.

(ii) If $w \in A_r$, then there exists $0 < \varepsilon < r-1$ such that $w \in A_{r-\varepsilon}$.

(iii) If $w \in A_r$ and $0 \leqslant \alpha \leqslant 1$, then $w^\alpha \in A_r$ and $[w^\alpha]_{A_r} \leqslant [w]_{A_r}^\alpha$.

(iv) If $w_1, w_2 \in A_r$, then $w_1^t w_2^{1-t} \in A_r$ for all $0 \leqslant t \leqslant 1$ and $[w_1^t w_2^{1-t}]_{A_r} \leqslant [w_1]_{A_r}^t [w_2]_{A_r}^{1-t}$.

(v) If $w \in A_r$, then there exists $\xi > 0$ such that $w^{1+\xi} \in A_r$.

(vi) $[w]_{A_r} \geqslant 1$.

We prove the following lemma which is a reformulation of Lemma 5 in [32] with some modifications and more clear explanation:

LEMMA 1. *Let* $1 < r < \infty$, $w \in A_r$ *and* $a^\delta \in A_r$ *for some* $\delta > 0$. *Then there exists* $\sigma_\delta > 0$ *such that* $\sigma_\delta < \delta$ *and* $wa^s \in A_{r-s}$ *for all* s, $0 < s \leqslant \sigma_\delta$. *Moreover,*

$$[wa^s]_{A_{r-s}} \leqslant [w]_{A_{r-\sigma_\delta}} [wa^{\sigma_\delta}]_{A_{r-\sigma_\delta}}.$$

Proof. Let $w \in A_r$ and $a^\delta \in A_r$ for some $\delta > 0$, then there exist $\sigma_0 > 0$, $\beta > 1$ such that $w \in A_{r-\sigma_0}$ and $w^\beta \in A_r$. Further, there exist $\sigma_1 > 0$, $\sigma_2 > 0$ such that $w^\beta \in A_{r-\sigma_1}$ and $a^\delta \in A_{r-\sigma_2}$. For some $s > 0$, write

$$wa^s = (w^\beta)^{\frac{1}{\beta}} (a^{s\beta'})^{\frac{1}{\beta'}}. \tag{11}$$

Now, $a^{s\beta'} = a^{\left(\frac{s\beta'}{\delta}\right)\delta} \in A_{r-\sigma_2}$ whenever $s \leqslant \frac{\delta}{\beta'}$. Choose $\sigma_\delta = \min\{\sigma_0, \sigma_1, \sigma_2, \frac{\delta}{\beta'}\}$. Then in view of (11), $wa^s \in A_{r-\sigma_\delta}$ whenever $0 < s \leqslant \sigma_\delta$. Hence, $wa^s \in A_{r-s}$ for

all s, $0 < s \leqslant \sigma_\delta$. Also, we have

$$\begin{aligned}
[wa^s]_{A_{r-s}} &\leqslant [w]_{A_{r-s}}^{1-\frac{s}{\sigma_\delta}} [wa^{\sigma_\delta}]_{A_{r-s}}^{\frac{s}{\sigma_\delta}} \\
&\leqslant [w]_{A_{r-\sigma_\delta}}^{1-\frac{s}{\sigma_\delta}} [wa^{\sigma_\delta}]_{A_{r-\sigma_\delta}}^{\frac{s}{\sigma_\delta}} \\
&\leqslant [w]_{A_{r-\sigma_\delta}} [wa^{\sigma_\delta}]_{A_{r-\sigma_\delta}}
\end{aligned}$$

and we are done. $\square$

The A_r class of weights is very important. It characterizes the boundedness of the maximal operator

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where B is ball in Ω, in Lebesgue spaces [21], and also in grand Lebesgue spaces (for $|\Omega| < \infty$) [8]. This also characterizes the Hilbert transform for $\Omega = \mathbb{R}$ in Lebesgue spaces [10] as well as on grand Lebesgue spaces [16], on $\Omega = J$, where J is some bounded intreval in $\mathbb{R}$. In [14] and [15] also, authors have characterized boundedness of strong maximal fractions, multilinear maximal fractions and singular integrals in weighted grand Lebesgue spaces with Muckenhoupt class of weights. Let us point it out that in [1], the so called Bukley's estimate has been given, we will use the estimate for the $L^r(\Omega, w)$ norm, which reads

$$\|M\|_{L^r(\Omega,w) \to L^r(\Omega,w)} \leqslant \overline{c} r'[w]_{A_r}^{1/(r-1)}, \tag{12}$$

where $\overline{c}$ is a constant.

In [6], an extrapolation theorem of Rubio de Francia type has been proved in the framework of Lebesgue spaces, which provides sharp bounds in terms of the A_r constant of a weight.

THEOREM B [6]. *Let ψ be a non-negative non-decreasing function on $(0,\infty)$, (f,g) be a pair of non-negative measurable functions defined on $\Omega \subseteq \mathbb{R}^n$. Let $1 \leqslant r_0 < \infty$ be fixed, and for every $w \in A_{r_0}$*

$$\left(\int_\Omega g^{r_0}(t) w(t) dt \right)^{\frac{1}{r_0}} \leqslant C_0 \, \psi([w]_{A_{r_0}}) \left(\int_\Omega f^{r_0}(t) w(t) dt \right)^{\frac{1}{r_0}}, \tag{13}$$

where C_0 does not depend on w. Then for all r, $1 < r < \infty$ and for every $w \in A_r$, we have

$$\left(\int_\Omega g^r(t) w(t) dt \right)^{\frac{1}{r}} \leqslant C_0 \, K(w) \left(\int_\Omega f^r(t) w(t) dt \right)^{\frac{1}{r}},$$

where

$$K(w) = \begin{cases} \psi \left([w]_{A_r} (2\|M\|_{L^r(\Omega,w)})^{r_0 - r} \right), & \text{if } r < r_0; \\ \psi \left([w]_{A_r}^{\frac{r_0-1}{r-1}} (2\|M\|_{L^{r'}(\Omega,w^{1-r'})})^{\frac{r-r_0}{r-1}} \right), & \text{if } r > r_0 \end{cases} \tag{14}$$

and C_0 is constant as in (13).

REMARK 4. In view of (12) and (14), the above constant $K(w)$ becomes

$$
K(w) = \begin{cases} \psi\left([w]_{A_r}^{\frac{r_0-1}{r-1}}(2r'\overline{c})^{r_0-r}\right), & \text{if } r < r_0; \\ \psi\left([w]_{A_r}(2r\overline{c})^{\frac{r-r_0}{r-1}}\right), & \text{if } r > r_0 \end{cases}
\tag{15}
$$

Now, we give our first main theorem.

THEOREM 2. *Let ψ be a non-negative non-decreasing function on $(0,\infty)$ and (f,g) be a pair of non-negative measurable functions defined on $\Omega \subset \mathbb{R}^n$, $|\Omega| \leqslant \infty$. Let $1 < r_0 < \infty$ be fixed and for every $w \in A_{r_0}$*

$$
\left(\int_\Omega g^{r_0}(t)w(t)dt\right)^{\frac{1}{r_0}} \leqslant C_0\,\psi([w]_{A_{r_0}})\left(\int_\Omega f^{r_0}(t)w(t)dt\right)^{\frac{1}{r_0}},
$$

where C_0 does not depend on w. Then for all r, $1 < r < \infty$ and for all $w \in A_r$, the following inequality holds

$$
\|g\|_{L_a^{r),\phi}(\Omega,w)} \leqslant C(w,a,\phi,r,\delta)\|f\|_{L_a^{r),\phi}(\Omega,w)},
$$

where $a \in L^r(\Omega, w)$ and $a^\delta \in A_r$ for some $\delta > 0$.

Proof. Let $1 < r < \infty$ and $w \in A_r$. Then there exists $\sigma_1 > 0$ such that $w \in A_{r-\sigma_1}$. By Lemma 1, there exists σ_2, $0 < \sigma_2 < \delta$ such that $wa^\varepsilon \in A_{r-\varepsilon}$ for all $0 < \varepsilon \leqslant \sigma_2$. Take $\sigma_\delta = \min\{\sigma_1,\sigma_2\}$, therefore, by Theorem B and (15), we have

$$
\left(\int_\Omega g^{r-\varepsilon}(t)w(t)a^\varepsilon(t)dt\right)^{\frac{1}{r-\varepsilon}} \leqslant C_0\,K(w,a,\varepsilon)\left(\int_\Omega f^{r-\varepsilon}(t)w(t)a^\varepsilon(t)dt\right)^{\frac{1}{r-\varepsilon}}
\tag{16}
$$

for all $0 < \varepsilon \leqslant \sigma_\delta$, where

$$
K(w,a,\varepsilon) = \begin{cases} \psi\left([wa^\varepsilon]_{A_{r-\varepsilon}}^{\frac{r_0-1}{r-\varepsilon-1}}(2(r-\varepsilon)'\overline{c})^{r_0-r+\varepsilon}\right), & \text{if } r-\varepsilon < r_0; \\ \psi\left([wa^\varepsilon]_{A_{r-\varepsilon}}(2(r-\varepsilon)\overline{c})^{\frac{r-\varepsilon-r_0}{r-\varepsilon-1}}\right), & \text{if } r-\varepsilon > r_0. \end{cases}
$$

Now, on using Theorem 1 (e) and (16), we have

$$
\begin{aligned}
\|g\|_{L_a^{r),\phi}(\Omega,w)} &\leqslant C(a,\sigma_\delta,\phi,r) \sup_{0<\varepsilon\leqslant\sigma} \phi(\varepsilon)^{\frac{1}{r-\varepsilon}}\|g\|_{L^{r-\varepsilon}(\Omega,wa^\varepsilon)} \\
&\leqslant C(a,\sigma_\delta,\phi,r) \sup_{0<\varepsilon\leqslant\sigma} C_0 K(w,a,\varepsilon)\phi(\varepsilon)^{\frac{1}{r-\varepsilon}}\|f\|_{L^{r-\varepsilon}(\Omega,wa^\varepsilon)} \\
&\leqslant C(a,\sigma_\delta,\phi,r) \sup_{0<\varepsilon\leqslant\sigma} C_0 K(w,a,\varepsilon)\|f\|_{L_a^{r),\phi}(\Omega,w)},
\end{aligned}
\tag{17}
$$

where $C(a, \sigma_\delta, \phi, r)$ is as in (10). Also, by Lemma 1, we have

$$\sup_{0 < \varepsilon \leqslant \sigma} K(w, a, \varepsilon) \leqslant \begin{cases} \psi\left([w]_{A_{r-\sigma}}^{\frac{r_0-1}{r-\sigma-1}}[wa^\sigma]_{A_{r-\sigma}}^{\frac{r_0-1}{r-\sigma-1}}(2r'\overline{c})^{r_0-r+\sigma}\right), & \text{if } r-\sigma < r_0; \\ \psi\left([w]_{A_{r-\sigma}}[wa^\sigma]_{A_{r-\sigma}}(2r\overline{c})^{\frac{r-r_0}{r-\sigma-1}}\right), & \text{if } r-\sigma > r_0 \end{cases}$$
$$\equiv K_1(w, a, \sigma, r).$$

Take

$$C(w, a, \phi, r, \delta) = C_0 \inf_{0 < \sigma < r-1} C(a, \sigma_\delta, \phi, r) K_1(w, a, \sigma, r),$$

and using it in (17), we get

$$\|g\|_{L_a^{r),\phi}(\Omega, w)} \leqslant C(w, a, \phi, r, \delta) \|f\|_{L_a^{r),\phi}(\Omega, w)}$$

and the result is proved. $\square$

REMARK 5. Theorem 2 extends a result of Samko and Umarkhadzhiev [27], which can be obtained by taking $a(x) = w^\beta(x)$, $\beta \neq 0$ and $\phi(t) = t^\theta$, $\theta > 0$ such that $\int_\Omega w^{1+\beta r} < \infty$.

4. Extrapolation theorem for the off-diagonal case

The main result in this section is the off diagonal version of Theorem 2. To deal with this, we first give $\mathscr{A}_{p,q}$, the Muckenhoupt-Wheeden class of weights [22] and some of its important properties.

DEFINITION 2. Let $1 < p, q < \infty$. We say that a weight $\rho \in \mathscr{A}_{p,q}$ if

$$[\rho]_{\mathscr{A}_{p,q}} := \sup_{B \subseteq \Omega} \left(\frac{1}{|B|} \int_B \rho^q(x) dx\right) \left(\frac{1}{|B|} \int_B \rho^{-p'}(x) dx\right)^{\frac{q}{p'}} < \infty.$$

For $p = q$, we set $\mathscr{A}_p := \mathscr{A}_{p,q}$.

As in the diagonal case, here too, we assume that the weight $\mathscr{A}_{p,q}$ satisfies the following properties when considered on general Ω since they hold when $\Omega = \mathbb{R}^n$:

(i) $[\rho]_{\mathscr{A}_p} = [\rho^p]_{A_p}$.

(ii) $\rho \in \mathscr{A}_{p,q} \Leftrightarrow \rho^q \in A_{1+\frac{q}{p'}}$ and $[\rho]_{\mathscr{A}_{p,q}} = [\rho^q]_{A_{1+\frac{q}{p'}}}$.

(iii) If $1 < p < r < \infty$ and $1 < q < s < \infty$ then $\mathscr{A}_{p,q} \subset \mathscr{A}_{r,s}$. Moreover, $[\rho]_{\mathscr{A}_{r,s}} \leqslant [\rho]_{\mathscr{A}_{p,q}}$.

(iv) $\rho \in \mathscr{A}_{p,q} \Leftrightarrow \rho^{-p'} \in A_{1+\frac{p'}{q}}$ and $[\rho]_{\mathscr{A}_{p,q}} = [\rho^{-p'}]_{A_{1+\frac{p'}{q}}}^{\frac{q}{p'}}$.

(v) If $w \in \mathscr{A}_{p,q}$, $1 < p < q < \infty$, then there exists $0 < \varepsilon_0 < q-1$, $0 < \eta_0 < p-1$ such that $w \in \mathscr{A}_{p-\eta_0,\, q-\varepsilon_0}$ with

$$\frac{1}{p-\eta_0} - \frac{1}{q-\varepsilon_0} = \frac{1}{p} - \frac{1}{q}$$

Using the above properties (iii) and (v), the following can be proved:

LEMMA 2. *Let* $1 < p < q < \infty$. *If* $w \in \mathscr{A}_{p,q}$, *then there exists* $0 < \varepsilon_0 < q-1$, $0 < \eta_0 < p-1$ *with*

$$\frac{1}{p-\eta_0} - \frac{1}{q-\varepsilon_0} = \frac{1}{p} - \frac{1}{q} =: \alpha$$

such that $w \in \mathscr{A}_{p-\eta,\, q-\varepsilon}$ *for all* $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \eta \leqslant \eta_0$ *with* $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$.

We prove the following crucial lemma:

LEMMA 3. *Let* $1 < p < q < \infty$. *If* $w \in \mathscr{A}_{p,q}$ *and* $a^\delta \in \mathscr{A}_{p,q}$ *for some* $\delta > 0$, *then there exist* ε_0 *and* η_0 *with* $0 < \varepsilon_0 < \delta$, $0 < \eta_0 < \delta$ *with*

$$\frac{1}{p-\eta_0} - \frac{1}{q-\varepsilon_0} = \frac{1}{p} - \frac{1}{q} =: \alpha$$

such that $\left(wa^{\frac{p}{q}\varepsilon} \right)^{\frac{1}{q-\varepsilon}} \in \mathscr{A}_{p-\eta,\, q-\varepsilon}$ *for all* $0 < \varepsilon < \varepsilon_0$, $0 < \eta < \eta_0$ *with* $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$. *Moreover,*

$$\left[\left(wa^{\frac{p}{q}\varepsilon} \right)^{\frac{1}{q-\varepsilon}} \right]_{\mathscr{A}_{p-\eta,\, q-\varepsilon}} \leqslant [w]_{A_{\frac{q-\varepsilon_0}{(p-\eta_0)'}+1}} \left[wa^{\frac{p}{q}\varepsilon_0} \right]_{A_{\frac{q-\varepsilon_0}{(p-\eta_0)'}+1}}. \tag{18}$$

Proof. Let $w \in \mathscr{A}_{p,q}$ and $a^\delta \in \mathscr{A}_{p,q}$ for some $\delta > 0$,. Then $w^q \in A_{1+\frac{q}{p'}}$ and $a^{\delta q} \in A_{1+\frac{q}{p'}}$. Therefore, for some $t > 1$, $(w^q)^t \in A_{1+\frac{q}{p'}}$ and $(a^{\delta q})^{\frac{p}{q}} \in A_{1+\frac{q}{p'}}$. Then by using property (v) given above, there exists $\varepsilon_1 > 0$, $\eta_1 > 0$ and $\varepsilon_2 > 0$, $\eta_2 > 0$ such that $w^{qt} \in A_{\frac{q-\varepsilon_1}{(p-\eta_1)'}+1}$ and $(a^{\frac{p}{q}})^{\delta q} \in A_{\frac{q-\varepsilon_2}{(p-\eta_2)'}+1}$ with

$$\frac{1}{p-\eta_1} - \frac{1}{q-\varepsilon_1} = \alpha = \frac{1}{p-\eta_2} - \frac{1}{q-\varepsilon_2}.$$

For given $\varepsilon > 0$, we can write

$$w^q (a^{\frac{p}{q}})^{\varepsilon q} = (w^{qt})^{\frac{1}{t}} \left((a^{\frac{p}{q}})^{t'\varepsilon q} \right)^{\frac{1}{t'}}. \tag{19}$$

Now, $(a^{\frac{p}{q}})^{t'\varepsilon q} \in A_{\frac{q-\varepsilon_2}{(p-\eta_2)'}+1}$ if $\varepsilon \leqslant \frac{\delta}{t'}$. Take $\varepsilon_0 = \min\left\{ \varepsilon_1, \varepsilon_2, \frac{\delta}{t'} \right\}$ and choose $\eta_0 = p - \frac{q-\varepsilon_0}{1+\alpha(q-\varepsilon_0)}$. Then clearly, $\eta_0 \leqslant \eta_1$, $\eta_0 \leqslant \eta_2$ and by (19), we have

$$\left(wa^{\frac{p\varepsilon}{q}} \right)^q \in A_{1+\frac{q-\varepsilon}{(p-\eta)'}}.$$

So, as $q > 1$, we have

$$wa^{\frac{p\varepsilon}{q}} \in A_{1+\frac{q-\varepsilon}{(p-\eta)'}}$$

and consequently

$$\left(wa^{\frac{p\varepsilon}{q}}\right)^{\frac{1}{q-\varepsilon}} \in \mathscr{A}_{p-\eta,\, q-\varepsilon}$$

for all $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \eta < \eta_0$ with $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$. Also, we have

$$
\left[\left(wa^{\frac{p\varepsilon}{q}}\right)^{\frac{1}{q-\varepsilon}}\right]_{\mathscr{A}_{p-\eta,\, q-\varepsilon}}
= \left[wa^{\frac{p\varepsilon}{q}}\right]_{A_{1+\frac{q-\varepsilon}{(p-\eta)'}}}
$$

$$
\leqslant [w]_{A_{1+\frac{q-\varepsilon}{(p-\eta)'}}}^{1-\frac{\varepsilon}{\varepsilon_0}} \, [wa^{\frac{p\varepsilon_0}{q}}]_{A_{1+\frac{q-\varepsilon}{(p-\eta)'}}}^{\frac{\varepsilon}{\varepsilon_0}}
$$

$$
\leqslant [w]_{A_{1+\frac{q-\varepsilon}{(p-\eta)'}}} \, [wa^{\frac{p\varepsilon_0}{q}}]_{A_{1+\frac{q-\varepsilon}{(p-\eta)'}}}
$$

$$
\leqslant [w]_{A_{1+\frac{q-\varepsilon_0}{(p-\eta_0)'}}} \, [wa^{\frac{p\varepsilon_0}{q}}]_{A_{1+\frac{q-\varepsilon_0}{(p-\eta_0)'}}}
$$

for all $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \eta < \eta_0$ with $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$. $\square$

We mention below the off diagonal case of Theorem B.

THEOREM C [6]. *Let ψ be a non-negative non-decreasing function on $(0,\infty)$, (f,g) be a pair of non-negative measurable functions defined on $\Omega \subseteq \mathbb{R}^n$. Let $1 \leqslant p_0 < \infty$ and $0 < q_0 < \infty$ be fixed and for every $w \in \mathscr{A}_{p_0,q_0}$*

$$\left(\int_\Omega (gw)^{q_0}(t)dt\right)^{\frac{1}{q_0}} \leqslant c\, \psi([w]_{\mathscr{A}_{p_0,q_0}})\left(\int_\Omega (fw)^{p_0}(t)dt\right)^{\frac{1}{p_0}}, \qquad (20)$$

where c does not depend on w. Then for all $1 < p < \infty$, $0 < q < \infty$ such that

$$\frac{1}{q_0} - \frac{1}{p_0} = \frac{1}{q} - \frac{1}{p}$$

and for every $w \in \mathscr{A}_{p,q}$, we have

$$\left(\int_\Omega (gw)^{q}(t)dt\right)^{\frac{1}{q}} \leqslant c\, K(w,p,q)\left(\int_\Omega (fw)^{p}(t)dt\right)^{\frac{1}{p}},$$

where

$$
K(w,p,q) =
\begin{cases}
\psi\left([w]_{\mathscr{A}_{p,q}}\left(2\|M\|_{L^{\gamma q}(\Omega, w^q)}\right)^{\gamma(q-q_0)}\right), & \text{if } q < q_0; \\[2ex]
\psi\left([w]_{\mathscr{A}_{p,q}}^{\frac{\gamma q_0 - 1}{\gamma q - 1}}\left(2\|M\|_{L^{\gamma p'}(\Omega, w^{-p'})}\right)^{\frac{\gamma(q-q_0)}{\gamma p - 1}}\right), & \text{if } q > q_0,
\end{cases}
\qquad (21)
$$

$\gamma = \frac{1}{q_0} + \frac{1}{p_0'}$, *and c is the constant same as in* (20).

REMARK 6. The above constant $K(w,p,q)$ given in (21) can be written in terms of the Bukley's estimate.

$$
K(w,p,q) = \begin{cases} \psi\left([w]_{\mathscr{A}_{p,q}}^{\frac{q_0 p'}{q p_0'}}\left(2\overline{c}\left(1+\frac{p'}{q}\right)\right)^{\gamma(q-q_0)}\right), & \text{if } q < q_0; \\ \psi\left([w]_{\mathscr{A}_{p,q}}\left(2\overline{c}\left(1+\frac{q}{p'}\right)\right)^{\frac{\gamma(q-q_0)p'}{q}}\right), & \text{if } q > q_0, \end{cases}
\tag{22}
$$

Now, we are ready to prove our main theorem of this section.

THEOREM 3. *Let ψ be a non-negative non-decreasing function on $(0,\infty)$, (f,g) be a pair of non-negative measurable functions defined on $\Omega \subseteq \mathbb{R}^n, |\Omega| \leqslant \infty$. Let $1 < p_0 < \infty$ and $1 < q_0 < \infty$ be fixed and for every $w \in \mathscr{A}_{p_0,q_0}$*

$$
\left(\int_\Omega (gw)^{q_0}(t)dt\right)^{\frac{1}{q_0}} \leqslant c\ \psi([w]_{\mathscr{A}_{p_0,q_0}})\left(\int_\Omega (fw)^{p_0}(t)dt\right)^{\frac{1}{p_0}},
\tag{23}
$$

where c does not depend on w. Then for all $1 < p < q < \infty$ such that

$$
\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q} =: \alpha
$$

and arbitrary $w \in \mathscr{A}_{p,q}$, the following inequality holds:

$$
\sup_{0<\varepsilon<q-1}\left(\varepsilon^\theta \int_\Omega |g|^{q-\varepsilon}(x)w(x)a^{\frac{p\varepsilon}{q}}(x)dx\right)^{\frac{1}{q-\varepsilon}} \leqslant C_2(a,w,\theta,\delta,p,q)\times
$$

$$
\times \sup_{0<\eta<p-1}\left(\phi^\theta(\eta)\int_\Omega |f(x)w^{-\alpha}(x)|^{p-\eta}w(x)a^\eta(x)dx\right)^{\frac{1}{p-\eta}},
$$

where $\phi(\eta) = \left(q + \frac{\eta-p}{1+\alpha(\eta-p)}\right)^{1-\alpha(p-\eta)}$, $a \in L^p(\Omega,w)$ and $a^\delta \in \mathscr{A}_{p,q}$ for some $\delta > 0$.

Proof. Let $1 < p < q < \infty$ be such that

$$
\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q} = \alpha
$$

and $a \in L^p(\Omega,w)$, $a^\delta \in \mathscr{A}_{p,q}$. Suppose that $w \in \mathscr{A}_{p,q}$ i.e., $w^q \in A_{1+\frac{q}{p'}}$ and $a^{\delta q} \in A_{1+\frac{q}{p'}}$. By property (v), there exist $0 < \varepsilon_1 < q-1$ and $0 < \eta_1 < p-1$ with

$$
\frac{1}{p-\eta_1} - \frac{1}{q-\varepsilon_1} = \alpha \text{ and } w^q \in A_{1+\frac{q-\varepsilon_1}{(p-\eta_1)'}}.
\tag{24}
$$

By Lemma 3, there exist $0 < \varepsilon_2 < \delta$ and $0 < \eta_2 < \delta$ with

$$
\frac{1}{p-\eta_2} - \frac{1}{q-\varepsilon_2} = \alpha \text{ and } wa^{\frac{p\varepsilon}{q}} \in A_{1+\frac{q-\varepsilon}{(p-\eta)'}}
\tag{25}
$$

for all $0 < \varepsilon \leqslant \varepsilon_2$, $0 < \eta \leqslant \eta_2$ with $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$. Take $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and choose $\eta_0 > 0$ such that $\frac{1}{p-\eta_0} - \frac{1}{q-\varepsilon_0} = \alpha$. Clearly, $\eta_0 = \min\{\eta_1, \eta_2\}$. From (24) and (25), we have

$$w^q \in A_{1 + \frac{q-\varepsilon_0}{(p-\eta_0)'}} \quad \text{and} \quad wa^{\frac{p\varepsilon}{q}} \in A_{1 + \frac{q-\varepsilon}{(p-\eta)'}}$$

for all $0 < \varepsilon \leqslant \varepsilon_0$, $0 < \eta \leqslant \eta_0$ with $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$. Now, for $\varepsilon_0 > 0$, using Theorem 1(e), we get

$$\sup_{0 < \varepsilon < q-1} \left(\varepsilon^\theta \int_\Omega |g|^{q-\varepsilon}(x) w(x) a^{\frac{p\varepsilon}{q}}(x) dx \right)^{\frac{1}{q-\varepsilon}}$$
$$\leqslant C_1(a, \varepsilon_0, q, \theta) \sup_{0 < \varepsilon \leqslant \varepsilon_0} \left(\varepsilon^\theta \int_\Omega |g|^{q-\varepsilon}(x) w(x) a^{\frac{p\varepsilon}{q}}(x) dx \right)^{\frac{1}{q-\varepsilon}}, \qquad (26)$$

where $C_1(a, \varepsilon_0, \theta, q) := \max\{1, C(a, \varepsilon_0, \theta, q)\}$, and $C(a, \varepsilon_0, \theta, q) =$

$$\begin{cases} \|a\|_{L^q(\Omega, w)}^{\frac{-q}{q-\varepsilon_0}} \left(\varepsilon_0^\theta \|a\|_{L^q(\Omega, w)}^q \right)^{\frac{1}{q-\varepsilon_0}} \varepsilon_0^{\frac{-\theta}{q-\varepsilon_0}}, & \text{if } \|a\|_{L^q(\Omega, w)}^q \leqslant \frac{1}{\varepsilon_0} e^{1 - \frac{1}{\varepsilon_0}}; \\[2mm] \|a\|_{L^q(\Omega, w)}^{\frac{-q}{q-\varepsilon_0}} e^{W(-1, -q\|a\|_{L^q(\Omega, w)}^q e^{-1})} \varepsilon_0^{\frac{-\theta}{q-\varepsilon_0}}, & \text{if } \frac{1}{\varepsilon_0} e^{1 - \frac{1}{\varepsilon_0}} < \|a\|_{L^q(\Omega, w)}^q < \frac{1}{q-1} e^{1 - \frac{1}{q-1}}; \\[2mm] \|a\|_{L^q(\Omega, w)}^{q\left(1 + \frac{-1}{q-\varepsilon_0}\right)} (q-1) \varepsilon_0^{\frac{-\theta}{q-\varepsilon_0}}, & \text{if } \|a\|_{L^q(\Omega, w)}^q \geqslant \frac{1}{q-1} e^{1 - \frac{1}{q-1}}. \end{cases}$$

and W is the Lambert's function. Since $\left(wa^{\frac{p\varepsilon}{q}} \right)^{\frac{1}{q-\varepsilon}} \in \mathscr{A}_{p-\eta, q-\varepsilon}$ for all $0 < \varepsilon < \varepsilon_0$, $0 < \eta < \eta_0$ with $\frac{1}{p-\eta} - \frac{1}{q-\varepsilon} = \alpha$, by Theorem C, (22) and (26), we get

$$\sup_{0 < \varepsilon < q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \left(\int_\Omega |g|^{q-\varepsilon}(x) w(x) a^{\frac{p\varepsilon}{q}}(x) dx \right)^{\frac{1}{q-\varepsilon}}$$
$$\leqslant c\, C_1(a, \varepsilon_0, \theta, q) \sup_{0 < \varepsilon \leqslant \varepsilon_0} K(a, w, p-\eta, q-\varepsilon) \times$$
$$\times \varepsilon^{\frac{\theta}{q-\varepsilon}} \left(\int_\Omega |f|^{p-\eta}(x) \left(wa^{\frac{p\varepsilon}{q}} \right)^{\frac{p-\eta}{q-\varepsilon}}(x) dx \right)^{\frac{1}{p-\eta}}$$
$$\leqslant c\, C_1(a, \varepsilon_0, \theta, q) \sup_{0 < \varepsilon \leqslant \varepsilon_0} \sup_{0 < \eta \leqslant \eta_0} K(a, w, p-\eta, q-\varepsilon) \times$$
$$\times \sup_{0 < \eta \leqslant \eta_0} \left(\phi^\theta(\eta) \int_\Omega |f(x) w^{-\alpha}(x)|^{p-\eta} w(x) a^\eta(x) dx \right)^{\frac{1}{p-\eta}}, \qquad (27)$$

where

$$K(a,w,p-\eta,q-\varepsilon) =$$

$$= \begin{cases} \psi\left(\left[\left(wa^{\frac{p\varepsilon}{q}}\right)^{\frac{1}{q-\varepsilon}}\right]^{\frac{(p-\eta)'q_0}{(q-\varepsilon)p_0'}}_{\mathscr{A}_{p-\eta,q-\varepsilon}} \left(2\overline{c}\left(1+\frac{(p-\eta)'}{q-\varepsilon}\right)\right)^{\gamma(q-\varepsilon-q_0)}\right), & \text{if } q-\varepsilon < q_0; \\[3em] \psi\left(\left[\left(wa^{\frac{p\varepsilon}{q}}\right)^{\frac{1}{q-\varepsilon}}\right]_{\mathscr{A}_{p-\eta,q-\varepsilon}} \left(2\overline{c}\left(1+\frac{q-\varepsilon}{(p-\eta)'}\right)\right)^{\frac{\gamma(q-\varepsilon-q_0)(p-\eta)'}{q-\varepsilon}}\right), & \text{if } q-\varepsilon > q_0, \end{cases}$$

c and $\overline{c}$ are constants same as in (23) and Bukley's estimate (12) respectively, $\phi(\eta) = \left(q + \frac{\eta-p}{1+\alpha(\eta-p)}\right)^{1-\alpha(p-\eta)}$ and $\gamma = \frac{1}{q} + \frac{1}{p'}$.

Now, by using the fact that ψ is a non-decreasing function and (18), we obtain

$$K_1(w,a,\varepsilon_0,\eta_0,p,q)$$

$$:= \sup_{\substack{0<\varepsilon\leqslant\varepsilon_0 \\ 0<\eta\leqslant\eta_0}} K(a,w,p-\eta,q-\varepsilon)$$

$$= \begin{cases} \psi\left(\left([w]_{A_{1+\frac{q-\varepsilon_0}{(p-\eta_0)'}}}[wa^{\frac{p\varepsilon_0}{q}}]_{A_{1+\frac{q-\varepsilon_0}{(p-\eta_0)'}}}\right)^{\frac{q_0(\gamma-1)}{p_0'}}\left(2\overline{c}\gamma\frac{q}{\gamma-1}\right)^{\gamma(q-q_0)}\right), & \text{if } q-\varepsilon < q_0; \\[3em] \psi\left([w]_{A_{1+\frac{q-\varepsilon_0}{(p-\eta_0)'}}}[wa^{\frac{p\varepsilon_0}{q}}]_{A_{1+\frac{q-\varepsilon_0}{(p-\eta_0)'}}}(2\overline{c}q\gamma)^{\frac{\gamma(q-q_0)q}{\gamma-1}}\right), & \text{if } q-\varepsilon > q_0. \end{cases}$$

$$(28)$$

By using (28) in (27), we have the following inequality

$$\sup_{0<\varepsilon<q-1} \varepsilon^{\frac{\theta}{q-\varepsilon}} \left(\int_\Omega |g|^{q-\varepsilon}(x)w(x)a^{\frac{p\varepsilon}{q}}(x)dx\right)^{\frac{1}{q-\varepsilon}}$$

$$\leqslant c\, C_1(a,\varepsilon_0,\theta,q)K_1(a,w,\eta_0,\varepsilon_0,p,q)\times$$

$$\times \sup_{0<\eta<p-1}\left(\phi^\theta(\eta)\int_\Omega |f(x)w^{-\alpha}(x)|^{p-\eta}w(x)a^\eta(x)dx\right)^{\frac{1}{p-\eta}}$$

$$\leqslant C_2(a,w,\theta,p,q,\delta)\sup_{0<\eta<p-1}\left(\phi^\theta(\eta)\int_\Omega |f(x)w^{-\alpha}(x)|^{p-\eta}w(x)a^\eta(x)dx\right)^{\frac{1}{p-\eta}},$$

where

$$C_2(a,w,\theta,\delta,p,q) := \inf_{0<\varepsilon_0,\eta_0\leqslant\delta} c\, C_1(a,\varepsilon_0,q,\theta)\, K_1(a,w,\eta_0,\varepsilon_0,p,q)$$

and the proof is complete. $\square$

COROLLARY 2. *Let ψ be a non-negative non-decreasing function on $(0,\infty)$, (f,g) be a pair of non-negative measurable functions defined on $\Omega \subseteq \mathbb{R}^n$. Let $1 < p_0 < \infty$ and $1 < q_0 < \infty$ be fixed and for every $w \in \mathscr{A}_{p_0,q_0}$*

$$\left(\int_\Omega (gw)^{q_0}(t)dt \right)^{\frac{1}{q_0}} \leqslant c \ \psi([w]_{\mathscr{A}_{p_0,q_0}}) \left(\int_\Omega (fw)^{p_0}(t)dt \right)^{\frac{1}{p_0}},$$

where c does not depend on w. Then for all $1 < p < q < \infty$ such that

$$\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q} =: \alpha$$

and arbitrary $w \in \mathscr{A}_{p,q}$, the following inequality holds:

$$\sup_{0<\varepsilon<q-1} \left(\varepsilon^\theta \int_\Omega |g|^{q-\varepsilon}(x)w(x)a^{\frac{p\varepsilon}{q}}(x)dx \right)^{\frac{1}{q-\varepsilon}}$$

$$\leqslant C_2(a,w,\theta,\delta,p,q) \sup_{0<\eta<p-1} \left(\eta^{\frac{p\theta}{q}} \int_\Omega |f(x)w^{-\alpha}(x)|^{p-\eta}w(x)a^\eta(x)dx \right)^{\frac{1}{p-\eta}},$$

where $a \in L^p(\Omega,w)$, $a^\delta \in \mathscr{A}_{p,q}$ for some $\delta > 0$.

Corollary 2 can easily be deducted from Theorem 3 in view of the following lemma:

LEMMA D [20]. *Let $1 < p < q < \infty$ and $\eta_0 \in (0, p-1)$. For $x \in (0, \eta_0)$, define*

$$\phi(x) = \left(q + \frac{x-p}{1+\alpha(x-p)} \right)^{1-\alpha(p-x)},$$

where $\alpha = \frac{1}{p} - \frac{1}{q}$. Then $\phi(x) \approx x^{\frac{p}{q}}$ near to 0.

5. Applications

In this section, we apply our extrapolation results Theorems 2 and 3 to prove a vector-valued inequality, boundedness of sublinear integral operators and boundedness of fractional integral transforms in the framework of generalized grand Lebesgue spaces $L_a^{p),\phi}(\Omega,w)$, $\Omega \subseteq \mathbb{R}$, $|\Omega| \leqslant \infty$. The first result in this direction is the following:

THEOREM 4. *Suppose that ψ is a non-negative non-decreasing function on $(0,\infty)$ and $\mathscr{F}$ is the collection of all pairs (f,g) of non-negative measurable functions defined on $\Omega \subseteq \mathbb{R}$. Suppose that $1 < p_0 < \infty$ be fixed, and for every $(f,g) \in \mathscr{F}$ and $w \in A_{p_0}$*

$$\left(\int_\Omega g^{p_0}(t)w(t)dt \right)^{\frac{1}{p_0}} \leqslant C_0 \ \psi([w]_{A_{p_0}}) \left(\int_\Omega f^{p_0}(t)w(t)dt \right)^{\frac{1}{p_0}},$$

where C_0 is a constant independent of w. Then for all $1 < p, q < \infty$ and for every $w \in A_p$, the vector-valued inequality

$$\left\| \left(\sum_{i=1}^{\infty} g_i^q \right)^{\frac{1}{q}} \right\|_{L_a^{p),\phi}(\Omega, w)} \leqslant C \left\| \left(\sum_{i=1}^{\infty} f_i^q \right)^{\frac{1}{q}} \right\|_{L_a^{p),\phi}(\Omega, w)} \tag{29}$$

holds for $\{(f_i, g_i)\} \subseteq \mathscr{F}$, where $a \in L^p(\Omega, w)$ and $a^\delta \in A_p$ for some $\delta > 0$.

Proof. Choose $1 < q < \infty$ and fix it. Let

$$F_{q,k} = \left(\sum_{i=1}^{k} f_i^q \right)^{\frac{1}{q}}$$

and

$$G_{q,k} = \left(\sum_{i=1}^{k} g_i^q \right)^{\frac{1}{q}}.$$

Clearly, $(F_{q,k}, G_{q,k})$ is a pair of non-negative measurable functions. Now, for any $w \in A_q$ applying Theorem B, we get

$$\int_{\Omega} g_i^q(t) w(t) dt \leqslant C_0^q K^q(w) \int_{\Omega} f_i^q(t) w(t) dt, \quad i = 1, 2, \ldots, k \tag{30}$$

where

$$K(w) = \begin{cases} \psi \left([w]_{A_q}^{\frac{p_0-1}{q-1}} (2q'\overline{c})^{p_0-q} \right), & \text{if } q < p_0; \\ \psi \left([w]_{A_q} (2q\overline{c})^{\frac{q-p_0}{q-1}} \right), & \text{if } q > p_0. \end{cases}$$

Adding the k inequalities in (30), we have

$$\left(\int_{\Omega} G_{q,k}^q(t) w(t) dt \right)^{\frac{1}{q}} \leqslant C_0 K(w) \left(\int_{\Omega} F_{q,k}^q(t) w(t) dt \right)^{\frac{1}{q}}.$$

Now, for $1 < p < \infty$ and $w \in A_p$, by following the proof of Theorem 2, we obtain

$$\|G_{q,k}\|_{L_a^{p),\phi}(\Omega, w)} \leqslant C(w, a, \phi, q, p, \delta) \|F_{q,k}\|_{L_a^{p),\phi}(\Omega, w)},$$

where $a \in L^p(\Omega, w)$ and $a^\delta \in A_p$ for some $\delta > 0$. This inequality is equivalent to (29) for all finite sums. Therefore by using Fatou property for the space $L_a^{p),\phi}(\Omega, w)$, the assertion follows. $\square$

The next result concerns with sublinear operators.

THEOREM 5. *Let ψ be a non-negative non-decreasing function on $(0,\infty)$ and T be a sublinear operator which is bounded on $L^{p_0}(\Omega,w)$ for a fixed p_0, $1 < p_0 < \infty$ and for every $w \in A_{p_0}$, i.e.,*

$$\left(\int_\Omega (Tf)^{p_0}(t)w(t)dt \right)^{\frac{1}{p_0}} \leqslant C_0 \; \psi([w]_{A_{p_0}}) \left(\int_\Omega f^{p_0}(t)w(t)dt \right)^{\frac{1}{p_0}},$$

where C_0 does not depend on w. Then for all $1 < p < \infty$ and for all $w \in A_p$, the inequality

$$\|Tf\|_{L_a^{p),\phi}(\Omega,w)} \leqslant C(w,a,\phi,p,\delta) \|f\|_{L_a^{p),\phi}(\Omega,w)},$$

holds where $a \in L^p(\Omega, w)$ and $a^\delta \in A_p$ for some $\delta > 0$.

Proof. Assertion follows easily by using Theorem 2 for the pair of non-negative measurable functions (Tf,f). $\square$

REMARK 7. Recall that the L^p-boundedness of the maximal operator M and Hilbert transform $\mathscr{H}$ are characterized by A_p class of weights. Consequently, by using Theorem 5, sufficient conditions for the boundedness of these operators respectively in the spaces $L_a^{p),\phi}(\Omega,w)$ and $L_a^{p),\phi}(\mathbb{R},w)$ can be written. Note that for the maximal operator, a sufficient condition was obtained in [32] for the case $|\Omega| < \infty$ and in [33] for $|\Omega| \leqslant \infty$.

Now, we consider the fractional Riesz potential operator

$$I_\alpha f(x) := \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

and the fractional maximal operator

$$M_\alpha f(x) := \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)| dy.$$

In [22], Muckenhoupt and Wheeden characterized the boundedness of these operators in terms of $\mathscr{A}_{p,q}$-class of weights, where $0 < \alpha < n$, on $L^p(\mathbb{R}^n,w)$. In the next theorem, we extend this boundedness to the generalized grand Lebesgue spaces.

THEOREM 6. *Let $1 < p < \infty$, $0 < \alpha < \frac{1}{p}$ and $\theta > 0$. Let $S_\alpha = I_\alpha$ or M_α. If $w \in \mathscr{A}_{p,q}$, where $q = \frac{p}{1-\alpha p}$, then there exists $C > 0$ such that*

$$\|S_\alpha f\|_{L_{a^{p/q}}^{q),\theta}(\Omega,w)} \leqslant C \|fw^{-\alpha}\|_{L_a^{p),\frac{p}{q}\theta}(\Omega,w)},$$

where $a \in L^p(\Omega,w)$ and $a^\delta \in \mathscr{A}_{p,q}$ for some $\delta > 0$.

Proof. It is obtained by Corollary 2 and the fact that when $w \in \mathscr{A}_{p_0,q_0}$, S_α are bounded from $L^{p_0}(\Omega,w^{p_0})$ to $L^{q_0}(\Omega,w^{q_0})$. $\square$

REMARK 8. In [29], results similar to the above theorem are given for $\Omega = \mathbb{R}^n$.

Finally, we consider the commutators $[g, I_\alpha]$ of fractional integral operators, defined by

$$[g, I_\alpha]f(x) := g(x)I_\alpha f(x) - I_\alpha(gf)(x)$$
$$= \int_\Omega \frac{g(x) - g(y)}{|x - y|^{n - \alpha}} f(y)dy,$$

where $0 < \alpha < n$, $g \in BMO(\Omega)$, i.e.,

$$\sup_B \frac{1}{|B|} \int_B |g(y) - g_B|dy < \infty,$$

where B is a ball in $\Omega \subseteq \mathbb{R}^n$ and g_B is the average of g over B.

REMARK 9. For $0 < \alpha < n$, the fractional averaging operator is defined as

$$A_\alpha f(x) := \frac{1}{|B|^{1 - \frac{\alpha}{n}}} \int_B f(y)dy.$$

In view of the estimate $|A_\alpha f(x)| \leqslant M_\alpha f(x)$, we have Theorem 6 for the fractional averaging operator as well.

Segovia and Torrea [30] proved the L^p-L^q boundedness of $[g, I_\alpha]$ as follows:

THEOREM E. *Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. Suppose $g \in BMO(\Omega)$ and $w \in \mathscr{A}_{p,q}$, then the commutator $[g, I_\alpha]$ is bounded from $L^p(\Omega, w^p)$ to $L^q(\Omega, w^q)$.*

Theorem E can be extended in the framework of generalized grand Lebesgue spaces using Corollary 2 as follows:

THEOREM 7. *Let $1 < p < \infty$, $0 < \alpha < \frac{1}{p}$ and $\theta > 0$. If $w \in \mathscr{A}_{p,q}$ with $q = \frac{p}{1 - \alpha p}$ and $g \in BMO(\Omega)$, then there exists $C > 0$ such that*

$$\|[g, I_\alpha]f\|_{L_a^{q),\theta}_{p/q}(\Omega, w)} \leqslant C \|fw^{-\alpha}\|_{L_a^{p),\frac{p}{q}\theta}(\Omega, w)},$$

where $a \in L^p(\Omega, w)$ and $a^\delta \in \mathscr{A}_{p,q}$ for some $\delta > 0$.

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