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ANOTHER IDENTITY RELATING TO HARDY'S INEQUALITY FOR ℓ_2

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Abstract. Let C denote the Cesàro operator on ℓ_2, I the identity and $\|x\|$ the ℓ_2-norm of x. Complementing an earlier result, an exact expression is derived for $\|(C-I)x\|^2$. Implications include the inequalities $\frac{1}{\sqrt{2}}\|x\| \leqslant \|(C-I)x\| \leqslant \|x\|$ and $\|(C-I)x\| \geqslant \|C^T-I)x\|$.

In this note we present a companion identity to one that was established in [2].

Denote by C the Cesàro (alias averaging) operator. For a (real) sequence $x=(x_n)$, write $X_n=\sum_{j=1}^n x_j$. Then $Cx=y$, where $y_n=X_n/n$.

Note that the transposed operator C^T is defined by $C^Tx=y$, where $y_n=\sum_{k=n}^\infty (x_k/k)$.

We denote by $\|x\|$ the ℓ_2-norm $(\sum_{n=1}^\infty x_n^2)^{1/2}$. For an operator A, we denote by $\|A\|$ the norm of A as an operator on ℓ_2. The nth unit vector will be denoted by e_n.

It was observed in [1] that CC^T equates to the matrix having $1/\max(j,k)$ in place (j,k). Hence $CC^T=C+C^T-\Delta_1$, where Δ_1 is the diagonal matrix with entries $\frac{1}{n}$. Equivalently,

$$(C-I)(C^T-I)=I-\Delta_1.$$

This, of course, implies that $\|C^T-I\|=\|C-I\|=1$, and hence the case $p=2$ in Hardy's inequality: $\|C\|\leqslant 2$. Further, it implies the following identity for x in ℓ_2:

$$\|(C^T-I)x\|^2=\sum_{n=2}^\infty\left(1-\frac{1}{n}\right)x_n^2, \tag{1}$$

An analogous identity relating to $(C-I)x$ was established in [2]: if $(C-I)x=z$, then

$$\sum_{n=2}^\infty \frac{n}{n-1}z_n^2=\sum_{n=1}^\infty x_n^2. \tag{2}$$

This again implies that $\|C-I\|\leqslant 1$, and also that $\|(C-I)x\|\geqslant (1/\sqrt{2})\|x\|$ for $x\in\ell_2$. (Equality occurs in the case $(C-I)(e_1-e_2)=e_2$.)

Here we present an identity for $\|(C-I)x\|^2$ itself, albeit a rather more complicated one. We remark first that there can be no identity of the form $\|(C-I)x\|^2=\sum_{n=1}^\infty \delta_n x_n^2$, since this would imply that $\|(C-I)(|x|)\|=\|(C-I)x\|$: the case $x=e_1-e_2$ is enough to show that this is not true. Our identity actually takes the form

$$\|(C-I)x\|^2=\sum_{n=2}^\infty\left(1-\frac{1}{n}\right)x_n^2+\sum_{n=1}^\infty c_nX_n^2$$

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Paper MIA-25-70

for a certain sequence (c_n). Note that, with (1), this will imply that $\|(C-I)x\| \geqslant \|(C^T-I)x\|$.

To identify the only possible candidate for c_n if this statement is to hold, take $x = e_n - e_{n+1}$. Then $X_n = 1$ and $X_r = 0$ for other r, so the right-hand side is $2 - \frac{1}{n} - \frac{1}{n+1} + c_n$. Meanwhile, $Cx = \frac{1}{n}e_n$, so the left-hand side is $(1-\frac{1}{n})^2 + 1$. We deduce that c_n can only be $1/[n^2(n+1)]$.

THEOREM 1. *Write $X_n = \sum_{j=1}^n x_j$. For all x in ℓ_2, we have*

$$\|(C-I)x\|^2 = \sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) x_n^2 + \sum_{n=1}^{\infty} \frac{X_n^2}{n^2(n+1)}. \tag{3}$$

Continue to write $Cx = y$ and $y - x = z$. It is essential to recognise that (3), like (2), applies strictly to *infinite* sequences. In fact, if $x_j = 1$ for $1 \leqslant j \leqslant n$, then $z_j = 0$ for $1 \leqslant j \leqslant n$. We clarify what (3) actually says for x of the form $(x_1, x_2, \ldots, x_n, 0, \ldots)$. For such x, we have $z_j = y_j = X_n/j$ for $j > n$, hence

$$\sum_{j=n+1}^{\infty} z_j^2 = X_n^2 \sum_{j=n+1}^{\infty} \frac{1}{j^2}.$$

Meanwhile,

$$\sum_{j=n+1}^{\infty} \frac{X_j^2}{j^2(j+1)} = X_n^2 \sum_{j=n+1}^{\infty} \frac{1}{j^2(j+1)}.$$

Now

$$\frac{1}{j^2} - \frac{1}{j^2(j+1)} = \frac{1}{j(j+1)}$$

and $\sum_{j=n+1}^{\infty} 1/[j(j+1)] = 1/(n+1)$, so (3) becomes

$$\sum_{j=1}^{n} z_j^2 + \frac{X_n^2}{n+1} = \sum_{j=2}^{n} \left(1 - \frac{1}{j}\right) x_j^2 + \sum_{j=1}^{n} \frac{X_j^2}{j^2(j+1)}. \tag{4}$$

We will prove that (4) holds for all x in ℓ_2 (not just x with finitely many non-zero terms). To deduce (3), we then need the following elementary lemma [2, Lemma 1].

LEMMA 1. *For $x \in \ell_2$, we have $X_n^2/n \to 0$ as $n \to \infty$.*

Proof of Theorem 1. For a given x in ℓ_2, we prove (4) by induction. For $n = 1$, both sides of (4) are $\frac{1}{2}x_1^2$ (note that $z_1 = 0$). Assume that (4) holds for $n-1$, where $n \geqslant 2$. To deduce that it holds for n, we require

$$z_n^2 + \frac{X_n^2}{n+1} - \frac{X_{n-1}^2}{n} = \left(1 - \frac{1}{n}\right) x_n^2 + \frac{X_n^2}{n^2(n+1)}. \tag{5}$$

Since $z_n = \frac{1}{n}(X_n - nx_n)$ and $X_{n-1} = X_n - x_n$, the left-hand side of (5) equals

$$\left(\frac{X_n}{n} - x_n\right)^2 + \frac{X_n^2}{n+1} - \frac{(X_n - x_n)^2}{n} = \left(\frac{1}{n^2} + \frac{1}{n+1} - \frac{1}{n}\right) X_n^2 + \left(1 - \frac{1}{n}\right) x_n^2$$
$$= \frac{X_n^2}{n^2(n+1)} + \frac{n-1}{n} x_n^2. \quad \square$$

The first term of the second series in (3) is $\frac{1}{2}x_1^2$, so an instant consequence is the following Corollary.

COROLLARY 1.1. *For $x \in \ell_2$, we have $\|(C-I)x\|^2 \geqslant \|(C^T - I)x\|^2 + \frac{1}{2}x_1^2$.*

Again, equality occurs for $x = e_1 - e_2$.

Another instant consequence of Theorem 1 is:

COROLLARY 1.2. *For $x \in \ell_2$, we have $\|(C-I)(|x|)\| \geqslant \|(C-I)x\|$.*

Clearly, (3) implies (again) that $\|(C-I)x\|^2 \geqslant \frac{1}{2}\|x\|^2$. The inequality $\|(C-I)x\| \leqslant \|x\|$ can be deduced from (3) using the fact that $X_n^2 \leqslant n\sum_{j=1}^n x_j^2$ (but of course this inequality follows more easily from (2)).

An alternative, but less self-contained, proof of Theorem 1 is by deduction from Theorem 1 of [2], which states that $CC^T = C^T \Delta_2 C$, where Δ_2 is the diagonal matrix with nth term $n/(n+1)$. We deduce that

$$(C^T - I)(C - I) - (C - I)(C^T - I) = C^T(I - \Delta_2)C,$$

hence $\|(C-I)x\|^2 - \|C^T - I)x\|^2 = \langle (I-\Delta_2)Cx, Cx\rangle = \sum_{n=1}^\infty y_n^2/(n+1)$.

Finally, as noted in [2, section 5], simple pointwise reasoning shows that Theorem 1 extends to the case where the x_j are themselves elements of a Hilbert space (in particular, complex numbers), in the following form: if X_n, y_n, z_n are defined as before, then

$$\sum_{n=1}^\infty \|z_n\|^2 = \sum_{n=2}^\infty \left(1 - \frac{1}{n}\right) \|x_n\|^2 + \sum_{n=1}^\infty \frac{\|X_n\|^2}{n^2(n+1)}.$$

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