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LOWER BOUND OF HAUSDORFF OPERATORS ON THE POWER WEIGHTED HARDY SPACES

LUONG DANG KY

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Abstract. Let $\alpha > -1$ and let φ be a measurable function on $(0,\infty)$ such that $\int_0^\infty t^\alpha |\varphi(t)|dt < \infty$. Denote by $H^1_{|\cdot|^\alpha}(\mathbb{R})$ the power weighted Hardy space associated with the power weight $|x|^\alpha$ and $\mathscr{H}_\varphi$ the Hausdorff operator associated with the kernel φ. Recently, it was showed in [11] that there is a constant $C > 0$ such that

$$\|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R})\to H^1_{|\cdot|^\alpha}(\mathbb{R})} \leqslant C\int_0^\infty t^\alpha|\varphi(t)|dt.$$

In this paper, we give a lower bound of $\|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R})\to H^1_{|\cdot|^\alpha}(\mathbb{R})}$ by proving that

$$\left|\int_0^\infty t^\alpha\varphi(t)dt\right| \leqslant \|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R})\to H^1_{|\cdot|^\alpha}(\mathbb{R})} \leqslant \int_0^\infty t^\alpha|\varphi(t)|dt.$$

1. Introduction and main result

Let φ be a measurable function on $(0,\infty)$. The *Hausdorff operator* $\mathscr{H}_\varphi$ associated with the kernel φ is defined for suitable functions f by

$$\mathscr{H}_\varphi f(x) = \int_0^\infty f\left(\frac{x}{t}\right)\frac{\varphi(t)}{t}dt, \quad x\in\mathbb{R}.$$

The Hausdorff operator is an interesting operator in harmonic analysis. There are many classical operators in analysis which are special cases of the Hausdorff operator if one chooses suitable kernel functions φ, such as the classical Hardy operator, its adjoint operator, the Cesàro type operators, the Riemann-Liouville fractional integral operator,... See the survey article [6] and the references therein. In the recent years, there is an increasing interest on the study of boundedness of the Hausdorff operator on the unweighted Hardy spaces (see, for example, [1, 3, 5, 6, 7, 8]), on the power weighted Hardy spaces (see, for example, [10, 11, 12, 15]), and on some spaces of holomorphic functions (see, for example, [4, 9, 14]).

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Let $1 \leqslant p < \infty$. A nonnegative locally integrable function $w : \mathbb{R} \to [0,\infty)$ belongs to the *Muckenhoupt class* $A_p(\mathbb{R})$, say $w \in A_p(\mathbb{R})$, if there exists a constant $C > 0$ so that

$$\frac{1}{|I|}\int_I w(x)dx \left(\frac{1}{|I|}\int_I [w(x)]^{-\frac{1}{p-1}}\right)^{p-1} \leqslant C, \quad \text{if } 1 < p < \infty,$$

and

$$\frac{1}{|I|}\int_I w(x)dx \leqslant C \operatorname*{ess-inf}_{x\in I} w(x), \quad \text{if } p = 1,$$

for all intervals $I \subset \mathbb{R}$. We say that $w \in A_\infty(\mathbb{R})$ if $w \in A_p(\mathbb{R})$ for some $1 \leqslant p < \infty$.

REMARK 1. Let $\alpha > -1$ and let $1 < p < \infty$ be such that $\alpha < p-1$. Then, the power weight $|x|^\alpha$ belongs to the Muckenhoup class $A_p(\mathbb{R})$.

Let $\alpha > -1$ and let Φ be a function in the Schwartz space $\mathscr{S}(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \Phi(x)dx \neq 0$. Set $\Phi_t(x) := t^{-1}\Phi(x/t)$. Following Strömberg and Torchinsky [13], we define the *power weighted Hardy space* $H^1_{|\cdot|^\alpha}(\mathbb{R})$ as the space of functions $f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$ such that

$$\|f\|_{H^1_{|\cdot|^\alpha}} := \|M_\Phi f\|_{L^1_{|\cdot|^\alpha}} = \int_{\mathbb{R}} M_\Phi f(x)|x|^\alpha dx < \infty,$$

where $M_\Phi f$ is the *smooth maximal function* of f defined by

$$M_\Phi f(x) = \sup_{t>0} |f * \Phi_t(x)|, \quad x \in \mathbb{R}.$$

Remark that $\|\cdot\|_{H^1_{|\cdot|^\alpha}}$ defines a norm on $H^1_{|\cdot|^\alpha}(\mathbb{R})$, whose size depends on the choice of Φ, but the space $H^1_{|\cdot|^\alpha}(\mathbb{R})$ does not depend on this choice.

Let $f \in L^2(\mathbb{R})$, we define the *Hilbert transform* H of f by

$$Hf(x) = \frac{1}{\pi}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{f(x-y)}{y}dy, \quad x \in \mathbb{R}.$$

When $\alpha > -1$, from the power weight $|x|^\alpha$ belongs to the Muckenhoupt class $A_\infty(\mathbb{R})$, it is classical (see [2, 13, 16]) that there exist $0 < c(\alpha,\Phi) < C(\alpha,\Phi) < \infty$ such that

$$c(\alpha,\Phi)\|f\|_{H^1_{|\cdot|^\alpha}} \leqslant \|f\|_{L^1_{|\cdot|^\alpha}} + \|Hf\|_{L^1_{|\cdot|^\alpha}} \leqslant C(\alpha,\Phi)\|f\|_{H^1_{|\cdot|^\alpha}}$$

for all $f \in H^1_{|\cdot|^\alpha}(\mathbb{R}) \cap L^2(\mathbb{R})$.

Let $\alpha > -1$ and let φ be a measurable function on $(0,\infty)$ such that $\int_0^\infty t^\alpha|\varphi(t)|dt < \infty$. Then, it was showed in [11] that there is a constant $C > 0$ such that

$$\|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R}) \to H^1_{|\cdot|^\alpha}(\mathbb{R})} \leqslant C\int_0^\infty t^\alpha|\varphi(t)|dt.$$

The main purpose of this paper is to give a lower bound of $\|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R}) \to H^1_{|\cdot|^\alpha}(\mathbb{R})}$.

Our main result is as follows.

THEOREM 1. *Let* $\alpha > -1$ *and let* φ *be a measurable function on* $(0,\infty)$ *such that* $\int_0^\infty t^\alpha |\varphi(t)|dt < \infty$. *Then*

(i) $\mathscr{H}_\varphi$ *is bounded on* $H^1_{|\cdot|^\alpha}(\mathbb{R})$, *moreover,*

$$\left|\int_0^\infty t^\alpha \varphi(t)dt\right| \leqslant \|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R})\to H^1_{|\cdot|^\alpha}(\mathbb{R})} \leqslant \int_0^\infty t^\alpha |\varphi(t)|dt.$$

(ii) $\mathscr{H}_\varphi$ *commutes with the Hilbert transform* H *on* $H^1_{|\cdot|^\alpha}(\mathbb{R})$.

COROLLARY 1. *Let* $\alpha > -1$ *and let* φ *be a nonnegative measurable function on* $(0,\infty)$ *such that* $\int_0^\infty t^\alpha \varphi(t)dt < \infty$. *Then* $\mathscr{H}_\varphi$ *is bounded on* $H^1_{|\cdot|^\alpha}(\mathbb{R})$, *moreover,*

$$\|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R})\to H^1_{|\cdot|^\alpha}(\mathbb{R})} = \int_0^\infty t^\alpha \varphi(t)dt.$$

REMARK 2. (i) Although the above norm $\|\cdot\|_{H^1_{|\cdot|^\alpha}}$ depends on the choice of Φ but the lower bound and upper bound estimates

$$\left|\int_0^\infty t^\alpha \varphi(t)dt\right| \leqslant \|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R})\to H^1_{|\cdot|^\alpha}(\mathbb{R})} \leqslant \int_0^\infty t^\alpha |\varphi(t)|dt$$

do not depend on this choice. Furthermore, the lower bound estimate

$$\left|\int_0^\infty t^\alpha \varphi(t)dt\right| \leqslant \|\mathscr{H}_\varphi\|_{(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)\to(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)}$$

holds for any norm $\|\cdot\|_*$ that is equivalent with $\|\cdot\|_{H^1_{|\cdot|^\alpha}}$ on $H^1_{|\cdot|^\alpha}(\mathbb{R})$. See Corollary 2 in Section 2.

(ii) Theorem 1 still holds when the norm $\|\cdot\|_{H^1_{|\cdot|^\alpha}}$ is replaced by the equivalent norm

$$\|f\|_* = \|f\|_{L^1_{|\cdot|^\alpha}} + \|Hf\|_{L^1_{|\cdot|^\alpha}},$$

where H is the Hilbert transform. See Theorem 3 in the appendix.

(iii) In the case of one dimension, our main result generalizes and improves some earlier results in [1, 3, 7, 8, 11].

Throughout the whole article, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $A \lesssim B$ means that $A \leqslant CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any $E \subset \mathbb{R}$, we denote by χ_E its characteristic function.

2. Proof of Theorem 1

Denote by $\mathbb{C}_+ = \{z = x+iy \in \mathbb{C} : y > 0\}$ the upper half-plane in the complex plane. Following Garcia-Cuerva [2], we define the Hardy space $\mathscr{H}^1_{|\cdot|^\alpha}(\mathbb{C}_+)$ as the set of all holomorphic functions F on $\mathbb{C}_+$ such that

$$\|F\|_{\mathscr{H}^1_{|\cdot|^\alpha}(\mathbb{C}_+)} := \sup_{y>0} \int_{-\infty}^{\infty} |F(x+iy)|\,|x|^\alpha dx < \infty.$$

Let P be the Poisson kernel on $\mathbb{R}$, that is, $P(x) = \frac{1}{x^2+1}$ for all $x \in \mathbb{R}$. Denote by $P_y(x) := \frac{y}{x^2+y^2}$ for all $x+iy \in \mathbb{C}_+$. The *Poisson maximal function* $M_P f$ of a function $f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$ is then defined by

$$M_P f(x) = \sup_{y>0} |P_y * f(x)|, \quad x \in \mathbb{R}.$$

The below three lemmas are well-known and can be found in [2, 13, 16].

LEMMA 1. *Let* $\alpha > -1$ *and* $f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$. *Then, the following conditions are equivalent:*

(i) $f \in H^1_{|\cdot|^\alpha}(\mathbb{R})$.

(ii) $Hf \in L^1_{|\cdot|^\alpha}(\mathbb{R})$.

(iii) $M_P f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$.

Moreover, in that case,

$$\|f\|_{H^1_{|\cdot|^\alpha}} \sim \|f\|_{L^1_{|\cdot|^\alpha}} + \|Hf\|_{L^1_{|\cdot|^\alpha}} \sim \|M_P f\|_{L^1_{|\cdot|^\alpha}}.$$

LEMMA 2. *Let* $\alpha > -1$ *and* $F \in \mathscr{H}^1_{|\cdot|^\alpha}(\mathbb{C}_+)$. *Then, the boundary value function* f *of* F*, which is defined by*

$$f(x) = \lim_{y\to 0} F(x+iy) \quad a.e.\ x \in \mathbb{R},$$

is in $H^1_{|\cdot|^\alpha}(\mathbb{R})$. *Moreover,*

$$\|f\|_{H^1_{|\cdot|^\alpha}} \sim \|f\|_{L^1_{|\cdot|^\alpha}} \sim \|F\|_{\mathscr{H}^1_{|\cdot|^\alpha}(\mathbb{C}_+)}$$

and $F(x+iy) = P_y * f(x)$ *for all* $x+iy \in \mathbb{C}_+$.

LEMMA 3. *Let* $\alpha > -1$ *and* $1 < p < \infty$. *Then,* $L^p(\mathbb{R}) \cap H^1_{|\cdot|^\alpha}(\mathbb{R})$ *is dense in* $H^1_{|\cdot|^\alpha}(\mathbb{R})$.

In order to prove Theorem 1, we need the following lemma.

LEMMA 4. *Let α and φ be as in Theorem 1. Then, for every $f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$,*

$$\|\mathscr{H}_\varphi f\|_{L^1_{|\cdot|^\alpha}} \leqslant \int_0^\infty t^\alpha |\varphi(t)| dt \, \|f\|_{L^1_{|\cdot|^\alpha}}.$$

Proof. Using the Fubini theorem, we obtain

$$\|\mathscr{H}_\varphi f\|_{L^1_{|\cdot|^\alpha}} \leqslant \int_{\mathbb{R}} |x|^\alpha dx \int_0^\infty \left| f\left(\frac{x}{t}\right)\right| \left|\frac{\varphi(t)}{t}\right| dt = \int_0^\infty t^\alpha |\varphi(t)| dt \, \|f\|_{L^1_{|\cdot|^\alpha}}. \qquad \square$$

Now we are ready to give the proof of Theorem 1.

Proof of Theorem 1. (i) For every $f \in H^1_{|\cdot|^\alpha}(\mathbb{R})$, Lemma 4 and the Fubini theorem give

$$\begin{aligned} M_\Phi(\mathscr{H}_\varphi f)(x) &= \sup_{r>0} \left| \int_{\mathbb{R}} dy \int_0^\infty \frac{1}{r} \Phi\left(\frac{x-y}{r}\right) f\left(\frac{y}{t}\right) \frac{\varphi(t)}{t} dt \right| \\ &= \sup_{r>0} \left| \int_0^\infty \Phi_{r/t} * f\left(\frac{x}{t}\right) \frac{\varphi(t)}{t} dt \right| \\ &\leqslant \mathscr{H}_{|\varphi|}(M_\Phi f)(x) \end{aligned}$$

for almost every $x \in \mathbb{R}$. Therefore, by Lemma 4 again,

$$\begin{aligned} \|\mathscr{H}_\varphi f\|_{H^1_{|\cdot|^\alpha}} &= \|M_\Phi(\mathscr{H}_\varphi f)\|_{L^1_{|\cdot|^\alpha}} \leqslant \|\mathscr{H}_{|\varphi|}(M_\Phi f)\|_{L^1_{|\cdot|^\alpha}} \\ &\leqslant \int_0^\infty t^\alpha |\varphi(t)| dt \, \|M_\Phi(f)\|_{L^1_{|\cdot|^\alpha}} = \int_0^\infty t^\alpha |\varphi(t)| dt \, \|f\|_{H^1_{|\cdot|^\alpha}}. \end{aligned}$$

This proves that $\mathscr{H}_\varphi$ is bounded on $H^1_{|\cdot|^\alpha}(\mathbb{R})$, moreover,

$$\|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}(\mathbb{R}) \to H^1_{|\cdot|^\alpha}(\mathbb{R})} \leqslant \int_0^\infty t^\alpha |\varphi(t)| dt. \tag{1}$$

Let $\delta \in (0,1)$ be arbitrary. Denote by $\varphi_\delta(t) := \varphi(t)\chi_{(\delta,1/\delta)}(t)$ for all $t \in (0,\infty)$. Then, by (1), we see that

$$\|\mathscr{H}_{\varphi_\delta}\|_{H^1_{|\cdot|^\alpha}(\mathbb{R}) \to H^1_{|\cdot|^\alpha}(\mathbb{R})} \leqslant \int_0^\infty t^\alpha |\varphi_\delta(t)| dt = \int_\delta^{1/\delta} t^\alpha |\varphi(t)| dt < \infty$$

and

$$\begin{aligned} \|\mathscr{H}_\varphi - \mathscr{H}_{\varphi_\delta}\|_{H^1_{|\cdot|^\alpha}(\mathbb{R}) \to H^1_{|\cdot|^\alpha}(\mathbb{R})} &\leqslant \int_0^\infty t^\alpha |\varphi(t) - \varphi_\delta(t)| dt \\ &= \int_0^\delta t^\alpha |\varphi(t)| dt + \int_{1/\delta}^\infty t^\alpha |\varphi(t)| dt < \infty. \end{aligned} \tag{2}$$

For any $0 < \varepsilon < 1$, we define the function $F_\varepsilon : \mathbb{C}_+ \to \mathbb{C}$ as follows

$$F_\varepsilon(z) = \frac{1}{(z+i)^{1+\alpha+\varepsilon}}.$$

Then, it is easy to establish that

$$\|F_\varepsilon\|_{\mathscr{H}^1_{|\cdot|^\alpha}(\mathbb{C}_+)} \sim \varepsilon^{-1},$$

where the constants $C > 0$ are independent of ε. Denote by f_ε the boundary value function of F_ε, that is, $f_\varepsilon(x) = \lim_{y\to 0} F_\varepsilon(x+iy)$. Then, by Lemma 2,

$$\|f_\varepsilon\|_{H^1_{|\cdot|^\alpha}} \sim \|F_\varepsilon\|_{\mathscr{H}^1_a(\mathbb{C}_+)} \sim \varepsilon^{-1}, \tag{3}$$

where the constants $C > 0$ are independent of ε.

For every $z = x+iy \in \mathbb{C}_+$, by the Fubini theorem and Lemma 2, we get

$$\begin{aligned} &P_y * \left(\mathscr{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_0^\infty t^\alpha \varphi_\delta(t)dt\right)(x) \\ &= \int_0^\infty \frac{1}{(\frac{z}{t}+i)^{1+\alpha+\varepsilon}} \frac{\varphi_\delta(t)}{t} dt - \frac{1}{(z+i)^{1+\alpha+\varepsilon}} \int_0^\infty t^\alpha \varphi_\delta(t)dt \\ &= \int_\delta^{1/\delta} [\phi_{\varepsilon,z}(t) - \phi_{\varepsilon,z}(1)] t^\alpha \varphi(t)dt, \end{aligned}$$

where $\phi_{\varepsilon,z}(t) := \frac{t^\varepsilon}{(z+ti)^{1+\alpha+\varepsilon}}$. For every $t \in (\delta, 1/\delta)$ and $z \in \mathbb{C}_+$, the Lagrange mean value theorem gives

$$\begin{aligned} |\phi_{\varepsilon,z}(t) - \phi_{\varepsilon,z}(1)| &\leqslant |t-1| \sup_{s\in(\delta,1/\delta)} |\phi'_{\varepsilon,z}(s)| \\ &\leqslant (\delta^{-1}-1)\left(\frac{\varepsilon\delta^{\varepsilon-1}}{|z+\delta i|^{1+\alpha+\varepsilon}} + \frac{(1+\alpha+\varepsilon)\delta^{-\varepsilon}}{|z+\delta i|^{2+\alpha+\varepsilon}}\right) \\ &\leqslant \frac{\varepsilon\delta^{-(3+\alpha)}}{|z+i|^{1+\alpha+\varepsilon}} + \frac{(2+\alpha)\delta^{-(5+\alpha)}}{|z+i|^{2+\alpha}}. \end{aligned}$$

Therefore, by Lemma 1 and (3),

$$\begin{aligned} &\frac{\left\|\mathscr{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_0^\infty t^\alpha \varphi_\delta(t)dt\right\|_{H^1_{|\cdot|^\alpha}}}{\|f_\varepsilon\|_{H^1_{|\cdot|^\alpha}}} \\ &\lesssim \frac{\left\|M_P\left(\mathscr{H}_{\varphi_\delta}(f_\varepsilon) - f_\varepsilon \int_0^\infty t^\alpha \varphi_\delta(t)dt\right)\right\|_{L^1_{|\cdot|^\alpha}}}{\|f_\varepsilon\|_{H^1_{|\cdot|^\alpha}}} \\ &\lesssim \int_\delta^{1/\delta} t^\alpha |\varphi(t)|dt \left[\frac{\int_{\mathbb{R}} \frac{\varepsilon\delta^{-(3+\alpha)}|x|^\alpha dx}{\sqrt{|x|^2+1}^{1+\alpha+\varepsilon}} + \int_{\mathbb{R}} \frac{(2+\alpha)\delta^{-(5+\alpha)}|x|^\alpha dx}{\sqrt{|x|^2+1}^{2+\alpha}}}{\varepsilon^{-1}}\right] \to 0 \end{aligned} \tag{4}$$

as $\varepsilon \to 0$. As a consequence, we obtain

$$\left|\int_\delta^{1/\delta} t^\alpha \varphi(t)dt\right| = \left|\int_0^\infty t^\alpha \varphi_\delta(t)dt\right| \leqslant \|\mathscr{H}_{\varphi_\delta}\|_{H^1_{|\cdot|^\alpha}\to H^1_{|\cdot|^\alpha}}.$$

This, together with (2), allows us to conclude that

$$\begin{aligned}\|\mathscr{H}_\varphi\|_{H^1_{|\cdot|^\alpha}\to H^1_{|\cdot|^\alpha}} &\geqslant \left|\int_0^\infty t^\alpha \varphi(t)dt\right| - 2\left[\int_0^\delta t^\alpha|\varphi(t)|dt + \int_{1/\delta}^\infty t^\alpha|\varphi(t)|dt\right]\\ &\to \left|\int_0^\infty t^\alpha \varphi(t)dt\right|\end{aligned}$$

as $\delta \to 0$ since $\int_0^\infty t^\alpha|\varphi(t)|dt < \infty$. Thus, by (1),

$$\left|\int_0^\infty t^\alpha \varphi(t)dt\right| \leqslant \|\mathscr{H}_\varphi\|_{H^1(\mathbb{R})\to H^1(\mathbb{R})} \leqslant \int_0^\infty t^\alpha|\varphi(t)|dt.$$

(ii) From $\alpha > -1$, it follows that there exists $p \in (1,\infty)$ such that

$$\alpha > \frac{1}{p} - 1.$$

Therefore,

$$\int_0^\infty t^{1/p-1}|\varphi_\delta(t)|dt = \int_\delta^{1/\delta} t^{1/p-1}|\varphi(t)|dt \leqslant \delta^{1/p-1-\alpha}\int_0^\infty t^\alpha|\varphi(t)|dt < \infty.$$

Hence, it follows from [4, Theorem 3.1] that

$$H(\mathscr{H}_{\varphi_\delta} f) = \mathscr{H}_{\varphi_\delta}(H(f))$$

for all $f \in L^p(\mathbb{R}) \cap H^1_{|\cdot|^\alpha}(\mathbb{R})$. Thus, by (2) and the boundedness of H on $H^1_{|\cdot|^\alpha}(\mathbb{R})$,

$$\begin{aligned}&\|H(\mathscr{H}_\varphi f) - \mathscr{H}_\varphi(H(f))\|_{H^1_{|\cdot|^\alpha}}\\ &\leqslant \|H(\mathscr{H}_\varphi f) - H(\mathscr{H}_{\varphi_\delta} f)\|_{H^1_{|\cdot|^\alpha}} + \|\mathscr{H}_\varphi(H(f)) - \mathscr{H}_{\varphi_\delta}(H(f))\|_{H^1_{|\cdot|^\alpha}}\\ &\leqslant 2\|H\|_{H^1_{|\cdot|^\alpha}\to H^1_{|\cdot|^\alpha}}\|f\|_{H^1_{|\cdot|^\alpha}}\left[\int_0^\delta t^\alpha|\varphi(t)|dt + \int_{1/\delta}^\infty t^\alpha|\varphi(t)|dt\right] \to 0\end{aligned}$$

as $\delta \to 0$ since $\int_0^\infty t^\alpha|\varphi(t)|dt < \infty$. Hence, $H(\mathscr{H}_\varphi f) = \mathscr{H}_\varphi(H(f))$ for all $f \in L^p(\mathbb{R}) \cap H^1_{|\cdot|^\alpha}(\mathbb{R})$. This, together with Lemma 3, allows us to conclude that

$$H(\mathscr{H}_\varphi f) = \mathscr{H}_\varphi(H(f))$$

for all $f \in H^1_{|\cdot|^\alpha}(\mathbb{R})$. This completes the proof of Theorem 1. □

COROLLARY 2. *Let α and φ be as in Theorem 1. Assume that $\|\cdot\|_*$ is a norm that is equivalent with $\|\cdot\|_{H^1_{|\cdot|^\alpha}}$ on $H^1_{|\cdot|^\alpha}(\mathbb{R})$. Then, $\mathscr{H}_\varphi$ is bounded on $(H^1_{|\cdot|^\alpha}(\mathbb{R}), \|\cdot\|_*)$, moreover,*

$$\|\mathscr{H}_\varphi\|_{(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)\to(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)} \geqslant \left|\int_0^\infty t^\alpha\varphi(t)dt\right|.$$

Proof. It follows directly from (2) and (4). □

3. Appendix

The main purpose of this section is to show that Theorem 1 still holds even when one replaces the norm $\|f\|_{H^1_{|\cdot|^\alpha}} = \|M_\Phi f\|_{L^1_{|\cdot|^\alpha}}$ by the norm $\|f\|_* = \|f\|_{L^1_{|\cdot|^\alpha}} + \|H(f)\|_{L^1_{|\cdot|^\alpha}}$ or some other maximal function norms on $H^1_{|\cdot|^\alpha}(\mathbb{R})$. Let ψ be a function in the Schwartz space $\mathscr{S}(\mathbb{R})$ satisfying $\int_{\mathbb{R}} \psi(x)dx \neq 0$; or be the Poisson kernel P on $\mathbb{R}$. Then, for $f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$, we define the *nontangential maximal function* $\mathscr{M}_\psi f$ by

$$\mathscr{M}_\psi f(x) = \sup_{|x-y|<t} |\psi_t * f(y)|, \quad x \in \mathbb{R}.$$

The following is well-known and can be found in [2, 13, 16].

THEOREM 2. *Let $\alpha > -1$ and $f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$. Then, the following four conditions are equivalent:*

(i) $f \in H^1_{|\cdot|^\alpha}(\mathbb{R})$.

(ii) $H(f) \in L^1_{|\cdot|^\alpha}(\mathbb{R})$.

(iii) $\mathscr{M}_\psi f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$.

(iv) $M_P f \in L^1_{|\cdot|^\alpha}(\mathbb{R})$.

Moreover, in that case,

$$\|f\|_{H^1_{|\cdot|^\alpha}} \sim \|f\|_{L^1_{|\cdot|^\alpha}} + \|H(f)\|_{L^1_{|\cdot|^\alpha}} \sim \|\mathscr{M}_\psi f\|_{L^1_{|\cdot|^\alpha}} \sim \|M_P f\|_{L^1_{|\cdot|^\alpha}}.$$

The main aim of this section is to establish the following.

THEOREM 3. *Let α and φ be as in Theorem 1. Assume that $\|\cdot\|_*$ is one of the four norms in Theorem 2. Then, $\mathscr{H}_\varphi$ is bounded on $(H^1_{|\cdot|^\alpha}(\mathbb{R}), \|\cdot\|_*)$, moreover,*

$$\left|\int_0^\infty t^\alpha \varphi(t)dt\right| \leqslant \|\mathscr{H}_\varphi\|_{(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)\to(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)} \leqslant \int_0^\infty t^\alpha |\varphi(t)|dt.$$

COROLLARY 3. *Let $\alpha > -1$ and φ be a nonnegative measurable function on $(0,\infty)$ such that $\int_0^\infty t^\alpha \varphi(t)dt < \infty$. Assume that $\|\cdot\|_*$ is one of the four norms in Theorem 2. Then, $\mathscr{H}_\varphi$ is bounded on $(H^1_{|\cdot|^\alpha}(\mathbb{R}), \|\cdot\|_*)$, moreover,*

$$\|\mathscr{H}_\varphi\|_{(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)\to(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)} = \int_0^\infty t^\alpha \varphi(t)dt.$$

Proof of Theorem 3. By Corollary 2 and Theorem 2, it suffices to prove that

$$\|\mathscr{H}_\varphi\|_{(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)\to(H^1_{|\cdot|^\alpha}(\mathbb{R}),\|\cdot\|_*)} \leqslant \int_0^\infty t^\alpha |\varphi(t)|dt. \tag{5}$$

Case 1: $\|f\|_* = \|f\|_{L^1_{|\cdot|^\alpha}} + \|H(f)\|_{L^1_{|\cdot|^\alpha}}$. By Lemma 4 and Theorem 1(ii),

$$\|\mathscr{H}_\varphi f\|_{L^1_{|\cdot|^\alpha}} \leqslant \int_0^\infty t^\alpha |\varphi(t)|dt\ \|f\|_{L^1_{|\cdot|^\alpha}}$$

and

$$\|H(\mathscr{H}_\varphi f)\|_{L^1_{|\cdot|^\alpha}} = \|\mathscr{H}_\varphi(H(f))\|_{L^1_{|\cdot|^\alpha}} \leqslant \int_0^\infty t^\alpha |\varphi(t)|dt\ \|H(f)\|_{L^1_{|\cdot|^\alpha}}$$

for all $f \in H^1_{|\cdot|^\alpha}(\mathbb{R})$. This implies that (5) holds.

Case 2: $\|f\|_* = \|\mathscr{M}_\psi f\|_{L^1_{|\cdot|^\alpha}}$. For every $f \in H^1_{|\cdot|^\alpha}(\mathbb{R})$, Lemma 4 and the Fubini theorem yield that

$$\begin{aligned}\mathscr{M}_\psi(\mathscr{H}_\varphi f)(x) &= \sup_{|y-x|<r} \left| \int_{\mathbb{R}} dz \int_0^\infty \frac{1}{r}\psi\left(\frac{y-z}{r}\right) f\left(\frac{z}{t}\right)\frac{\varphi(t)}{t}dt \right| \\ &= \sup_{|y-x|<r} \left| \int_0^\infty \psi_{r/t} * f\left(\frac{y}{t}\right)\frac{\varphi(t)}{t}dt \right| \\ &\leqslant \mathscr{H}_{|\varphi|}(\mathscr{M}_\psi f)(x)\end{aligned}$$

for almost every $x \in \mathbb{R}$. Therefore, by Lemma 4 again,

$$\begin{aligned}\|\mathscr{H}_\varphi f\|_* &= \|\mathscr{M}_\psi(\mathscr{H}_\varphi f)\|_{L^1_{|\cdot|^\alpha}} \leqslant \|\mathscr{H}_{|\varphi|}(\mathscr{M}_\psi f)\|_{L^1_{|\cdot|^\alpha}} \\ &\leqslant \int_0^\infty t^\alpha |\varphi(t)|dt \|\mathscr{M}_\psi f\|_{L^1_{|\cdot|^\alpha}} = \int_0^\infty t^\alpha |\varphi(t)|dt\ \|f\|_*,\end{aligned}$$

which implies that (5) holds.

Case 3: $\|f\|_* = \|M_P f\|_{L^1_{|\cdot|^\alpha}}$. For every $f \in H^1_{|\cdot|^\alpha}(\mathbb{R})$, Lemma 4 and the Fubini theorem yield that

$$\begin{aligned}M_P(\mathscr{H}_\varphi f)(x) &= \sup_{r>0} \left| \int_{\mathbb{R}} dy \int_0^\infty \frac{1}{r} P\left(\frac{x-y}{r}\right) f\left(\frac{y}{t}\right)\frac{\varphi(t)}{t}dt \right| \\ &= \sup_{r>0} \left| \int_0^\infty P_{r/t} * f\left(\frac{x}{t}\right)\frac{\varphi(t)}{t}dt \right| \\ &\leqslant \mathscr{H}_{|\varphi|}(M_P f)(x)\end{aligned}$$

for almost every $x \in \mathbb{R}$. Therefore, by Lemma 4 again,

$$\begin{aligned}\|\mathscr{H}_\varphi f\|_* = \|M_P(\mathscr{H}_\varphi f)\|_{L^1_{|\cdot|^\alpha}} &\leqslant \|\mathscr{H}_{|\varphi|}(M_P f)\|_{L^1_{|\cdot|^\alpha}} \\ &\leqslant \int_0^\infty t^\alpha |\varphi(t)|dt\, \|M_P(f)\|_{L^1_{|\cdot|^\alpha}} = \int_0^\infty t^\alpha |\varphi(t)|dt\ \|f\|_*.\end{aligned}$$

This proves that (5) holds, and thus ends the proof of Theorem 3. $\square$

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Luong Dang Ky
Department of Education
Quy Nhon University
170 An Duong Vuong, Quy Nhon, Binh Dinh, Viet Nam
e-mail: luongdangky@qnu.edu.vn