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SOME BOUNDS FOR CENTRAL MOMENTS AND SPREADS OF MATRICES

RAVINDER KUMAR*, VIDHI BHATIA AND HARIKRIPAL

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Abstract. In this paper, we obtain some inequalities for the central moments of discrete and continuous distributions, which, in turn, gives some lower bounds for the spread of a matrix when all of its eigenvalues are real. Likewise, we obtain lower bounds for the span of a polynomial equation.

1. Introduction

Let $r_1, r_2, \ldots, r_n$ be real numbers. Then the arithmetic mean $\overline{r}$ of the numbers $r_1, r_2, \ldots, r_n$ is defined as

$$\overline{r} = \frac{1}{n}\Sigma_{i=1}^{n} r_i \tag{1}$$

and their k-th central moment is defined as

$$m_k = \frac{1}{n}\Sigma_{i=1}^{n}(r_i - \overline{r})^k. \tag{2}$$

Let $m = \min_i r_i$, and $M = \max_i r_i$. Then the k-th central moment of a random variable R in $[m, M]$ for the discrete and continuous cases, respectively, are defined as

$$\mu_k = \Sigma_{i=1}^{n} p_i (r_i - \mu_1')^k; \;\; \mu_1' = \Sigma_{i=1}^{n} p_i r_i \tag{3}$$

and

$$\mu_k = \int_m^M (r - \mu_1')^k \psi(r)\,dr; \;\; \mu_1' = \int_m^M r\psi(r)\,dr$$

where p_i's and $\psi(r)$ are corresponding probability functions and probability densities such that

$$\sum_{i=1}^{n} p_i = 1, \;\; \int_m^M \psi(r)\,dr = 1.$$

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* Corresponding author.

In [11], Popoviciu proved that

$$\mu_2 \leqslant \frac{1}{4}(M-m)^2. \tag{4}$$

It is shown in [14] that

$$\mu_4 \leqslant \frac{1}{12}(M-m)^4. \tag{5}$$

In [15], Sharma et al. proved the following inequality:

$$\frac{1}{4}\left(\mu_4+3\mu_2^2\right) \leqslant \frac{1}{16}(M-m)^4. \tag{6}$$

(4), (5), and (6) provide lower bounds for the range $M-m$ of the random variable in terms of its central moments. In literature, these inequalities are used to derive lower bounds for the spread of a matrix, and span of a polynomial equation. For related work see [1, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15].

In the present context we also need the following two inequalities, [2, 15]:

$$\mu_2 \leqslant -\alpha\beta \tag{7}$$

and

$$\mu_2\mu_4 \leqslant \frac{4}{243}(M-m)^6 \tag{8}$$

where

$$\alpha = m-\mu_1' \text{ and } \beta = M-\mu_1'. \tag{9}$$

We here discuss some further improvements, refinements, and applications of the above inequalities to the theory of polynomial equations and matrix analysis.

In Section 2, the first two results give upper bounds for the fourth central moments (see Theorem 1 and Corollary 1, below). We present inequalities involving $M-m$, μ_2 and μ_4 (Corollary 2 and Theorem 2); an inequality involving $M-m$, m_2 and m_3 (Corollary 3); an inequality involving $M-m$ and μ_6 (Theorem 3). In Section 3, we obtain several lower bounds for the spread of a matrix in terms of the traces of a matrix (Theorem 4); and it is also shown that some inequalities for the spreads can be extended to positive linear functionals (Theorem 5). In Section 4, we give the application of Section 2 to the theory of polynomial equation where we present several lower bounds for the span of a polynomial (Theorem 6). We also include an example (Example 1).

2. Inequalities for central moments

We prove the following results for the case when R is a discrete random variable and taking finitely many values $r_1, r_2, \ldots, r_n$ with probabilities $p_1, p_2, \ldots, p_n$, respectively. The arguments are similar to the case when R is a continuous random variable.

THEOREM 1. *Let α and β be defined as in (9). Then for $m \leqslant r_i \leqslant M$, $i = 1,2,\ldots,n$,*

$$\mu_4 \leqslant -\alpha\beta\left(\alpha^2+\beta^2+\alpha\beta\right). \tag{10}$$

Proof. We have, for $\alpha \leqslant s \leqslant \beta$,

$$(s-\alpha)(s-\beta)\left(s+\frac{1}{2}(\alpha+\beta)\right)^2 \leqslant 0. \tag{11}$$

Applying (11) to n numbers s_i's, $\alpha \leqslant s_i \leqslant \beta$; we obtain

$$s_i^4 \leqslant \left(\frac{3}{4}(\alpha+\beta)^2-\alpha\beta\right)s_i^2+\left(\frac{1}{4}(\alpha+\beta)^3-\alpha\beta(\alpha+\beta)\right)s_i-\frac{1}{4}\alpha\beta(\alpha+\beta)^2 \tag{12}$$

for all $i=1,2,\ldots,n$. By multiplying (12) by $p_i \geqslant 0$ and summing over i from 1 to n, we get

$$\begin{aligned}\sum_{i=1}^{n} p_i s_i^4 \leqslant \left(\frac{3}{4}(\alpha+\beta)^2-\alpha\beta\right)\sum_{i=1}^{n} p_i s_i^2+\left(\frac{1}{4}(\alpha+\beta)^3-\alpha\beta(\alpha+\beta)\right)\sum_{i=1}^{n} p_i s_i \\ -\frac{1}{4}\alpha\beta(\alpha+\beta)^2.\end{aligned} \tag{13}$$

By substituting $s_i=r_i-\mu_1'$ in (13), and then using (3), a simple calculation shows that

$$\mu_4 \leqslant \left(\frac{3}{4}(\alpha+\beta)^2-\alpha\beta\right)\mu_2-\frac{1}{4}\alpha\beta(\alpha+\beta)^2. \tag{14}$$

By combining (7) and (14), we deduce the desired inequality (10). □

We present an improvement of (5) in the following corollary.

COROLLARY 1. *With the conditions as in Theorem 1, we have*

$$\mu_4 \leqslant \frac{1}{16}(M-m)^4. \tag{15}$$

Proof. Using the arithmetic-geometric mean inequality, we have

$$-\alpha\beta \leqslant \frac{1}{4}(M-m)^2. \tag{16}$$

Inserting the values of α and β from (9) in (10), and then combining with (16), we find that

$$\mu_4 \leqslant \frac{1}{4}(M-m)^2\left((m-\mu_1')^2+(M-\mu_1')^2)+(m-\mu_1')(M-\mu_1')\right). \tag{17}$$

The right-hand side of (17) achieves its minimum in the interval $m \leqslant \mu_1' \leqslant M$ at $\mu_1' = \frac{1}{2}(m+M)$, where its value is $\frac{1}{16}(M-m)^4$. This completes the proof. □

By using the inequality $\mu_4 \geqslant \mu_2^2$, (15) may be written as

$$\mu_2^2 \leqslant \mu_4 \leqslant \frac{1}{16}(M-m)^4$$

which gives the refinement of (4).

We present a refinement of (6) in the following corollary.

COROLLARY 2. *With the conditions as in Theorem 1, we have*

$$\frac{1}{4}\left(\mu_4+3\mu_2^2\right)\leqslant\frac{1}{2}\left(\mu_4+\mu_2^2\right)\leqslant\frac{1}{16}\left(M-m\right)^4. \tag{18}$$

Proof. By (10), we have

$$\mu_4+\alpha^2\beta^2\leqslant-\alpha\beta\left(\alpha^2+\beta^2\right). \tag{19}$$

The right-hand side inequality in (18) now follows on combining (7) with (19), and using the arithmetic-geometric mean inequality:

$$-\alpha\beta\left(\alpha^2+\beta^2\right)\leqslant\frac{1}{8}\left(\alpha-\beta\right)^4=\frac{1}{8}\left(M-m\right)^4.$$

The left-hand side inequality in (18) holds because $\mu_4\geqslant\mu_2^2$. □

We present a lower bound for the range $M-m$ in terms of μ_2 and μ_4 in the following inequality.

THEOREM 2. *For $m\leqslant r_i\leqslant M,\ i=1,2,\ldots,n$, we have*

$$\mu_2^3+\mu_2\mu_4\leqslant\frac{1}{32}(M-m)^6. \tag{20}$$

Proof. We find from (10) that

$$\mu_2\mu_4\leqslant-\alpha\beta(\alpha^2+\beta^2+\alpha\beta)\mu_2. \tag{21}$$

By combining (7) and (21), a simple calculation shows that

$$\mu_2^3+\mu_2\mu_4\leqslant\alpha^2\beta^2(\alpha^2+\beta^2). \tag{22}$$

Also, since $0\leqslant\mu_2\leqslant-\alpha\beta\leqslant\frac{1}{4}(M-m)^2$, therefore (22) gives

$$\mu_2^3+\mu_2\mu_4\leqslant\frac{1}{16}(M-m)^4(\alpha^2+\beta^2). \tag{23}$$

The right-hand side of (23) achieves its minimum in the interval $m\leqslant\mu_1'\leqslant M$ at $\mu_1'=\frac{m+M}{2}$ where its value is $\frac{1}{32}(M-m)^6$. This proves the theorem. □

By using the inequality $\mu_4\geqslant\mu_2^2$, (20) may be written as

$$\mu_2^3\leqslant\frac{1}{2}\left(\mu_2^3+\mu_2\mu_4\right)\leqslant\frac{1}{64}(M-m)^6$$

which proves the refinement of (4) for both discrete and continuous distributions.

Using a similar argument, one can find from (4) and (15) that

$$\mu_2^3\leqslant\mu_2\mu_4\leqslant\frac{1}{64}\left(M-m\right)^6 \tag{24}$$

which gives an improvement of (8).

We present a relation between $m_2,\ m_3$ and $M-m$ in the following corollary.

COROLLARY 3. *With the conditions as in Theorem 2, we have*

$$m_3^2 + 2m_2^3 \leqslant \frac{1}{32}(M-m)^6. \tag{25}$$

Proof. Pearson [10] proves the relation between the skewness and the kurtosis of a distribution, that is,

$$\frac{m_3^2}{m_2^3} + 1 \leqslant \frac{m_4}{m_2^2}. \tag{26}$$

By combining (20) and (26), we immediately get (25). □

We present a lower bound for the range $M-m$ in terms of μ_6 in the following theorem.

THEOREM 3. *With the conditions as in Theorem 1, we have*

$$\mu_6 \leqslant \frac{1}{16}(M-m)^6. \tag{27}$$

Proof. We have, for $\alpha \leqslant s \leqslant \beta$,

$$(s-\alpha)(s-\beta)\left(s^2 + \tfrac{1}{2}(\alpha+\beta)s - \frac{1}{4}(\alpha-\beta)^2\right)^2 \leqslant 0. \tag{28}$$

Applying (28) to n numbers s_i's, $\alpha \leqslant s_i \leqslant \beta$, and then resulting inequalities multiplying by $p_i \geqslant 0$ and summing over i from 1 to n; a simple calculation shows that

$$\begin{aligned}\sum_{i=1}^{n} p_i s_i^6 \leqslant{}& \frac{1}{4}(5\alpha^2+5\beta^2-2\alpha\beta)\sum_{i=1}^{n} p_i s_i^4 \\ &- \frac{1}{16}\left(5\alpha^4+5\beta^4+22\alpha^2\beta^2-8\alpha\beta(\alpha^2+\beta^2)\right)\sum_{i=1}^{n} p_i s_i^2 - \frac{1}{16}\alpha\beta\,(\alpha-\beta)^4.\end{aligned} \tag{29}$$

By substituting $s_i = r_i - \mu_1'$ in (29), and then using (3), we get

$$\begin{aligned}\mu_6 \leqslant{}& \frac{1}{4}(5\alpha^2+5\beta^2-2\alpha\beta)\mu_4 - \frac{1}{16}\left(5\alpha^4+5\beta^4+22\alpha^2\beta^2-8\alpha\beta(\alpha^2+\beta^2)\right)\mu_2 \\ &- \frac{1}{16}\alpha\beta\,(\alpha-\beta)^4.\end{aligned} \tag{30}$$

Since $\mu_2 \geqslant 0$ and $\frac{1}{16}\left(5\alpha^4+5\beta^4+22\alpha^2\beta^2-8\alpha\beta(\alpha^2+\beta^2)\right) \geqslant 0$, therefore (30) can be written as

$$\mu_6 \leqslant \frac{1}{4}(5\alpha^2+5\beta^2-2\alpha\beta)\mu_4 - \frac{1}{16}\alpha\beta\,(\alpha-\beta)^4. \tag{31}$$

By combining (15), (16), and (31), and then insert the values of α and β from (9) in the resulting inequality; we obtain

$$\mu_6 \leqslant \frac{1}{64}(M-m)^4(5\alpha^2+5\beta^2-6\alpha\beta). \tag{32}$$

The right-hand side of (32) achieves its minimum in the interval $m \leqslant \mu_1' \leqslant M$ at $\mu_1' = \frac{m+M}{2}$ where its value is $\frac{(M-m)^6}{16}$. This proves the theorem. □

Note that (30) gives an upper bound for μ_6 in terms of m, M, $\bar{r}$, μ_2, and μ_4.

3. Bounds for spreads of matrices

Let $\mathbb{M}_n$ denotes the algebra of all $n \times n$ complex matrices. We denote the trace of an $n \times n$ matrix by $tr(A)$. Let $A \in \mathbb{M}_n$, and let $\lambda_1(A)$, $\lambda_2(A)$,..., $\lambda_n(A)$ be the eigenvalues of A. The term spread of a matrix A was introduced by Mirsky [9], and defined by: $spd(A) = \max_{i,j} \left|\lambda_i(A) - \lambda_j(A)\right|$. For real eigenvalues, $spd(A) = \lambda_{\max}(A) - \lambda_{\min}(A)$, where $\lambda_{\min}(A) = \min_i \lambda_i(A)$ and $\lambda_{\max}(A) = \max_i \lambda_i(A)$. We now present several lower bounds for $spd(A)$.

THEOREM 4. *If the eigenvalues of $A \in \mathbb{M}(n)$ are all real, then*

$$spd(A) \geqslant \left(\frac{16}{n} tr\left(B^4\right)\right)^{\frac{1}{4}}, \tag{33}$$

$$spd(A) \geqslant \left[32\left(\left(\frac{1}{n} tr\left(B^2\right)\right)^3 + \frac{1}{n^2} tr\left(B^2\right) tr\left(B^4\right)\right)\right]^{\frac{1}{6}} \tag{34}$$

and

$$spd(A) \geqslant \left(\frac{16}{n} tr\left(B^6\right)\right)^{\frac{1}{6}} \tag{35}$$

where $B = A - \frac{1}{n} tr(A) I$.

Proof. Let $\lambda_1(A)$, $\lambda_2(A)$,..., $\lambda_n(A)$ be the eigenvalues of A. Then the arithmetic mean of the eigenvalues $\lambda_i(A)$ can be written as $\overline{\lambda}(A) = \frac{1}{n}\sum_{i=1}^n \lambda_i(A) = \frac{1}{n} tr(A)$; and their second, fourth, and sixth central moment can be expressed in terms of $tr\left(B^2\right)$, $tr\left(B^4\right)$, and $tr\left(B^6\right)$ respectively, that is,

$$m_2 = \frac{1}{n} tr\left(B^2\right),\ m_4 = \frac{1}{n} tr\left(B^4\right) \text{ and } m_6 = \frac{1}{n} tr\left(B^6\right). \tag{36}$$

The inequalities (33), (34), and (35) follow by Corollary 1, Theorem 2 and Theorem 3 where m_2, m_4 and m_6 are substituted by (36) and $\bar{r} = \overline{\lambda}(A) = \dfrac{1}{n} tr(A)$ in (15), (20), and (27), respectively. □

Likewise, from the inequality (24), we have

$$spd(A) \geqslant \left(\frac{64}{n^2} tr\left(B^2\right) tr\left(B^4\right)\right)^{\frac{1}{6}}. \tag{37}$$

Under the conditions of Theorem 4, it is shown respectively in [14] and [15] that

$$spd(A) \geqslant \left(\frac{12}{n} tr\left(B^4\right)\right)^{\frac{1}{4}} \tag{38}$$

and

$$spd(A) \geqslant \left(\frac{243}{4n^2} tr\left(B^2\right) tr\left(B^4\right)\right)^{\frac{1}{6}}. \tag{39}$$

Note that the inequalities (33) and (37) provide improvement of (38) and (39), respectively.

A linear functional $\varphi : \mathbb{M}(n) \to \mathbb{C}$ is called positive if $\varphi(A) \geqslant 0$ whenever $A \geqslant O$ and unital if $\varphi(I) = 1$, see [3, 14]. We now show that the above inequalities can be extended for positive linear functionals.

THEOREM 5. *Let $\varphi : \mathbb{M}_n \to \mathbb{C}$ be a unital positive linear functional and let A be any Hermitian element of $\mathbb{M}_n$. Then*

$$spd(A) \geqslant 2\left(\varphi\left(B^4\right)\right)^{\frac{1}{4}} \tag{40}$$

where $B = A - \varphi(A) I$.

Proof. By the spectral theorem, for $k = 1, 2, \ldots,$ we have

$$A^k = \sum_{i=1}^n \lambda_i^k(A) P_i \quad \text{and} \quad B^k = \sum_{i=1}^n (\lambda_i(A) - \varphi(A))^k P_i \tag{41}$$

where P_i are corresponding projections, $P_i \geqslant O$ and $\sum_{i=1}^n P_i = I$.

By applying φ, we find from (41) that

$$\varphi\left(A^k\right) = \sum_{i=1}^n \lambda_i^k(A) \varphi(P_i) \quad \text{and} \quad \varphi\left(B^k\right) = \sum_{i=1}^n (\lambda_i(A) - \varphi(A))^k \varphi(P_i)$$

with $\sum_{i=1}^n \varphi(P_i) = 1$.

Also, we observe that $\varphi(A)$, $\varphi\left(B^2\right)$, $\varphi\left(B^4\right)$, and $\varphi\left(B^6\right)$ are respectively the arithmetic mean, second, fourth, and sixth central moments of the eigenvalues $\lambda_i(A)$ with respective weights $\varphi(P_i)$, $i = 1, 2, \ldots, n$. Therefore, we can apply Corollary 1, and the inequality (40) follows immediately from (15). $\square$

On using similar arguments one can easily obtain from Theorem 2 and Theorem 3 that

$$spd(A) \geqslant \left[32\left(\left(\varphi\left(B^2\right)\right)^3 + \varphi\left(B^2\right)\varphi\left(B^4\right)\right)\right]^{\frac{1}{6}}$$

and

$$spd(A) \geqslant \left(16\varphi\left(B^6\right)\right)^{\frac{1}{6}}$$

respectively.

4. Bounds for spans of polynomial equations

In [14], Sharma et al. have considered some bounds for the span of a polynomial equation when all its roots are real. We here present some more bounds for the span in terms of the coefficients of the polynomial equation.

Let $r_1,\ r_2,\ldots,r_n$ denote the roots of the monic polynomial equation

$$f(r)=r^n+a_2r^{n-2}+a_3r^{n-3}+\ldots+a_{n-1}r+a_n=0. \tag{42}$$

We assume that all the roots of $f(r)$ are real. On using the relations between roots and coefficients of a polynomial, one can see that the arithmetic mean $\overline{r}$ of the r_i's equals zero. The second, fourth, and sixth central moment can respectively be written as

$$m_2=-\frac{2}{n}a_2,\ \ m_4=\frac{2}{n}\left(a_2^2-2a_4\right) \tag{43}$$

and

$$m_6=\frac{1}{n}\left(-2a_2^3+3a_3^2+6a_2a_4-6a_6\right). \tag{44}$$

Now we apply the results of Section 2 to the case in which distribution is uniform. Note that for a uniform distribution, $p_i=\frac{1}{n}$ for all $i=1,2,\ldots,n$, and therefore in this case $\mu_1'=\overline{r}$ and $\mu_k=m_k$.

THEOREM 6. *If roots of the polynomial* (42) *are all real, then*

$$spn\,(f)\geqslant\left(\frac{32}{n}(a_2^2-2a_4)\right)^{\frac{1}{4}}, \tag{45}$$

$$spn\,(f)\geqslant 2\left(-\frac{2(n+2)}{n^3}a_2^3+\frac{4}{n^2}a_2a_4\right)^{\frac{1}{6}} \tag{46}$$

and

$$spn\,(f)\geqslant\left(\frac{16}{n}\left(-2a_2^3+3a_3^2+6a_2a_4-6a_6\right)\right)^{\frac{1}{6}}. \tag{47}$$

Proof. Apply Corollary 1, Theorem 2 and Theorem 3; the inequalities (45), (46), and (47) follow on substituting the values of m_2, m_4, and m_6 from (43) and (44), $\overline{r}=0$ in (15), (20) and (27), respectively. □

On using similar arguments one can easily obtain from the right-hand side inequality in (18) that

$$spn\,(f)\geqslant\left(\frac{16}{n^2}(n+2)a_2^2-\frac{32}{n}a_4\right)^{\frac{1}{4}}. \tag{48}$$

EXAMPLE 1. Let

$$f(r) = r^5 - 138r^3 + 916r^2 - 1921r + \frac{3321}{4}.$$

So, $a_2 = -138$, $a_3 = 916$ and $a_4 = -1921$. From the Popoviciu inequality (4), we have $spn(f) \geqslant \sqrt{-\frac{8}{5}a_2} \cong 14.859$, and from the Sharma et al. bound, [15], that is,

$$spn(f) \geqslant \left(\frac{8}{n^2}(n+6)a_2^2 - \frac{16}{n}a_4\right)^{\frac{1}{4}}$$

we have $spn(f) \geqslant 16.448$ while from (48) we have a better estimate $spn(f) \geqslant 17.676$.

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Ravinder Kumar
Assistant Professor, Department of Mathematics
Dr B R Ambedkar National Institute of Technology Jalandhar
e-mail: thakurrk@nitj.ac.in
ravithakur557@gmail.com

Vidhi Bhatia
Ph.D. Research Scholar, Department of Mathematics
Dr B R Ambedkar National Institute of Technology Jalandhar
e-mail: vidhib.ma.19@nitj.ac.in

Harikripal
Project Fellow, Department of Mathematics
Dr B R Ambedkar National Institute of Technology Jalandhar
e-mail: harikripal2@gmail.com