

## ON ESTIMATES FOR NORM OF SOME INTEGRAL OPERATORS IN WEIGHTED LEBESGUE SPACES

K. D. KULIEV, G. T. KULIEVA AND M. K. ESHIMOVA

(Communicated by L. E. Persson)

*Abstract.* In this work, upper estimates for norm of an integral operator with Oinarov kernel in weighted Lebesgue spaces are obtained. The results given are very important in the study of the oscillation and non-oscillation properties of solutions of differential equations, as well as spectral properties.

### 1. Introduction

Let  $(a, b) \subset \mathbb{R}$  and  $u, v$  be weight functions in  $(a, b)$ , i.e., positive measurable functions defined a.e. in  $(a, b)$ . Let  $p, q > 1$  and introduce weighted Lebesgue spaces

$$L_{p,v} = \left\{ f : \|f\|_{p,v}^p := \int_a^b |f(t)|^p v(t) dt < \infty \right\}$$

and similarly  $L_{q,u}$ . In this paper we consider the integral operator

$$H : L_{p,v} \rightarrow L_{q,u}, (Hf)(x) := \int_a^x k(x,t) f(t) dt, \quad (1.1)$$

where  $k(x, t)$  is called kernel of the operator, which is nonnegative measurable function defined a.e. in  $(a, b) \times (a, b)$ .

The problem of boundedness of this operator in Lebesgue spaces began to be studied in the last decades of the last century. Let us now give some scientific conclusions concerning operators of this type. For example, F. J. Martin-Reyes and E. Sawyer [11] and V. D. Stepanov [14] considered the Riemann-Liouville fractional integral operator, i.e., (1.1) of the kernel  $k(x, t) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}$ ,  $\alpha \geq 1$  and  $\Gamma(\alpha)$  is the Gamma function. S. Bloom and R. Kerman [2] and R. Oinarov [12, 13] gave equivalent conditions for boundedness of (1.1) for kernels  $k(x, t)$  is a continuous nonnegative function increasing in the first argument, decreasing in the second argument and satisfying the condition: there exists a number  $h \geq 1$  such that

$$k(x, s) \leq h(k(x, t) + k(t, s))$$

*Mathematics subject classification* (2020): 53C12, 57R25, 57R35.

*Keywords and phrases:* Integral operator, norm, weight function, Lebesgue space, Hardy type inequality, kernel.

for all  $a < s \leq t \leq x < b$ . Functions  $k(x, t)$  satisfying the above conditions are also called *Oinarov's kernel*. In the works on this topic, the main focus was on finding equivalent conditions for the boundedness of the operator in (1.1), where estimates for the operator norm are very rare. There are some works in which the main purpose is to study another object, but the authors also gave estimates for the norm of the operator (1.1), see e.g. [3], [4], [6] and [7]. However, in the theory of differential equations it is very important to obtain the exact estimates.

Recently, in 2021 A. Kalybay and A. Baiarystanov [5] were obtained upper and lower estimates for the norm of the operator (1.1) in the case  $1 < p \leq q < \infty$  as

$$A \leq \|H\|_{L_{p,v} \rightarrow L_{q,u}} \leq (h+1)^3 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A, \quad (1.2)$$

where  $A = \max\{A_1, A_2\}$  and

$$A_1 = \sup_{a < x < b} \left( \int_x^b k^q(t, x) u(t) dt \right)^{\frac{1}{q}} \left( \int_a^x v^{1-p'}(t) dt \right)^{\frac{1}{p'}},$$

$$A_2 = \sup_{a < x < b} \left( \int_x^b u(t) dt \right)^{\frac{1}{q}} \left( \int_a^x k^{p'}(x, t) v^{1-p'}(t) dt \right)^{\frac{1}{p'}}.$$

But, the calculations show that for certain values of  $p$  and  $q$  it is possible to obtain a better estimate.

If we look at the right hand side of (1.2) we see two expressions  $(h+1)^3$  and  $p^{\frac{1}{q}} (p')^{\frac{1}{p'}}$ , one related to  $h$  and the other to the powers  $p$  and  $q$ , i.e., a change in one expression does not affect the other. In this paper, we give another upper estimates for the operator norm, where in the expression these parameters are continuously dependent each other, which is useful to find a better estimate for the operator norm for some choice of  $p$  and  $q$ , see Example 2.3.

The paper is organized as follows. The first section is introduction. In the second section we give the main results. The proofs of the results are given in the third section.

## 2. Main results

The first result of this paper states that the upper estimate for the operator norm (2.1) can be given by a solution of some nonlinear equation. Unfortunately, such equations cannot always be solved analytically. However, now the time when computer technologies are advancing, such equations can be solved approximately.

The first result of the paper reads:

**THEOREM 2.1.** *Let  $1 < p \leq q < \infty$ . Then the norm of the integral operator in (1.1) satisfies*

$$A \leq \|H\| \leq C, \quad (2.1)$$

where  $C$  is a positive solution of the equation

$$C^{q'} - hq^{\frac{1}{q-1}} (q')^{\frac{1}{q-1}} A^{\frac{1}{q-1}} C = hq^{\frac{1}{q-1}} (p')^{\frac{q'}{p'}} A^{q'} \quad (2.2)$$

and  $A = \max\{A_1, A_2\}$ .

REMARK 2.2. Equation (2.2) has a unique positive solution, since the function

$$h(x) = \frac{x^{q'}}{q^{\frac{1}{q-1}}(p')^{\frac{q'}{p'}}A^{q'} + q^{\frac{1}{q-1}}(q')^{\frac{1}{q-1}}A^{\frac{1}{q-1}}x}$$

is continuous and monotone increasing function of  $x$  in the half line  $(0, \infty)$ ,  $h(0) = 0$  and  $h(\infty) = \infty$ .

EXAMPLE 2.3. Let  $1 < p \leq q = 2$  and  $h \geq 1$ . Then Equation (2.2) and its positive solution take the forms

$$C^2 - 4AhC = 2h(p')^{\frac{2}{p'}}A^2$$

and

$$C = \left( 2h + \sqrt{4h^2 + 2h(p')^{\frac{2}{p'}}} \right) A,$$

respectively. Then corresponding estimates take the form

$$A \leq \|H\| \leq \left( 2h + \sqrt{4h^2 + 2h(p')^{\frac{2}{p'}}} \right) A.$$

REMARK 2.4. Similarly, the above results can be written for so-called conjugate Hardy operator

$$H^* : L_{p,v} \rightarrow L_{q,u}, (H^* f)(x) := \int_x^b k(t,x)g(t)dt \tag{2.3}$$

to the Hardy operator, i.e., the following estimates hold true for  $H^*$ :

$$A^* \leq \|H^*\| \leq C,$$

where  $C$  is a positive solution of (2.2) and  $A^* = \max\{A_1^*, A_2^*\}$ , where

$$A_1^* = \sup_{a < x < b} \left( \int_a^x k^p(x,t)u(t)dt \right)^{\frac{1}{p}} \left( \int_a^x v^{1-p'}(t)dt \right)^{\frac{1}{p'}},$$

$$A_2^* = \sup_{a < x < b} \left( \int_a^x u(t)dt \right)^{\frac{1}{p}} \left( \int_x^b k^{p'}(t,x)v^{1-p'}(t)dt \right)^{\frac{1}{p'}}.$$

Our next result reads as:

THEOREM 2.5. Let  $1 < p \leq q < \infty$ . Then the norm of the integral operator in (1.1) satisfies

$$A \leq \|H\| \leq eh^{q-1}qq'A, \tag{2.4}$$

where  $A = \max\{A_1, A_2\}$ .

### 3. Proofs

In this section we present proofs of the main theorems.

*Proof of Theorem 2.1.* Here we first assume that the integral operator (1.1) is bounded, that is,  $A < \infty$ . Then it can be described in the form of this inequality

$$\left( \int_a^b \left( \int_a^x k(x,t)f(t)dt \right)^q u(x)dx \right)^{\frac{1}{q}} \leq C \left( \int_a^b f^p(x)v(x)dx \right)^{\frac{1}{p}},$$

which is so-called generalized Hardy inequality, where  $C = \|H\|_{L_{p,v} \rightarrow L_{q,u}}$  is the best constant (the smallest constant in which the inequality holds) of the inequality.

We do not give a lower estimate for the norm, which is proved without change, as in Oinarov's theorem [9, Theorem 2.10] or in [5, Theorem 1]. Let denote

$$I = \int_a^b \left( \int_a^x k(x,t)f(t)dt \right)^q u(x)dx,$$

then consequently using Fubini's theorem and Hölder's inequality we get

$$\begin{aligned} I &= q \int_a^b \left( \int_a^x k(x,t)f(t) \left( \int_a^t k(x,s)f(s)ds \right)^{q-1} dt \right) u(x)dx \\ &= q \int_a^b f(t) \left( \int_t^b k(x,t)u(x) \left( \int_a^t k(x,s)f(s)ds \right)^{q-1} dx \right) dt \\ &\leq q \|f\|_{p,v} \left( \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t)u(x) \left( \int_a^t k(x,s)f(s)ds \right)^{q-1} dx \right)^{p'} dt \right)^{1/p'} \\ &= q \|f\|_{p,v} J^{1/p'}. \end{aligned} \tag{3.1}$$

We now proceed to the proof by estimating  $J$ . To do this, we estimate its inner integral separately. Using Hölder's inequality with exponents  $\frac{[q]}{q-1}$  and  $\frac{[q]}{1-[q]}$  we have

$$\begin{aligned} &\int_t^b k(x,t)u(x) \left( \int_a^t k(x,s)f(s)ds \right)^{q-1} dx \\ &= \int_t^b \left[ k^{\{q\}}(x,t)u(x) \left( \int_a^t k(x,s)f(s)ds \right)^{[q]} \right]^{\frac{q-1}{[q]}} \times [k^q(x,t)u(x)]^{\frac{1-[q]}{[q]}} dx \\ &\leq \left( \int_t^b k^{\{q\}}(x,t)u(x) \left( \int_a^t k(x,s)f(s)ds \right)^{[q]} dx \right)^{\frac{q-1}{[q]}} \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{1-[q]}{[q]}}. \end{aligned}$$

We now derive the following estimate from the condition of the Oinarov kernel and then Newton's binomial formulae

$$\begin{aligned}
&\leq h^{q-1} \left( \int_t^b k^{\{q\}}(x,t)u(x) \left( k(x,t) \int_a^t f(s)ds + \int_a^t k(t,s)f(s)ds \right)^{[q]} dx \right)^{\frac{q-1}{[q]}} \\
&\quad \times \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{1-[q]}{[q]}} \\
&= h^{q-1} \left( \int_t^b k^{\{q\}}(x,t)u(x) \left( \sum_{n=0}^{[q]} C_{[q]}^n k^n(x,t) \left( \int_a^t f(s)ds \right)^n \right. \right. \\
&\quad \left. \left. \times \left( \int_a^t k(t,s)f(s)ds \right)^{[q]-n} \right) dx \right)^{\frac{q-1}{[q]}} \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{1-[q]}{[q]}} \\
&= h^{q-1} \left( \sum_{n=0}^{[q]} C_{[q]}^n \left( \int_t^b k^{\{q\}+n}(x,t)u(x)dx \right) \left( \int_a^t f(s)ds \right)^n \right. \\
&\quad \left. \times \left( \int_a^t k(t,s)f(s)ds \right)^{[q]-n} dx \right)^{\frac{q-1}{[q]}} \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{1-[q]}{[q]}}. \tag{3.2}
\end{aligned}$$

Using the Hölder inequality to the first integral of the sum for  $0 < n < [q]$  we have

$$\begin{aligned}
\int_t^b k^{\{q\}+n}(x,t)u(x)dx &= \int_t^b (k^q(x,t)u(x))^{\frac{n-1+\{q\}}{q-1}} (k(x,t)u(x))^{\frac{[q]-n}{q-1}} dx \\
&\leq \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{n-1+\{q\}}{q-1}} \left( \int_t^b k(x,t)u(x)dx \right)^{\frac{[q]-n}{q-1}}.
\end{aligned}$$

Then (3.2) is estimated as follows:

$$\begin{aligned}
&\leq h^{q-1} \left( \sum_{n=0}^{[q]} C_{[q]}^n \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{n-1+\{q\}}{q-1}} \left( \int_t^b k(x,t)u(x)dx \right)^{\frac{[q]-n}{q-1}} \left( \int_a^t f(s)ds \right)^n \right. \\
&\quad \left. \times \left( \int_a^t k(t,s)f(s)ds \right)^{[q]-n} dx \right)^{\frac{q-1}{[q]}} \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{1-[q]}{[q]}} \\
&= h^{q-1} \left( \sum_{n=0}^{[q]} C_{[q]}^n \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{n}{q-1}} \left( \int_t^b k(x,t)u(x)dx \right)^{\frac{[q]-n}{q-1}} \right. \\
&\quad \left. \times \left( \int_a^t f(s)ds \right)^n \left( \int_a^t k(t,s)f(s)ds \right)^{[q]-n} dx \right)^{\frac{q-1}{[q]}}
\end{aligned}$$

$$\begin{aligned}
&= h^{q-1} \left( \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{1}{q-1}} \left( \int_a^t f(s)ds \right) \right. \\
&\quad \left. + \left( \int_t^b k(x,t)u(x)dx \right)^{\frac{1}{q-1}} \left( \int_a^t k(t,s)f(s)ds \right) \right)^{q-1}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_t^b k(x,t)u(x) \left( \int_a^t k(x,s)f(s)ds \right)^{q-1} dx \\
&\leq h^{q-1} \left[ \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{1}{q-1}} \left( \int_a^t f(s)ds \right) \right. \\
&\quad \left. + \left( \int_t^b k(x,t)u(x)dx \right)^{\frac{1}{q-1}} \left( \int_a^t k(t,s)f(s)ds \right) \right]^{q-1}.
\end{aligned}$$

From this we get

$$\begin{aligned}
J &= \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t)u(x) \left( \int_a^t k(x,s)f(s)ds \right)^{q-1} dx \right)^{p'} dt \\
&\leq h^{(q-1)p'} \int_a^b v^{1-p'}(t) \left[ \left( \int_t^b k^q(x,t)u(x)dx \right)^{\frac{1}{q-1}} \left( \int_a^t f(s)ds \right) \right. \\
&\quad \left. + \left( \int_t^b k(x,t)u(x)dx \right)^{\frac{1}{q-1}} \left( \int_a^t k(t,s)f(s)ds \right) \right]^{(q-1)p'} dt.
\end{aligned}$$

Using the Minkowski inequality, we obtain

$$\begin{aligned}
&\leq h^{(q-1)p'} \left( \left[ \int_a^b v^{1-p'}(t) \left( \int_t^b k^q(x,t)u(x)dx \right)^{p'} \left( \int_a^t f(s)ds \right)^{(q-1)p'} dt \right]^{\frac{1}{(q-1)p'}} \right. \\
&\quad \left. + \left[ \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t)u(x)dx \right)^{p'} \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'} dt \right]^{\frac{1}{(q-1)p'}} \right)^{(q-1)p'} \\
&= h^{(q-1)p'} \left( I_1^{\frac{1}{(q-1)p'}} + I_2^{\frac{1}{(q-1)p'}} \right)^{(q-1)p'}.
\end{aligned}$$

where

$$I_1 = \int_a^b v^{1-p'}(t) \left( \int_t^b k^q(x,t)u(x)dx \right)^{p'} \left( \int_a^t f(s)ds \right)^{(q-1)p'} dt,$$

$$I_2 = \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t)u(x)dx \right)^{p'} \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'} dt.$$

From this and we have that

$$I \leq q \|f\|_{p,v} J^{1/p'} \leq q \|f\|_{p,v} h^{q-1} \left( I_1^{\frac{1}{(q-1)p'}} + I_2^{\frac{1}{(q-1)p'}} \right)^{q-1},$$

i.e.,

$$I^{\frac{1}{q-1}} \leq h q^{\frac{1}{q-1}} \|f\|_{p,v}^{\frac{1}{q-1}} \left( I_1^{\frac{1}{(q-1)p'}} + I_2^{\frac{1}{(q-1)p'}} \right).$$

Further estimate  $I_1$  and  $I_2$ , separately. To estimate  $I_1$ , we use the following Hardy inequality

$$\begin{aligned} I_1^{\frac{1}{(q-1)p'}} &= \left( \int_a^b v^{1-p'}(t) \left( \int_t^b k^q(x,t)u(x)dx \right)^{p'} \left( \int_a^t f(s)ds \right)^{(q-1)p'} dt \right)^{\frac{1}{(q-1)p'}} \\ &\leq C_{p,(q-1)p'} \left( \int_a^b f^p(t)v(t)dt \right)^{\frac{1}{p}}, \end{aligned} \quad (3.3)$$

which holds, since its characteristic condition is satisfied

$$\begin{aligned} &\sup_{s \in (a,b)} \left( \int_s^b v^{1-p'}(t) \left( \int_t^b k^q(x,t)u(x)dx \right)^{p'} dt \right)^{\frac{1}{(q-1)p'}} \left( \int_a^s v^{1-p'}(t)dt \right)^{\frac{1}{p'}} \\ &\leq \sup_{s \in (a,b)} \left( \int_s^b v^{1-p'}(t) \left( A_1^q \left( \int_a^t v^{1-p'}(x)dx \right)^{-\frac{q}{p'}} \right)^{p'} dt \right)^{\frac{1}{(q-1)p'}} \left( \int_a^s v^{1-p'}(t)dt \right)^{\frac{1}{p'}} \\ &\leq A_1^{q'} \sup_{s \in (a,b)} \left( \int_s^b v^{1-p'}(t) \left( \int_a^t v^{1-p'}(x)dx \right)^{-q} dt \right)^{\frac{1}{(q-1)p'}} \left( \int_a^s v^{1-p'}(t)dt \right)^{\frac{1}{p'}} \\ &\leq A_1^{q'} \left( \frac{1}{q-1} \right)^{\frac{1}{(q-1)p'}} < \infty. \end{aligned}$$

For the best constant  $C_{p,(q-1)p'}$  in (3.3) the following estimate holds:

$$C_{p,(q-1)p'} \leq ((q-1)p')^{\frac{1}{(q-1)p'}} (p')^{\frac{1}{p'}} A_1^{q'} \left( \frac{1}{q-1} \right)^{\frac{1}{(q-1)p'}} = (p')^{\frac{q'}{p'}} A_1^{q'}.$$

Then

$$I_1^{\frac{1}{(q-1)p'}} \leq (p')^{\frac{q'}{p'}} A_1^{q'} \|f\|_{p,v}.$$

Let us estimate  $I_2$ :

$$\begin{aligned} I_2 &= \int_a^b v^{1-p'}(t) \left( \int_t^b k(x,t)u(x)dx \right)^{p'} \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'} dt \\ &= \int_a^b \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'} d \left( - \int_t^b \left( \int_s^b k(x,t)u(x)dx \right)^{p'} v^{1-p'}(s)ds \right) \\ &= \int_a^b \left( \int_t^b \left( \int_s^b k(x,t)u(x)dx \right)^{p'} v^{1-p'}(s)ds \right) d \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'} \end{aligned}$$

using the Minkowski inequality in the inner integral, we get

$$\leq \int_a^b \left( \int_t^b u(x) \left( \int_s^b k^{p'}(x,t)v^{1-p'}(s)ds \right)^{\frac{1}{p'}} dx \right)^{p'} d \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'}.$$

We estimate the inner integral of the last inequality in the form

$$\begin{aligned} &\leq \int_a^b \left( \int_t^b u(x)A_2 \left( \int_x^b u(s)ds \right)^{-\frac{1}{q}} dx \right)^{p'} d \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'} \\ &= (q')^{p'} A_2^{p'} \int_a^b \left( \int_t^b u(x)dx \right)^{\frac{p'}{q}} d \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'} \\ &= (q')^{p'} A_2^{p'} \left( \left[ \int_a^b \left( \int_t^b u(x)dx \right)^{\frac{p'}{q}} d \left( \int_a^t k(t,s)f(s)ds \right)^{(q-1)p'} \right]^{\frac{q'}{p'}} \right)^{\frac{p'}{q'}} \\ &\leq (q')^{p'} A_2^{p'} \left( \int_a^b u(x) \left( \int_a^x k(x,s)f(s)ds \right)^q dx \right)^{\frac{p'}{q'}} = (q')^{p'} A_2^{p'} I_1^{\frac{p'}{q'}}. \end{aligned}$$

Then

$$I_2^{\frac{1}{(q-1)p'}} \leq (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} I_1^{\frac{1}{q}}.$$

From the above estimates we have finally obtained this

$$I_1^{\frac{1}{q-1}} \leq hq^{\frac{1}{q-1}} \|f\|_{p,v}^{\frac{1}{q-1}} \left( (p')^{\frac{q'}{p'}} A_1^{q'} \|f\|_{p,v} + (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} I_1^{\frac{1}{q}} \right).$$

Using  $I_1^{\frac{1}{q}} \leq \|H\| \|f\|_{p,v}$  we get

$$I_1^{\frac{1}{q-1}} \leq hq^{\frac{1}{q-1}} \|f\|_{p,v}^{\frac{q}{q-1}} \left( (p')^{\frac{q'}{p'}} A_1^{q'} + (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} \|H\| \right).$$



$$\left( \frac{I^{\frac{1}{q}}}{\|f\|_{p,v}} \right)^{q'} \leq hq^{\frac{1}{q-1}} \left( (p')^{\frac{q'}{p'}} A_1^{q'} + (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} \|H\| \right).$$

So we have the estimate for the best constant

$$\|H\|^{q'} \leq hq^{\frac{1}{q-1}} \left( (p')^{\frac{q'}{p'}} A_1^{q'} + (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} \|H\| \right),$$

i.e.

$$\|H\|^{q'} - hq^{\frac{1}{q-1}} (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} \|H\| \leq hq^{\frac{1}{q-1}} (p')^{\frac{q'}{p'}} A_1^{q'}. \quad (3.4)$$

Consequently, we obtain

$$\frac{\|H\|^{q'}}{q^{\frac{1}{q-1}} (p')^{\frac{q'}{p'}} A_1^{q'} + q^{\frac{1}{q-1}} (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} \|H\|} \leq h.$$

Let now consider the function

$$f(x) = \frac{x^{q'}}{q^{\frac{1}{q-1}} (p')^{\frac{q'}{p'}} A_1^{q'} + q^{\frac{1}{q-1}} (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} x}$$

corresponding to the left hand side of the estimate. It is easy to see that this function is monotone increasing and continuous in  $(0, \infty)$ ,  $f(0) = 0$  and  $f(\infty) = \infty$ , which implies that the equation has exactly one positive solution. If  $C$  is a solution of the equation, i.e.,

$$\frac{C^{q'}}{q^{\frac{1}{q-1}} (p')^{\frac{q'}{p'}} A_1^{q'} + q^{\frac{1}{q-1}} (q')^{\frac{1}{q-1}} A_2^{\frac{1}{q-1}} C} = h$$

then

$$\|H\| \leq C.$$

The proof is complete.  $\square$

*Proof of Theorem 2.5.* Substituting  $C = xh^{q-1}qq'A$  in (2.2) we obtain the following equation

$$x^{q'} - x = \frac{(p')^{\frac{q'}{p'}}}{h^{q-1}q(q')^{q'}}. \quad (3.5)$$

It is easy to check that the equation has only one positive solution which belongs to  $(1, \infty)$ . We know that in general the equation can not be analytically solved, but from the other side it is enough to find an upper estimate to the solution. For this aim we consider the equation

$$y^{q'} - y = \frac{e^{\frac{q'}{e}}}{q(q')^{q'}}, \quad (3.6)$$

the right hand side of which is greater than the right side of (3.5), since  $(p')^{\frac{1}{p'}} \leq e^{\frac{1}{e}}$  for all  $p' > 1$ . Then the corresponding solutions satisfy

$$x \leq y.$$

Further, we show that the solution of (3.6) lies in the interval  $(1, e)$ .

Let us consider the function

$$f(y) = y^{q'} - y - \frac{e^{\frac{q'}{e}}}{q(q')^{q'}} \quad (3.7)$$

for  $y \in (1, \infty)$ . It is known that  $f(1) < 0$  and

$$f(e) = e^{q'} - e - \frac{e^{\frac{q'}{e}}(q' - 1)}{(q')^{q'+1}} > 0. \quad (3.8)$$

The positivity of  $f(e)$  follows from the properties of the function

$$\varphi(t) = e^t - e - \frac{e^{\frac{t}{e}}(t - 1)}{t^{t+1}},$$

i.e.,  $\varphi(1) = 0$  and the function is monotone increasing. Therefore,  $f(e) = \varphi(q') > 0$  for all  $q' > 1$ .

So, finally we have the existence of the zero point  $y$  of the function  $f$  in the interval  $(1, e)$ , which implies that the solution  $x$  of (3.5) is less than  $e$ .

The proof is complete.  $\square$

#### REFERENCES

- [1] K. ANDERSEN, H. HEINIG, *Weighted norm inequalities for certain integral operators*, SIAM J. Math. Anal. 14 (1983), pp. 834–844.
- [2] S. BLOOM, R. KERMAN, *Weighted norm inequalities for operators of Hardy type*, Proc. Amer. Math. Soc. 113 (1991) (1), pp. 135–141.
- [3] P. DRÁBEK, K. KULIEV, M. MARLETTA, *Some criteria for discreteness of spectrum of half-linear fourth order Sturm-Liouville problem*, Nonlinear Differential Equations and Applications NoDEA, 2017/4/1, Vol. 24, Issue 2, 11.
- [4] S. KAIJSER, L. NIKOLOVA, L.-E. PERSSON, A. WEDESTIG, *Hardy-type inequalities via convexity*, Mathematical Inequalities and applications, Vol. 8, 2005/3, 403–417.
- [5] A. KALYBAY, A. BAIARYSTANOV, *Exact estimate of norm of integral operator with Oinarov condition*, Kaz. Math. Jour. 21 (2021), 6–14.
- [6] A. KUFNER, K. KULIEV, R. OINAROV, *Some criteria for boundedness and compactness of the Hardy operator with some special kernels*, Jour. Ineq & Appl, 2013/12, Issue 1, 1–15.
- [7] A. KUFNER, K. KULIEV, L.-E. PERSSON, *Some higher order Hardy inequalities*, Jour. Ineq & Appl, 2012/69, Issue 1, 1–14.
- [8] A. KUFNER, L. MALIGRANDA, L.-E. PERSSON, *The Hardy inequality-about its history and some related results*, Pilsen, 2007.
- [9] A. KUFNER, L.-E. PERSSON, *Weighted inequalities of Hardy type*, World scientific: New Jersey, London, Singapore, Hong Kong, 2003.

- [10] F. J. MARTIN-REYES, *Weights, one-sided operators, singular integrals and ergodic theorems*, in: *Nonlinear Analysis, Function Spaces and Applications*, Vol. 5 (Eds: M. Krbeč, A. Kufner, B. Opic and J. Rakosnik), Prometheus, Prague, 1994, 103–137.
- [11] F. J. MARTIN-REYES AND E. SAWYER, *Weighted Inequalities for Riemann-Liouville Fractional Integrals of Order One and Greater*, *Proc. Amer. Math. Soc.* 106, 3 (1989), pp. 727–733.
- [12] R. OINAROV, *Weighted inequalities for a class of integral operators* (Russian), *Dokl. Akad. Nauk. SSSR* 319 (1991) (5), 1076–1078; translation in *Soviet Math. Dokl.* 44 (1992) (1), 291–298.
- [13] R. OINAROV, *Two-sided estimates for the norm of some classes of integral operators* (Russian), *Trudy Mat. Inst. Steklov* 204 (1993), 240–250; translation in *Proc. Steklov Inst. Math.* 204 (1994) (3), 205–214.
- [14] V. D. STEPANOV, *Weighted inequalities for a class of Volterra convolution operators*, *J. London Math. Soc.* 45 (1992), No. 2, 232–242.

(Received August 2, 2021)

*K. D. Kuliev*

*Uzbek-Finnish Pedagogical Institute  
140100, Samarkand, Uzbekistan*

*and*

*Institute of Mathematics named after V. I. Romanovsky of the  
Academy Sciences of Republic of Uzbekistan  
100174, Tashkent, Uzbekistan*

*e-mail: komilkuliev@gmail.com*

*G. T. Kulieva*

*Samarkand State University  
140104, Samarkand, Uzbekistan*

*e-mail: kulievag@mail.ru*

*M. K. Eshimova*

*Institute of Mathematics named after V. I. Romanovsky of the  
Academy Sciences of Republic of Uzbekistan  
100174, Tashkent, Uzbekistan*

*e-mail: eshimova\_math@mail.ru*