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RADII PROBLEMS FOR THE FUNCTION $az^2J''_\nu(z)+bzJ'_\nu(z)+cJ_\nu(z)$

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Abstract. In this paper, for three different normalizations of the function

$$N_\nu(z)=az^2J''_\nu(z)+bzJ'_\nu(z)+cJ_\nu(z),$$

where J_ν is Bessel functions of the first kind of order ν, the radius of parabolic starlikeness and uniform convexity are determined. We also give some simple results according to special cases of the parameters.

1. Introduction

There is an extensive literature in geometric function theory that deals with the geometric properties of different kinds of special functions like Bessel function [1, 5, 8], Struve function [3, 13], Lommel function [3] and Ramanujan function [6]. Bessel function is one of the important special functions, which are especially important for solving many problems of wave propagation and static potentials. This function plays vital role in various branches of science and engineering. In the last few decades, several researchers are interested to investigate various geometric properties of special functions involving Bessel functions. For further information related to Bessel functions and several geometric properties (convex, starlike, uniformly convex, parabolic starlike and so forth), we refer to [1, 2, 3, 5, 9, 10, 13] and the references cited therein.

Let $U(z_0,r)=\{z\in\mathbb{C}:|z-z_0|<r\}$ denote the disk of radius r and center z_0. We let $U(r)=U(0,r)$ and $U=U(0,1)=\{z\in\mathbb{C}:|z|<1\}$. Let $(a_n)_{n\geqslant 2}$ be a sequence of complex numbers with

$$d=\limsup_{n\to\infty}|a_n|^{\frac{1}{n}}\geqslant 0,\ \text{and}\ r_f=\frac{1}{d}.$$

If $d=0$ then $r_f=+\infty$. As usual, with $\mathscr{A}$ we denote the class of analytic functions $f:U(r_f)\to\mathbb{C}$ of the form

$$f(z)=z+\sum_{n\geqslant 2}a_nz^n. \tag{1}$$

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Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of univalent functions. Goodman [7] introduced the class $\mathscr{UCV}$ of uniformly convex functions. A function $f \in \mathscr{A}$ is uniformly convex if for every circular arc γ contained in U with center also in U, the image $arcf(\gamma)$ is convex. Ma and Minda [11] in 1992 and Rønning [14] in 1993 independently proved that the function $f \in \mathscr{A}$ is uniformly convex if and only if, for $z \in U$,

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|.$$

The real number

$$r^{ucv}(f) = \sup\left\{r \in (0, r_f): \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|, \ z \in U(r)\right\}$$

is called $\mathscr{UCV}$-radius of the function f. In same paper, Rønning [14] introduced the class $\mathscr{S}_p$ of functions $f \in \mathscr{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

The real number

$$r^{\mathscr{S}_p}(f) = \sup\left\{r \in (0, r_f): \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right|, \ z \in U(r)\right\}$$

is called $\mathscr{S}_p$-radius of the function f. The functions belonging to this class are called parabolically starlike. It is very clear that, every uniformly convex and parabolic starlike function also convex and starlike, respectively.

The Bessel function of the first kind of order ν is defined by (see [16])

$$J_\nu(z) = \sum_{n \geqslant 0} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu} \quad (z \in \mathbb{C}), \tag{2}$$

where $z \in \mathbb{C}$ and $\nu \in \mathbb{C}$ such that $\nu \neq -1, -2, \ldots$ We know that it has all its zeros real for $\nu > -1$ and has exactly two purely imaginary conjugate complex zeros, and all the other zeros are real for $\nu \in (-2, -1)$. Recently Deniz and Szász in [5] determined the radius of uniform convexity for three kinds of normalized Bessel functions of the first kind in case of both $\nu > -1$ and $\nu \in (-2, -1)$. Later Bohra and Ravichandran [4] extended the results of Deniz and Szász and also obtained radius of parabolic starlikeness of these functions.

In this paper, we consider the following function

$$N_\nu(z) = az^2 J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z)$$

that studied by Mercer [12]. Here, as in [12], $(c = 0 \text{ and } b \neq a)$ or $(c > 0 \text{ and } b > a)$. Mercer used this function when studying the zeros of the second order derivatives of Bessel functions. From (2), this function has an infinite series representation given by

$$N_\nu(z) = \sum_{n \geqslant 0} \frac{Q(2n+\nu)(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu} \quad (z \in \mathbb{C}), \tag{3}$$

where $Q(\nu) = a\nu(\nu-1) + b\nu + c \ (a,b,c \in \mathbb{R})$.

Note that N_ν does not belong to $\mathscr{A}$. Firstly, to prove the main results we need normalizations of the function N_ν. In this paper we consider the following normalized forms

$$f_\nu(z) = \left[\frac{2^\nu\Gamma(\nu+1)}{Q(\nu)}N_\nu(z)\right]^{\frac{1}{\nu}}, \tag{4}$$

$$g_\nu(z) = \frac{2^\nu\Gamma(\nu+1)z^{1-\nu}}{Q(\nu)}N_\nu(z), \tag{5}$$

$$h_\nu(z) = \frac{2^\nu\Gamma(\nu+1)z^{1-\frac{\nu}{2}}}{Q(\nu)}N_\nu(\sqrt{z}). \tag{6}$$

In 1992, Mercer [12] proved that the kth positive zero of N_ν increases according to ν for $\nu > 0$. In 1995, Ismail and Muldoon [8] proved that following results:

(i) For $\nu > 0$, the zeros of $N_\nu(z)$ are either real or purely imaginary,

(ii) For $\nu \geqslant \max\{0, \nu_0\}$, where ν_0 is the largest real root of the quadratic polynomial $Q(\nu) = a\nu(\nu-1) + b\nu + c$, the zeros of $N_\nu(z)$ are real,

(iii) If $\nu > 0$, $Q(\nu)/(b-a) > 0$ and $a/(b-a) < 0$, the zeros of $N_\nu(z)$ are all real except for a single pair which are conjugate purely imaginary

where $a,b,c \in \mathbb{R}$ such that $(c = 0$ and $b \neq a)$ or $(c > 0$ and $b > a)$.

Baricz, Çağlar and Deniz [1] obtained sufficient and necessary conditions for the starlikeness of a normalized form of N_ν by using results of Mercer [12], Ismail and Muldoon [8] and Shah and Trimble [15]. Recently, Kazımoğlu and Deniz [9, 10] studied radii of starlikeness and convexity of order β for the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$.

In this paper, we obtained the radii of parabolic starlikeness and uniform convexity of the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$ for the cases (ii) and (iii), separately. The key tools in their proofs are some new Mittag-Leffler expansions for quotients of the function N_ν, special properties of the zeros of the function N_ν and its derivative.

In order to prove the main results, we need the following lemma given in [5].

LEMMA 1. *i. If $a > b > r \geqslant |z|$, and $\lambda \in [0,1]$, then*

$$\left|\frac{z}{b-z} - \lambda\frac{z}{a-z}\right| \leqslant \frac{r}{b-r} - \lambda\frac{r}{a-r}. \tag{7}$$

The followings are very simple consequences of this inequality

$$\operatorname{Re}\left(\frac{z}{b-z} - \lambda\frac{z}{a-z}\right) \leqslant \frac{r}{b-r} - \lambda\frac{r}{a-r} \tag{8}$$

and

$$\operatorname{Re}\left(\frac{z}{b-z}\right) \leqslant \left|\frac{z}{b-z}\right| \leqslant \frac{r}{b-r}. \tag{9}$$

ii. If $b > a > r \geqslant |z|$, *then*

$$\left|\frac{1}{(a+z)(b-z)}\right| \leqslant \frac{1}{(a-r)(b+r)}. \tag{10}$$

2. Main results

In the rest of this paper, the quadratic polynomial $Q(\nu) = a\nu(\nu-1)+b\nu+c$ provides the conditions $a,b,c \in \mathbb{R}$ $(c=0 \text{ and } a \neq b)$ or $(c>0 \text{ and } a<b)$. Our first theorem gives the $\mathscr{S}_p$-radius of the functions $f_\nu(z)$, $g_\nu(z)$ and $h_\nu(z)$.

THEOREM 1. *Let* ν_0 *is the largest real root of the quadratic polynomial* $Q(\nu) = a\nu(\nu-1)+b\nu+c$ *and* $\nu \geqslant \max\{0,\nu_0\}$. *Then the following statements are true:*

(i) *For* $\nu \neq 0$, *the* $\mathscr{S}_p$*-radius of the function* f_ν *is the smallest positive root of the equation*

$$\frac{1}{\nu}\left(\frac{ar^3J'''_\nu(r)+\left(2a+b-\frac{a\nu}{2}\right)r^2J''_\nu(r)+\left(b+c-\frac{b\nu}{2}\right)rJ'_\nu(r)-\frac{c\nu}{2}J_\nu(r)}{ar^2J''_\nu(r)+brJ'_\nu(r)+cJ_\nu(r)}\right)=0.$$

(ii) *The* $\mathscr{S}_p$*-radius of the function* g_ν *is the smallest positive root of the equation*

$$\left(\frac{1}{2}-\nu\right)+\frac{ar^3J'''_\nu(r)+(2a+b)r^2J''_\nu(r)+(b+c)rJ'_\nu(r)}{ar^2J''_\nu(r)+brJ'_\nu(r)+cJ_\nu(r)}=0.$$

(iii) *The* $\mathscr{S}_p$*-radius of the function* h_ν *is the smallest positive root of the equation*

$$(1-\nu)+\frac{ar\sqrt{r}J'''_\nu(\sqrt{r})+(2a+b)rJ''_\nu(\sqrt{r})+(b+c)\sqrt{r}J'_\nu(\sqrt{r})}{arJ''_\nu(\sqrt{r})+b\sqrt{r}J'_\nu(\sqrt{r})+cJ_\nu(\sqrt{r})}=0.$$

Proof. We know that zeros of the function $N_\nu(z)$ are real for $\nu \geqslant \max\{0,\nu_0\}$, where ν_0 is the largest real root of the quadratic polynomial $Q(\nu) = a\nu(\nu-1)+b\nu+c$. Thus, the function $N_\nu(z)$ admits a Weierstrass decomposition [1] of the form

$$N_\nu(z) = \frac{Q(\nu)z^\nu}{2^\nu\Gamma(\nu+1)}\prod_{n\geqslant 1}\left(1-\frac{z^2}{\lambda_{\nu,n}^2}\right) \tag{11}$$

where $\lambda_{\nu,n}$ denotes the nth positive zeros of the function N_ν. The zeros of the function $N_\nu(z)$ satisfy the inequality $\lambda_{\nu,1} < \lambda_{\nu,2} < \cdots$ for $\nu \geqslant \max\{0,\nu_0\}$ and the infinite product (11) is uniformly convergent on each compact subset of $\mathbb{C}$ (see [9]). Also from (11), we have

$$\frac{zN'_\nu(z)}{N_\nu(z)} = \nu - \sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2 - z^2}. \tag{12}$$

Therefore, using (4) and (12), we obtain

$$\frac{zf_\nu'(z)}{f_\nu(z)}=\frac{1}{\nu}\frac{zN_\nu'(z)}{N_\nu(z)}=1-\frac{1}{\nu}\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2-z^2}. \tag{13}$$

Similarly, from (5), (6) and (12), we get

$$\frac{zg_\nu'(z)}{g_\nu(z)}=(1-\nu)+\frac{zN_\nu'(z)}{N_\nu(z)}=1-\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2-z^2}, \tag{14}$$

$$\frac{zh_\nu'(z)}{h_\nu(z)}=\left(1-\frac{\nu}{2}\right)+\frac{1}{2}\frac{\sqrt{z}N_\nu'(\sqrt{z})}{N_\nu(\sqrt{z})}=1-\sum_{n\geqslant 1}\frac{z}{\lambda_{\nu,n}^2-z}. \tag{15}$$

If we replace z by z^2 in the inequality (9) and we put $b=\lambda_{\nu,n}^2$, it follows that

$$\frac{|z|^2}{\lambda_{\nu,n}^2-|z|^2}\geqslant \operatorname{Re}\left(\frac{z^2}{\lambda_{\nu,n}^2-z^2}\right)$$

holds for $\nu\geqslant\max\{0,\nu_0\}$ $(\nu\neq 0)$ and $|z|<\lambda_{\nu,n}$, $n\in\mathbb{N}$. Therefore, from (13) we obtain

$$\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)=1-\frac{1}{\nu}\sum_{n\geqslant 1}\operatorname{Re}\left(\frac{2z^2}{\lambda_{\nu,n}^2-z^2}\right)\geqslant 1-\frac{1}{\nu}\sum_{n\geqslant 1}\frac{2|z|^2}{\lambda_{\nu,n}^2-|z|^2}=\frac{|z|f_\nu'(|z|)}{f_\nu(|z|)}.$$

On the other hand, using the reverse triangle inequality $||z_1|-|z_2||\leqslant|z_1-z_2|$, we get

$$\left|\frac{zf_\nu'(z)}{f_\nu(z)}-1\right|=\left|\frac{1}{\nu}\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2-z^2}\right|\leqslant\frac{1}{\nu}\sum_{n\geqslant 1}\frac{2|z|^2}{\lambda_{\nu,n}^2-|z|^2}=1-\frac{|z|f_\nu'(|z|)}{f_\nu(|z|)}$$

and, so

$$\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)-\left|\frac{zf_\nu'(z)}{f_\nu(z)}-1\right|\geqslant\frac{2|z|f_\nu'(|z|)}{f_\nu(|z|)}-1, \tag{16}$$

with equality when $z=|z|=r$. By using the similar calculations for the functions g_ν and h_ν under the condition $\nu\geqslant\max\{0,\nu_0\}$, we have

$$\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right)-\left|\frac{zg_\nu'(z)}{g_\nu(z)}-1\right|\geqslant\frac{2|z|g_\nu'(|z|)}{g_\nu(|z|)}-1 \tag{17}$$

and

$$\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)-\left|\frac{zh_\nu'(z)}{h_\nu(z)}-1\right|\geqslant\frac{2|z|h_\nu'(|z|)}{h_\nu(|z|)}-1, \tag{18}$$

with equality when $z=|z|=r$.

The minimum principle for harmonic functions implies that the inequalities (16), (17) and (18) are valid if and only if $|z|<r_f$, $|z|<r_g$ and $|z|<r_h$, respectively, where r_f, r_g and r_h are the smallest positive roots of the following three equations

$$\frac{rf_\nu'(r)}{f_\nu(r)}=\frac{1}{2},\quad\frac{rg_\nu'(r)}{g_\nu(r)}=\frac{1}{2}\ \text{and}\ \frac{rh_\nu'(r)}{h_\nu(r)}=\frac{1}{2}.$$

Also, these equations are equivalent to, respectively,

$$ar^3 J_\nu'''(r) + \left(2a + b - \frac{a\nu}{2}\right) r^2 J_\nu''(r) + \left(b + c - \frac{b\nu}{2}\right) r J_\nu'(r) - \frac{c\nu}{2} J_\nu(r) = 0,$$

$$\left(\frac{1}{2} - \nu\right) + \frac{ar^3 J_\nu'''(r) + (2a+b) r^2 J_\nu''(r) + (b+c) r J_\nu'(r)}{ar^2 J_\nu''(r) + br J_\nu'(r) + c J_\nu(r)} = 0$$

and

$$(1-\nu) + \frac{ar\sqrt{r} J_\nu'''(\sqrt{r}) + (2a+b) r J_\nu''(\sqrt{r}) + (b+c)\sqrt{r} J_\nu'(\sqrt{r})}{ar J_\nu''(\sqrt{r}) + b\sqrt{r} J_\nu'(\sqrt{r}) + c J_\nu(\sqrt{r})} = 0.$$

We note that

$$\lim_{r \searrow 0} 1 - \frac{1}{\nu} \sum_{n \geqslant 1} \frac{4r^2}{\lambda_{\nu,n}^2 - r^2} = 1 > 0$$

and

$$\lim_{r \nearrow \lambda_{\nu,1}} 1 - \frac{1}{\nu} \sum_{n \geqslant 1} \frac{4r^2}{\lambda_{\nu,n}^2 - r^2} = -\infty.$$

Hence $\frac{r f_\nu'(r)}{f_\nu(r)} = \frac{1}{2}$ has a root in $(0, \lambda_{\nu,1})$. Similarly, it can be verified for the other two equations. This completes the proof of theorem. □

Using the following representation of the function $N_{1/2}(z)$, in terms of elementary trigonometric functions

$$N_{1/2}(z) = \frac{4(b-a) z\cos z + \left[a\left(3 - 4z^2\right) - 2b + 4c\right]\sin z}{2\sqrt{2\pi}\sqrt{z}},$$

we have

$$f_{1/2}(z) = \frac{\left[4(a-b) z\cos z + \left(4az^2 - 3a + 2b - 4c\right)\sin z\right]^2}{(a - 2b - 4c)^2 z},$$

$$g_{1/2}(z) = \frac{4(a-b) z\cos z + \left(4az^2 - 3a + 2b - 4c\right)\sin z}{a - 2b - 4c}$$

and

$$h_{1/2}(z) = \frac{4(a-b) z\cos\sqrt{z} + (4az - 3a + 2b - 4c)\sqrt{z}\sin\sqrt{z}}{a - 2b - 4c}.$$

We now state the following results for the functions $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$.

COROLLARY 1. *The following statements are true.*

(i) *The $\mathscr{S}_p$-radius of the function $f_{1/2}$ is the smallest positive root of the equation*

$$\frac{4\left(4ar^2 - 2a + b - 4c\right) r\cos r + \left(4ar^2 + 16br^2 + 9a - 6b + 12c\right)\sin r}{16(a-b) r\cos r + 4\left(4ar^2 - 3a + 2b - 4c\right)\sin r} = 0.$$

(*ii*) *The* $\mathscr{S}_p$*-radius of the function* $g_{1/2}$ *is the smallest positive root of the equation*

$$\frac{2\left(4ar^2-a-4c\right)r\cos r+\left(4ar^2+8br^2+3a-2b+4c\right)\sin r}{8\left(a-b\right)r\cos r+2\left(4ar^2-3a+2b-4c\right)\sin r}=0.$$

(*iii*) *The* $\mathscr{S}_p$*-radius of the function* $h_{1/2}$ *is the smallest positive root of the equation*

$$\frac{\left(4ar+a-2b-4c\right)\sqrt{r}\cos\sqrt{r}+4\left(a+b\right)r\sin\sqrt{r}}{8\left(a-b\right)\sqrt{r}\cos\sqrt{r}+2\left(4ar-3a+2b-4c\right)\sin\sqrt{r}}=0.$$

In Corollary 1, according to special cases of a,b and c the following table created.

Table 1: *Radii of parabolic starlikeness for* f_ν, g_ν *and* h_ν *for* $\nu=1/2$

	$b=3$ and $c=0$			$a=1$ and $c=0$			$a=1$ and $b=2$		
	$a=2$	$a=3$	$a=4$	$b=2$	$b=3$	$b=4$	$c=2$	$c=3$	$c=4$
$r^{sp}\left(f_{\frac{1}{2}}\right)$	0.2122	0.1640	0.1220	0.2409	0.2747	0.2942	0.4218	0.4730	0.5127
$r^{sp}\left(g_{\frac{1}{2}}\right)$	0.2860	0.2207	0.1640	0.3251	0.3711	0.3977	0.5753	0.6476	0.7038
$r^{sp}\left(h_{\frac{1}{2}}\right)$	0.1381	0.0818	0.0450	0.1789	0.2339	0.2691	0.5785	0.7417	0.8843

Now, we present the radii of parabolic starlikeness of functions f_ν, g_ν and h_ν when the zeros of $N_\nu(z)$ are all real except for a single pair which are conjugate purely imaginary (i.e. when $\nu>0$, $Q(\nu)/(b-a)>0$ and $a/(b-a)<0$).

Here and in the sequel I_ν denotes the modified Bessel function of the fist kind of order ν. Note that $I_\nu(z)=i^{-\nu}J_\nu(iz)$ and $I_\nu(\sqrt{z})=(-1)^{-\frac{\nu}{2}}J_\nu(\sqrt{-z})$.

THEOREM 2. *Let* $\nu>0$, $Q(\nu)/(b-a)>0$, $a/(b-a)<0$ *and* $\frac{Q(\nu+2)}{Q(\nu)}<0$. *Then the following statements are true:*

(*i*) *The* $\mathscr{S}_p$*-radius of the function* f_ν *is the smallest positive root of the equation*

$$\frac{1}{\nu}\left(\frac{ar^3I_\nu'''(r)+\left(2a+b-\frac{a\nu}{2}\right)r^2I_\nu''(r)+\left(b+c-\frac{b\nu}{2}\right)rI_\nu'(r)-\frac{c\nu}{2}I_\nu(r)}{ar^2I_\nu''(r)+brI_\nu'(r)+cI_\nu(r)}\right)=0.$$

(*ii*) *The* $\mathscr{S}_p$*-radius of the function* g_ν *is the smallest positive root of the equation*

$$\left(\frac{1}{2}-\nu\right)+\frac{ar^3I_\nu'''(r)+(2a+b)r^2I_\nu''(r)+(b+c)rI_\nu'(r)}{ar^2I_\nu''(r)+brI_\nu'(r)+cI_\nu(r)}=0.$$

(*iii*) *The* $\mathscr{S}_p$*-radius of the function* h_ν *is the smallest positive root of the equation*

$$(1-\nu)+\frac{ar\sqrt{r}I_\nu'''(\sqrt{r})+(2a+b)rI_\nu''(\sqrt{r})+(b+c)\sqrt{r}I_\nu'(\sqrt{r})}{arI_\nu''(\sqrt{r})+b\sqrt{r}I_\nu'(\sqrt{r})+cI_\nu(\sqrt{r})}=0.$$

Proof. By using a result of Ismail and Muldoon [8] on zeros of the function $N_\nu(z)$, the conditions $\nu>0$, $Q(\nu)/(b-a)>0$ and $a/(b-a)<0$ imply that $\lambda_{\nu,1}=i\alpha$, $\alpha>0$ and $\lambda_{\nu,n}>0$ for $n\in\{2,3,\ldots\}$. If this situation is used in the equations (13), (14) and (15), we have

$$\frac{zf_\nu'(z)}{f_\nu(z)}=1-\frac{1}{\nu}\left(\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\alpha^2+z^2}+2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)}\right), \tag{19}$$

$$\frac{zg_\nu'(z)}{g_\nu(z)}=1-\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\alpha^2+z^2}-2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)} \tag{20}$$

and

$$\frac{zh_\nu'(z)}{h_\nu(z)}=1-\frac{\alpha^2Q(\nu+2)}{4(\nu+1)Q(\nu)}\frac{z}{\alpha^2+z}-\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^2}{(\alpha^2+z)\left(\lambda_{\nu,n}^2-z\right)}. \tag{21}$$

In above equalities, we used (see [9])

$$\frac{1}{\alpha^2}=\sum_{n\geqslant 2}\frac{1}{\lambda_{\nu,n}^2}-\frac{Q(\nu+2)}{4(\nu+1)Q(\nu)}.$$

In [10], the following equations have been proved

$$\begin{aligned}\inf_{z\in U(r)}\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)&\\=1+\frac{1}{\nu}\left(\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2}-2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}\right)&\\=\left.\frac{zf_\nu'(z)}{f_\nu(z)}\right|_{z=ir},&\end{aligned} \tag{22}$$

$$\begin{aligned}\inf_{z\in U(r)}\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right)&\\=1+\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2}-2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}&\\=\left.\frac{zg_\nu'(z)}{g_\nu(z)}\right|_{z=ir}&\end{aligned} \tag{23}$$

and

$$\begin{aligned}\inf_{z\in U(r)}\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)&\\=1+\frac{\alpha^2Q(\nu+2)}{4(\nu+1)Q(\nu)}\frac{r}{\alpha^2-r}-\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^2}{(\alpha^2-r)\left(\lambda_{\nu,n}^2+r\right)}&\\=\left.\frac{zh_\nu'(z)}{h_\nu(z)}\right|_{z=-r}.&\end{aligned} \tag{24}$$

On the other hand, we replace z by z^2, $a=\alpha^2$ and $b=\lambda_{\nu,n}^2$ in the inequality (10) and taking into account that $\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}<0$, we get

$$\begin{aligned}
&\left|\frac{zf_\nu'(z)}{f_\nu(z)}-1\right| \\
&=\frac{1}{\nu}\left|\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\alpha^2+z^2}+2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)}\right| \\
&\leqslant \frac{1}{\nu}\left(\frac{-\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\left|\frac{z^2}{\alpha^2+z^2}\right|+2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\left|\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)}\right|\right) \\
&\leqslant \frac{-1}{\nu}\left(\frac{\alpha^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2}-2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}\right) \\
&=1-\frac{irf_\nu'(ir)}{f_\nu(ir)},\quad |z|\leqslant r<\alpha.
\end{aligned} \tag{25}$$

Thus, from inequalities (22) and (25), we have

$$\inf_{|z|<r}\left[\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right)-\left|\frac{zf_\nu'(z)}{f_\nu(z)}-1\right|\right]=\frac{2irf_\nu'(ir)}{f_\nu(ir)}-1 \tag{26}$$

for every $r\in(0,\alpha)$. When the similar processes are also applied to the functions g_ν and h_ν, we obtain

$$\inf_{|z|<r}\left[\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right)-\left|\frac{zg_\nu'(z)}{g_\nu(z)}-1\right|\right]=\frac{2irg_\nu'(ir)}{g_\nu(ir)}-1 \tag{27}$$

and

$$\inf_{|z|<r}\left[\operatorname{Re}\left(\frac{zh_\nu'(z)}{h_\nu(z)}\right)-\left|\frac{zh_\nu'(z)}{h_\nu(z)}-1\right|\right]=\frac{-2rh_\nu'(-r)}{h_\nu(-r)}-1. \tag{28}$$

Now, we consider the functions $\varphi:(0,\alpha)\longrightarrow\mathbb{R}$, $\psi:(0,\alpha)\longrightarrow\mathbb{R}$ and $\phi:\left(0,\alpha^2\right)\longrightarrow\mathbb{R}$ defined by

$$\varphi_\nu(r)=1-\frac{4r^2}{\nu}\left(\frac{1}{\alpha^2-r^2}-\sum_{n\geqslant 2}\frac{1}{\lambda_{\nu,n}^2+r^2}\right),$$

$$\psi_\nu(r)=1-4r^2\left(\frac{1}{\alpha^2-r^2}-\sum_{n\geqslant 2}\frac{1}{\lambda_{\nu,n}^2+r^2}\right)$$

and

$$\phi_\nu(r)=1-2r\left(\frac{1}{\alpha^2-r}-\sum_{n\geqslant 2}\frac{1}{\lambda_{\nu,n}^2+r}\right).$$

Since the functions φ, ψ and ϕ are strictly decreasing and following limits are true

$$\lim_{r\searrow 0}\varphi(r)=1>0,\ \lim_{r\nearrow\alpha}\varphi(r)=-\infty,\ \lim_{r\searrow 0}\psi(r)=1>0,\ \lim_{r\nearrow\alpha}\psi(r)=-\infty$$

and

$$\lim_{r\searrow 0}\phi(r)=1>0,\ \lim_{r\nearrow \alpha^2}\phi(r)=-\infty,$$

it follows that the equations

$$\frac{irf_\nu'(ir)}{f_\nu(ir)}=\frac{1}{2},\ \frac{irg_\nu'(ir)}{g_\nu(ir)}=\frac{1}{2}\ \text{and}\ \frac{-rh_\nu'(-r)}{h_\nu(-r)}=\frac{1}{2}$$

has a unique root $r^{\mathcal{S}p}(f_\nu)$, $r^{\mathcal{S}p}(g_\nu)\in(0,\alpha)$ and $r^{\mathcal{S}p}(h_\nu)\in\left(0,\alpha^2\right)$.

This completes the proof. □

Taking $\nu=1/2$ in Theorem 2, we have the following results.

COROLLARY 2. *The following statements are true.*

(i) *The $\mathcal{S}_p$-radius of the function $f_{1/2}$ is the smallest positive root of the equation*

$$\frac{4\left(4ar^2+2a-b+4c\right)r\cosh r+\left(4ar^2+16br^2-9a+6b-12c\right)\sinh r}{16(b-a)r\cosh r+4\left(4ar^2+3a-2b+4c\right)\sinh r}=0.$$

(ii) *The $\mathcal{S}_p$-radius of the function $g_{1/2}$ is the smallest positive root of the equation*

$$\frac{2\left(4ar^2+a+4c\right)r\cosh r+\left(4ar^2+8br^2-3a+2b-4c\right)\sinh r}{8(b-a)r\cosh r+2\left(4ar^2+3a-2b+4c\right)\sinh r}=0.$$

(iii) *The $\mathcal{S}_p$-radius of the function $h_{1/2}$ is the smallest positive root of the equation*

$$\frac{(4ar-a+2b+4c)\sqrt{r}\cosh\sqrt{r}+4(a+b)r\sinh\sqrt{r}}{8(b-a)\sqrt{r}\cosh\sqrt{r}+2(4ar+3a-2b+4c)\sinh\sqrt{r}}=0.$$

The following table related with radii of parabolic starlikeness of $f_{1/2}$, $g_{1/2}$ and $h_{1/2}$ for special cases of a, b and c in Corollary 2.

Table 2: *Radii of parabolic starlikeness for f_ν, g_ν and h_ν for $\nu=1/2$*

	$b=-5$ and $c=0$			$a=-5$ and $c=1$			$a=-2$ and $b=-1$		
	$a=4$	$a=5$	$a=6$	$b=4$	$b=5$	$b=6$	$c=4$	$c=5$	$c=6$
$r^{\mathcal{S}p}(f_{\frac{1}{2}})$	0.7156	0.5602	0.4795	0.5466	0.6485	0.7763	0.6103	0.7132	0.8166
$r^{\mathcal{S}p}(g_{\frac{1}{2}})$	0.8745	0.7102	0.6182	0.6976	0.8093	0.9399	0.7820	0.8998	1.0132
$r^{\mathcal{S}p}(h_{\frac{1}{2}})$	1.0765	0.7500	0.5846	0.7313	0.9500	1.2281	0.9251	1.1930	1.4740

For convenience in the rest of the paper, we shall use the following notation

$$\begin{aligned}M_\nu(z)&=i^{-\nu}N_\nu(iz)=i^{-\nu}\left[az^2J_\nu''(iz)+bzJ_\nu'(iz)+cJ_\nu(iz)\right]\\&=az^2I_\nu''(z)+bzI_\nu'(z)+cI_\nu(z).\end{aligned}$$

The second principal result we established concerns the radii of uniform convexity and reads as follows.

THEOREM 3. *The $\mathcal{UCV}$-radius of the function f_ν is*

(*i*) *the smallest positive root $r^{ucv}(f_\nu)$ of the equation*

$$1+\frac{2rN_\nu''(r)}{N_\nu'(r)}+\left(\frac{1}{\nu}-1\right)\frac{2rN_\nu'(r)}{N_\nu(r)}=0$$

for $\nu \geqslant \max\{0,\nu_0\}$ $(\nu \neq 0)$, where ν_0 is the largest real root of the quadratic polynomial $Q(\nu)=a\nu(\nu-1)+b\nu+c$.

(*ii*) *the smallest positive root r_f' of the equation*

$$1+\frac{2rM_\nu''(r)}{M_\nu'(r)}+\left(\frac{1}{\nu}-1\right)\frac{2rM_\nu'(r)}{M_\nu(r)}=0$$

in the interval $(0,\alpha')$, where $0<\nu\leqslant 1$, $\frac{Q(\nu+2)}{Q(\nu)}<0$, $Q(\nu)/(b-a)>0$ and $a/(b-a)<0$.

Proof. (*i*) From [9], we know that

$$\frac{zf_\nu''(z)}{f_\nu'(z)}=\frac{zN_\nu''(z)}{N_\nu'(z)}+\left(\frac{1}{\nu}-1\right)\frac{zN_\nu'(z)}{N_\nu(z)}.$$

Using the Weierstrass decomposition of $N_\nu(z)$ and $N_\nu'(z)$ given by

$$N_\nu(z)=\frac{Q(\nu)z^\nu}{2^\nu\Gamma(\nu+1)}\prod_{n\geqslant 1}\left(1-\frac{z^2}{\lambda_{\nu,n}^2}\right) \text{ and } N_\nu'(z)=\frac{Q(\nu)\nu z^{\nu-1}}{2^\nu\Gamma(\nu+1)}\prod_{n\geqslant 1}\left(1-\frac{z^2}{\lambda_{\nu,n}'^2}\right),$$

where $\lambda_{\nu,n}$ and $\lambda_{\nu,n}'$ are the nth positive roots of N_ν and N_ν', respectively, we have

$$1+\frac{zf_\nu''(z)}{f_\nu'(z)}=1-\left(\frac{1}{\nu}-1\right)\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2-z^2}-\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}'^2-z^2}. \tag{29}$$

We now consider the cases $\max\{0,\nu_0\}\leqslant\nu\leqslant 1$ $(\nu\neq 0)$ and $\nu>1$ separately.

Case 1. Let $\max\{0,\nu_0\}\leqslant\nu\leqslant 1$ $(\nu\neq 0)$. Then, using (9), we get

$$\operatorname{Re}\left(\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2-z^2}\right)\leqslant\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}^2-r^2},\quad |z|\leqslant r<\lambda_{\nu,1}'<\lambda_{\nu,1}$$

and

$$\operatorname{Re}\left(\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}'^2-z^2}\right)\leqslant\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}'^2-r^2},\quad |z|\leqslant r<\lambda_{\nu,1}'<\lambda_{\nu,1}.$$

Thus, we have

$$\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right) \geqslant 1-\left(\frac{1}{\nu}-1\right)\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}^2-r^2}-\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}'^2-r^2}$$
$$=1+\frac{rf_\nu''(r)}{f_\nu'(r)} \tag{30}$$

where $|z|=r$ and $r\in\left(0,\lambda_{\nu,1}'\right)$.

On the other hand, using the triangle inequality $|z_1+z_2|\leqslant|z_1|+|z_2|$ in (29) together with the fact that $\frac{1}{\nu}-1\geqslant 0,$ we obtain

$$\left|\frac{zf_\nu''(z)}{f_\nu'(z)}\right| \leqslant \left(\frac{1}{\nu}-1\right)\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}^2-r^2}+\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}'^2-r^2}$$
$$=-\frac{rf_\nu''(r)}{f_\nu'(r)}. \tag{31}$$

When (30) and (31) are considered together, we conclude that

$$\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)-\left|\frac{zf_\nu''(z)}{f_\nu'(z)}\right| \geqslant 1+\frac{2rf_\nu''(r)}{f_\nu'(r)}.$$

Case 2. Let $\nu>1.$ In the case, using the inequalities (8) and (7), respectively, we get

$$\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right) = 1-\sum_{n\geqslant 1}\operatorname{Re}\left(\frac{2z^2}{\lambda_{\nu,n}'^2-z^2}-\left(\frac{1}{\nu}-1\right)\frac{2z^2}{\lambda_{\nu,n}^2-z^2}\right)$$
$$\geqslant 1-\sum_{n\geqslant 1}\left(\frac{2r^2}{\lambda_{\nu,n}'^2-r^2}-\left(\frac{1}{\nu}-1\right)\frac{2r^2}{\lambda_{\nu,n}^2-r^2}\right)$$
$$=1+\frac{rf_\nu''(r)}{f_\nu'(r)} \tag{32}$$

and

$$\frac{zf_\nu''(z)}{f_\nu'(z)} = \left|\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}'^2-z^2}-\left(1-\frac{1}{\nu}\right)\sum_{n\geqslant 1}\frac{2z^2}{\lambda_{\nu,n}^2-z^2}\right|$$
$$\leqslant \sum_{n\geqslant 1}\left|\frac{2z^2}{\lambda_{\nu,n}'^2-z^2}-\left(1-\frac{1}{\nu}\right)\frac{2z^2}{\lambda_{\nu,n}^2-z^2}\right|$$
$$\leqslant \sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}'^2-r^2}-\left(1-\frac{1}{\nu}\right)\sum_{n\geqslant 1}\frac{2r^2}{\lambda_{\nu,n}^2-r^2}$$
$$=-\frac{rf_\nu''(r)}{f_\nu'(r)}. \tag{33}$$

Thus, from (32) and (33), we get

$$\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)-\left|\frac{zf_\nu''(z)}{f_\nu'(z)}\right| \geqslant 1+\frac{2rf_\nu''(r)}{f_\nu'(r)}.$$

Equality holds in last inequality if and only if $z=r$. Thus it follows that

$$\inf_{|z|<r}\left[\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)-\left|\frac{zf_\nu''(z)}{f_\nu'(z)}\right|\right]=1+\frac{2rf_\nu''(r)}{f_\nu'(r)},$$

where $r\in\left(0,\lambda_{\nu,1}'\right)$. Now, we define the function $\Phi_\nu(r):\left(0,\lambda_{\nu,1}'\right)\to\mathbb{R}$ by

$$\Phi_\nu(r)=1+\frac{2rf_\nu''(r)}{f_\nu'(r)}=1-2\left(\sum_{n\geqslant1}\frac{2r^2}{\lambda_{\nu,n}'^2-r^2}-\left(1-\frac{1}{\nu}\right)\sum_{n\geqslant1}\frac{2r^2}{\lambda_{\nu,n}^2-r^2}\right).$$

It can be easily seen that

$$\begin{aligned}\Phi_\nu'(r) &= \left(1-\frac{1}{\nu}\right)\sum_{n\geqslant1}\frac{8r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2-r^2\right)^2}-\sum_{n\geqslant1}\frac{8r\lambda_{\nu,n}'^2}{\left(\lambda_{\nu,n}'^2-r^2\right)^2}\\ &< \sum_{n\geqslant1}\frac{8r\lambda_{\nu,n}^2}{\left(\lambda_{\nu,n}^2-r^2\right)^2}-\sum_{n\geqslant1}\frac{8r\lambda_{\nu,n}'^2}{\left(\lambda_{\nu,n}'^2-r^2\right)^2}\\ &= 8r\sum_{n\geqslant1}\frac{(\lambda_{\nu,n})^2\left(\lambda_{\nu,n}'^2-r^2\right)^2-\left(\lambda_{\nu,n}'\right)^2\left(\lambda_{\nu,n}^2-r^2\right)^2}{\left(\lambda_{\nu,n}^2-r^2\right)^2\left(\lambda_{\nu,n}'^2-r^2\right)^2}<0,\end{aligned}$$

since $(\lambda_{\nu,n})^2\left(\lambda_{\nu,n}'^2-r^2\right)^2<\left(\lambda_{\nu,n}'\right)^2\left(\lambda_{\nu,n}^2-r^2\right)^2$. Thus, Φ_ν is a strictly decreasing function, $\lim_{r\searrow0}\Phi_\nu(r)=1>0$ and $\lim_{r\nearrow\lambda_{\nu,1}'}\Phi_\nu(r)=-\infty$. This means that

$$\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)-\left|\frac{zf_\nu''(z)}{f_\nu'(z)}\right|>0$$

for all $r^{ucv}(f_\nu)\in\left(0,\lambda_{\nu,1}'\right)$ where $r^{ucv}(f_\nu)$ is the unique root of $1+\frac{2rf_\nu''(r)}{f_\nu'(r)}=0$ or

$$1+\frac{2rN_\nu''(r)}{N_\nu'(r)}+\left(\frac{1}{\nu}-1\right)\frac{2rN_\nu'(r)}{N_\nu(r)}=0.$$

(ii) By using a result of Ismail and Muldoon [8], on zeros of the function $N_\nu(z)$, the conditions $0<\nu\leqslant1$, $Q(\nu)/(b-a)>0$ and $a/(b-a)<0$ implies that $\lambda_{\nu,1}=i\alpha$, $\lambda_{\nu,1}'=i\alpha'$ and $\alpha,\alpha'>0$, $\lambda_{\nu,n},\lambda_{\nu,n}'>0$ for $n\in\{2,3,\dots\}$. Thus, from (29), we have

$$\begin{aligned}1+\frac{zf_\nu''(z)}{f_\nu'(z)} &= 1+\left(\frac{1}{\nu}-1\right)\left[2\frac{\alpha^2z^2}{\alpha^2+z^2}\left(\sum_{n\geqslant2}\frac{1}{\lambda_{\nu,n}^2}-\frac{Q(\nu+2)}{4(\nu+1)Q(\nu)}\right)-2\sum_{n\geqslant2}\frac{z^2}{\lambda_{\nu,n}^2-z^2}\right]\\ &\quad+2\frac{\alpha'^2z^2}{\alpha'^2+z^2}\left(\sum_{n\geqslant2}\frac{1}{\lambda_{\nu,n}'^2}-\frac{(\nu+2)Q(\nu+2)}{4\nu(\nu+1)Q(\nu)}\right)-2\sum_{n\geqslant2}\frac{z^2}{\lambda_{\nu,n}'^2-z^2}.\end{aligned}\tag{34}$$

Here, we used [9]

$$\frac{1}{\alpha^2}=\sum_{n\geqslant 2}\frac{1}{\lambda_{\nu,n}^2}-\frac{Q(\nu+2)}{4(\nu+1)Q(\nu)} \quad \text{and} \quad \frac{1}{\alpha'^2}=\sum_{n\geqslant 2}\frac{1}{\lambda'^2_{\nu,n}}-\frac{(\nu+2)Q(\nu+2)}{4\nu(\nu+1)Q(\nu)}.$$

In [10], for $0<\nu\leqslant 1,$ the following inequality has been proved

$$\begin{aligned}
&\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)\\
&\geqslant 1+\left(\frac{1}{\nu}-1\right)\left[\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2}-2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}\right]\\
&\quad+\frac{\alpha'^2(\nu+2)Q(\nu+2)}{2\nu(\nu+1)Q(\nu)}\frac{r^2}{\alpha'^2-r^2}-2\sum_{n\geqslant 2}\frac{\alpha'^2+\lambda'^2_{\nu,n}}{\lambda'^2_{\nu,n}}\frac{r^4}{(\alpha'^2-r^2)\left(\lambda'^2_{\nu,n}+r^2\right)}\\
&=1+\frac{irf_\nu''(ir)}{f_\nu'(ir)},\quad |z|\leqslant r<\alpha'.
\end{aligned}\tag{35}$$

On the other hand, if in inequalities (9) and (10) we replace z by $z^2,$ a by $\alpha,$ b by $\lambda_{\nu,n}$ and taking into account that $\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}<0,$ we get the following inequality:

$$\begin{aligned}
&\left|\frac{zf_\nu''(z)}{f_\nu'(z)}\right|\\
&\leqslant\left|-\left(\frac{1}{\nu}-1\right)\left[\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\alpha^2+z^2}+2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)}\right]\right.\\
&\quad\left.-\frac{\alpha'^2(\nu+2)Q(\nu+2)}{2\nu(\nu+1)Q(\nu)}\frac{z^2}{\alpha'^2+z^2}-2\sum_{n\geqslant 2}\frac{\alpha'^2+\lambda'^2_{\nu,n}}{\lambda'^2_{\nu,n}}\frac{z^4}{(\alpha'^2+z^2)\left(\lambda'^2_{\nu,n}-z^2\right)}\right|\\
&\leqslant\left(\frac{1}{\nu}-1\right)\left[-\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\left|\frac{z^2}{\alpha^2+z^2}\right|+2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\left|\frac{z^4}{(\alpha^2+z^2)\left(\lambda_{\nu,n}^2-z^2\right)}\right|\right]\\
&\quad-\frac{\alpha'^2(\nu+2)Q(\nu+2)}{2\nu(\nu+1)Q(\nu)}\left|\frac{z^2}{\alpha'^2+z^2}\right|+2\sum_{n\geqslant 2}\frac{\alpha'^2+\lambda'^2_{\nu,n}}{\lambda'^2_{\nu,n}}\left|\frac{z^4}{(\alpha'^2+z^2)\left(\lambda'^2_{\nu,n}-z^2\right)}\right|\\
&\leqslant\left(\frac{1}{\nu}-1\right)\left[-\frac{\alpha^2Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\alpha^2-r^2}+2\sum_{n\geqslant 2}\frac{\alpha^2+\lambda_{\nu,n}^2}{\lambda_{\nu,n}^2}\frac{r^4}{(\alpha^2-r^2)\left(\lambda_{\nu,n}^2+r^2\right)}\right]\\
&\quad-\frac{\alpha'^2(\nu+2)Q(\nu+2)}{2\nu(\nu+1)Q(\nu)}\frac{r^2}{\alpha'^2-r^2}+2\sum_{n\geqslant 2}\frac{\alpha'^2+\lambda'^2_{\nu,n}}{\lambda'^2_{\nu,n}}\frac{r^4}{(\alpha'^2-r^2)\left(\lambda'^2_{\nu,n}+r^2\right)}\\
&=-\frac{irf_\nu''(ir)}{f_\nu'(ir)},\quad |z|\leqslant r<\alpha'.
\end{aligned}\tag{36}$$

Thus, from inequalities (35) and (36) we conclude that

$$\inf_{|z|<r}\left[\operatorname{Re}\left(1+\frac{zf_\nu''(z)}{f_\nu'(z)}\right)-\left|\frac{zf_\nu''(z)}{f_\nu'(z)}\right|\right]=1+\frac{2irf_\nu''(ir)}{f_\nu'(ir)}$$

for $r\in(0,\alpha')$. Now, we consider the function $\Upsilon_\nu:(0,\alpha')\longrightarrow\mathbb{R}$, defined by

$$\Upsilon_\nu(r)=1+\frac{2irf_\nu''(ir)}{f_\nu'(ir)}$$
$$=1-\left(\frac{1}{\nu}-1\right)\left[4\frac{r^2}{\alpha^2-r^2}-\sum_{n\geqslant 2}\frac{4r^2}{\lambda_{\nu,n}^2+r^2}\right]-4\frac{r^2}{\alpha'^2-r^2}+\sum_{n\geqslant 2}\frac{4r^2}{\lambda_{\nu,n}'^2+r^2}.$$

Since $\Upsilon_\nu(r)=1+\frac{2irf_\nu''(ir)}{f_\nu'(ir)}$ is strictly decreasing, $\lim_{r\searrow 0}\Upsilon_\nu(r)=1>0$ and $\lim_{r\nearrow\alpha'}\Upsilon_\nu(r)=-\infty$ it follows that the equation $1+\frac{2irf_\nu''(ir)}{f_\nu'(ir)}=0$ has a unique root $r^{ucv}(f_\nu)\in(0,\alpha')$. □

Taking $\nu=1/2$ in Theorem 3, we have the following corollary.

COROLLARY 3. *The following statements are true.*

(*i*) *The $\mathcal{UCV}$-radius of the function $f_{1/2}$ is the smallest positive root of the equation*

$$1+\frac{2\left(4ar^2-a-4c\right)r\cos r+\left(4ar^2+8br^2+A\right)\sin r}{4(a-b)r\cos r+(4ar^2-A)\sin r}$$
$$+\frac{4\left(4(2a+b)r^2+2a-b+4c\right)r\cos r+\left(-16ar^4+8(a+b+2c)r^2-3A\right)\sin r}{2(4ar^2-a-4c)r\cos r+(4ar^2+8br^2+A)\sin r}$$
$$=0.$$

(*ii*) *Let* $\frac{Q(5/2)}{Q(1/2)}<0$, $Q(1/2)/(b-a)>0$ *and* $a/(b-a)<0$. *Then the $\mathcal{UCV}$-radius of the function $f_{1/2}$ is the smallest positive root of the equation*

$$1+\frac{2\left(4ar^2+a+4c\right)r\cosh r+\left(4ar^2+8br^2-A\right)\sinh r}{4(-a+b)r\cosh r+(4ar^2+A)\sinh r}$$
$$+\frac{4\left(4(2a+b)r^2-2a+b-4c\right)r\cosh r+\left(16ar^4+8(a+b+2c)r^2+3A\right)\sinh r}{2(4ar^2+a+4c)r\cos r+(4ar^2+8br^2-A)\sin r}$$
$$=0,$$

where $A=3a-2b+4c$.

THEOREM 4. *The $\mathcal{UCV}$-radius of the function g_ν is*

(*i*) *the smallest positive root $r^{ucv}(g_\nu)$ of the equation*

$$1+2\frac{r^2N_\nu''(r)+2(1-\nu)rN_\nu'(r)+(\nu^2-\nu)N_\nu(r)}{rN_\nu'(r)+(1-\nu)N_\nu(r)}=0$$

for $\nu\geqslant\max\{0,\nu_0\}$, *where* ν_0 *is the largest real root of the quadratic polynomial* $Q(\nu)=a\nu(\nu-1)+b\nu+c$.

(ii) *the smallest positive root r_g' of the equation*

$$1+2\frac{r^2M_\nu''(r)+2(1-\nu)rM_\nu'(r)+\left(\nu^2-\nu\right)M_\nu(r)}{rM_\nu'(r)+(1-\nu)M_\nu(r)}=0$$

in the interval $(0,\theta)$, where $\nu>0$, $\frac{Q(\nu+2)}{Q(\nu)}<0$, $Q(\nu)\diagup(b-a)>0$ and $a\diagup(b-a)<0$.

Proof. (i) In [9], we find that

$$1+\frac{zg_\nu''(z)}{g_\nu'(z)}=1-\sum_{n\geqslant 1}\frac{2z^2}{\delta_{\nu,n}^2-z^2} \tag{37}$$

and

$$\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)\geqslant 1-\sum_{n\geqslant 1}\frac{2r^2}{\delta_{\nu,n}^2-r^2}=1+\frac{rg_\nu''(r)}{g_\nu'(r)}, \tag{38}$$

where $\delta_{\nu,n}$ are the nth positive zeros of the function $g_\nu'(z)$. Equality (37) also implies that

$$\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right|\leqslant\left|\sum_{n\geqslant 1}\frac{2z^2}{\delta_{\nu,n}^2-z^2}\right|\leqslant\sum_{n\geqslant 1}\frac{2r^2}{\delta_{\nu,n}^2-r^2}=-\frac{rg_\nu''(r)}{g_\nu'(r)},\quad r\in(0,\delta_{\nu,1}). \tag{39}$$

Now summarizing (38) and (39) we get

$$\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)-\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right|\geqslant 1-\sum_{n\geqslant 1}\frac{4r^2}{\delta_{\nu,n}^2-r^2}=1+\frac{2rg_\nu''(r)}{g_\nu'(r)},$$

where $|z|=r$. If we define

$$\Psi_\nu(r)=1+\frac{2rg_\nu''(r)}{g_\nu'(r)},\quad r\in(0,\delta_{\nu,1})$$

then Ψ_ν is strictly decreasing, $\lim_{r\searrow 0}\Psi_\nu(r)=1>0$ and $\lim_{r\nearrow\delta_{\nu,1}}\Psi_{\nu,n}(r)=-\infty$. Consequently, the equation

$$1+\frac{2rg_\nu''(r)}{g_\nu'(r)}=0$$

which is equivalent to

$$1+2\frac{r^2N_\nu''(r)+2(1-\nu)rN_\nu'(r)+(\nu^2-\nu)N_\nu(r)}{rN_\nu'(r)+(1-\nu)N_\nu(r)}=0$$

has a unique root say $r^{ucv}(g_\nu)$ in $(0,\delta_{\nu,1})$. Thus

$$\operatorname{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right)-\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right|>0$$

for all $z \in (0, r^{ucv}(g_\nu))$. This completes the proof of part (i).

(ii) The result of Ismail and Muldoon [8] on zeros of the function $N_\nu(z)$ under the conditions $\nu > 0$, $Q(\nu)/(b-a) > 0$ and $a/(b-a) < 0$ implies that $\delta_{\nu,1} = i\theta$, $\theta > 0$ and $\delta_{\nu,n} > 0$ for $n = 2,3,\ldots$. From (37), we have

$$1 + \frac{zg_\nu''(z)}{g_\nu'(z)} = 1 + 2\frac{z^2}{\theta^2 + z^2} - 2\sum_{n\geqslant 2} \frac{z^2}{\delta_{\nu,n}^2 - z^2}$$
$$= 1 + 2\frac{\theta^2 z^2}{\theta^2 + z^2}\frac{1}{\theta^2} - 2\sum_{n\geqslant 2} \frac{z^2}{\delta_{\nu,n}^2 - z^2}.$$

In [10], authors find that

$$\mathrm{Re}\left(1 + \frac{zg_\nu''(z)}{g_\nu'(z)}\right) \geqslant 1 + ir\frac{g_\nu''(ir)}{g_\nu'(ir)},$$

for $|z| \leqslant r < \theta$. Since $\frac{3\theta^2 Q(\nu+2)}{2(\nu+1)Q(\nu)} < 0$, the inequality (10) implies that

$$\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right| = \left| -\frac{3\theta^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{z^2}{\theta^2+z^2} - 2\sum_{n\geqslant 2}\frac{\theta^2 + \delta_{\nu,n}^2}{\delta_{\nu,n}^2}\frac{z^4}{(\theta^2+z^2)\left(\delta_{\nu,n}^2 - z^2\right)}\right|$$
$$\leqslant -\frac{3\theta^2 Q(\nu+2)}{2(\nu+1)Q(\nu)}\frac{r^2}{\theta^2 - r^2} + 2\sum_{n\geqslant 2}\frac{\theta^2 + \delta_{\nu,n}^2}{\delta_{\nu,n}^2}\frac{r^4}{(\theta^2 - r^2)\left(\delta_{\nu,n}^2 + r^2\right)}$$
$$= -ir\frac{g_\nu''(ir)}{g_\nu'(ir)}.$$

Hence, we have

$$\mathrm{Re}\left(1 + \frac{zg_\nu''(z)}{g_\nu'(z)}\right) - \left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right| \geqslant 1 + 2ir\frac{g_\nu''(ir)}{g_\nu'(ir)}.$$

Consider the function

$$\Omega_\nu(r) = 1 + 2ir\frac{g_\nu''(ir)}{g_\nu'(ir)}.$$

It can be easily seen that $\Omega_\nu(r)$ is strictly decreasing,

$$\lim_{r\searrow 0}\Omega_\nu(r) = 1 > 0 \quad \text{and} \quad \lim_{r\nearrow \theta}\Omega_\nu(r) = -\infty.$$

Thus, the equation $\Omega_\nu(r) = 0$ has a unique root r_g' in $(0,\theta)$. By using the relation $M_\nu(z) = i^{-\nu}N_\nu(iz)$, the equation $\Omega_\nu(r) = 0$ is equivalent to

$$1 + 2\frac{r^2 M_\nu''(r) + 2(1-\nu)rM_\nu'(r) + \left(\nu^2 - \nu\right)M_\nu(r)}{rM_\nu'(r) + (1-\nu)M_\nu(r)} = 0. \quad \square$$

Taking $\nu = 1/2$ in Theorem 4, we have the following corollary.

COROLLARY 4. *The following statements are true.*

(i) *The $\mathcal{UCV}$-radius of the function $g_{1/2}$ is the smallest positive root of the equation*

$$\frac{\left[4\left(7a+2b\right)r^{2}-3a+2b-4c\right]\cos r+\left[a\left(-8r^{4}+br^{2}+3\right)+8\left(2b+c\right)r^{2}-2b+4c\right]\sin r}{\left[\left(4ar^{2}+a-2b-4c\right)\cos r+4\left(a+b\right)r\sin r\right]r}=0.$$

(ii) *Let $\frac{Q(5/2)}{Q(1/2)}<0$, $Q(1/2)/(b-a)>0$ and $a/(b-a)<0$. Then the $\mathcal{UCV}$-radius of the function $g_{1/2}$ is the smallest positive root of the equation*

$$\frac{\left[4\left(7a+2b\right)r^{2}+3a-2b+4c\right]\cosh r+\left[8ar^{4}+2\left(3a+8b+4c\right)r^{2}-3a+2b-4c\right]\sinh r}{\left[\left(4ar^{2}-a+2b+4c\right)\cosh r+4\left(a+b\right)r\sinh r\right]r}=0.$$

THEOREM 5. *The $\mathcal{UCV}$-radius of the function h_ν is*

(i) *the smallest positive root $r^{ucv}(h_\nu)$ of the equation*

$$1+\frac{rN_{\nu}''(\sqrt{r})+(3-2\nu)\sqrt{r}N_{\nu}'(\sqrt{r})+\left(\nu^{2}-2\nu\right)N_{\nu}(\sqrt{r})}{\sqrt{r}N_{\nu}'(\sqrt{r})+(2-\nu)N_{\nu}(\sqrt{r})}=0$$

for $\nu\geqslant\max\{0,\nu_0\}$, where ν_0 is the largest real root of the quadratic polynomial $Q(\nu)=a\nu(\nu-1)+b\nu+c$.

(ii) *the smallest positive root r_h' of the equation*

$$1+\frac{rM_{\nu}''(\sqrt{r})+(3-2\nu)\sqrt{r}M_{\nu}'(\sqrt{r})+\left(\nu^{2}-2\nu\right)M_{\nu}(\sqrt{r})}{\sqrt{r}M_{\nu}'(\sqrt{r})+(2-\nu)M_{\nu}(\sqrt{r})}=0$$

in the interval $(0,\kappa)$, where $\nu>0$, $\frac{Q(\nu+2)}{Q(\nu)}<0$, $Q(\nu)/(b-a)>0$ and $a/(b-a)<0$.

Proof. (i) In [9], we know that

$$1+\frac{zh_{\nu}''(z)}{h_{\nu}'(z)}=1-\sum_{n\geqslant 1}\frac{z}{\gamma_{\nu,n}^{2}-z} \tag{40}$$

and

$$\operatorname{Re}\left(1+\frac{zh_{\nu}''(z)}{h_{\nu}'(z)}\right)\geqslant 1-\sum_{n\geqslant 1}\frac{r}{\gamma_{\nu,n}^{2}-r}=1+\frac{rh_{\nu}''(r)}{h_{\nu}'(r)}, \tag{41}$$

where $\gamma_{\nu,n}$ are the nth positive zeros of the function $h_\nu'(z)$. Equality (40) also implies that

$$\left|\frac{zh_{\nu}''(z)}{h_{\nu}'(z)}\right|\leqslant\left|\sum_{n\geqslant 1}\frac{z}{\gamma_{\nu,n}^{2}-z}\right|\leqslant\sum_{n\geqslant 1}\frac{r}{\gamma_{\nu,n}^{2}-r}=-\frac{rh_{\nu}''(r)}{h_{\nu}'(r)},\quad r\in(0,\gamma_{\nu,1}). \tag{42}$$

Thus, from (41) and (42) we conclude that

$$\operatorname{Re}\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)-\left|\frac{zh_\nu''(z)}{h_\nu'(z)}\right|\geqslant 1-\sum_{n\geqslant 1}\frac{2z}{\gamma_{\nu,n}^2-z}=1+\frac{2rh_\nu''(r)}{h_\nu'(r)},\quad |z|=r.$$

Define

$$\Theta_\nu(r)=1+\frac{2rh_\nu''(r)}{h_\nu'(r)},\quad r\in\left(0,\gamma_{\nu,1}^2\right).$$

Then, it can be easily seen that $\Theta_\nu(r)$ is strictly decreasing, $\lim_{r\searrow 0}\Theta_\nu(r)=1>0$ and $\lim_{r\nearrow\gamma_{\nu,1}^2}\Theta_\nu(r)=-\infty$. Consequently, the equation $1+\frac{2rh_\nu''(r)}{h_\nu'(r)}=0$ which is equivalent to

$$1+2\frac{rh_\nu''(r)}{h_\nu'(r)}=1+\frac{rN_\nu''(\sqrt{r})+(3-2\nu)\sqrt{r}N_\nu'(\sqrt{r})+\left(\nu^2-2\nu\right)N_\nu(\sqrt{r})}{\sqrt{r}N_\nu'(\sqrt{r})+(2-\nu)N_\nu(\sqrt{r})}=0$$

has a unique root say $r^{ucv}(h_\nu)$ in $\left(0,\gamma_{\nu,1}^2\right)$.

(ii) From a result of Ismail and Muldoon [8], on zeros of the function $N_\nu(z)$, the conditions $\nu>0$, $Q(\nu)/(b-a)>0$ and $a/(b-a)<0$ we say that $\gamma_{\nu,1}=i\kappa$, $\kappa>0$ and $\gamma_{\nu,n}>0$ for $n=2,3,\ldots$. Also from (40), we get

$$1+\frac{zh_\nu''(z)}{h_\nu'(z)}=1+\frac{z}{\kappa^2+z}-\sum_{n\geqslant 2}\frac{z}{\gamma_{\nu,n}^2-z}=1+\frac{\kappa^2 z}{\kappa^2+z}\frac{1}{\kappa^2}-\sum_{n\geqslant 2}\frac{z}{\gamma_{\nu,n}^2-z}.$$

In [10], the following inequality has been proved

$$\operatorname{Re}\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)\geqslant 1-r\frac{h_\nu''(-r)}{h_\nu'(-r)}$$

for $|z|\leqslant r<\kappa^2$.

On the other hand, similarly to Theorem 4, we obtain the following inequality

$$\left|\frac{zh_\nu''(z)}{h_\nu'(z)}\right|\leqslant r\frac{h_\nu''(-r)}{h_\nu'(-r)},$$

where $|z|\leqslant r<\kappa^2$. Consequently, we have

$$\operatorname{Re}\left(1+\frac{zh_\nu''(z)}{h_\nu'(z)}\right)-\left|\frac{zh_\nu''(z)}{h_\nu'(z)}\right|\geqslant 1-2r\frac{h_\nu''(-r)}{h_\nu'(-r)}.$$

We consider the function

$$\Delta_\nu(r)=1-2r\frac{h_\nu''(-r)}{h_\nu'(-r)},\quad r\in\left(0,\kappa^2\right).$$

This function is strictly decreasing, $\lim_{r\searrow 0}\Delta_\nu(r)=1>0$ and $\lim_{r\nearrow\kappa^2}\Delta_\nu(r)=-\infty$. Thus, it follows that the equation $\Delta_\nu(r)=0$ has a unique root r_h' in $\left(0,\kappa^2\right)$. Also, the equation $\Delta_\nu(r)=1-2r\frac{h_\nu''(-r)}{h_\nu'(-r)}=0$ is equivalent to

$$1+\frac{rM_\nu''(\sqrt{r})+(3-2\nu)\sqrt{r}M_\nu'(\sqrt{r})+\left(\nu^2-2\nu\right)M_\nu(\sqrt{r})}{\sqrt{r}M_\nu'(\sqrt{r})+(2-\nu)M_\nu(\sqrt{r})}=0.\quad\square$$

Taking $\nu = 1/2$ in Theorem 5, we have the following corollary.

COROLLARY 5. *The following statements are true.*

(*i*) *The* $\mathcal{UCV}$ *-radius of the function* $h_{1/2}$ *is the smallest positive root of the equation*

$$\frac{2(10ar+2br+a-2b-4c)\sqrt{r}\cos\sqrt{r}+(-4ar+11a+14b+4c)r\sin\sqrt{r}}{(4ar+5a-6b-4c)\sqrt{r}\cos\sqrt{r}+(8ar+4br-3a+2b-4c)\sin\sqrt{r}}=0.$$

(*ii*) *Let* $\frac{Q(5/2)}{Q(1/2)}<0$, $Q(1/2)/(b-a)>0$ *and* $a/(b-a)<0$. *Then the* $\mathcal{UCV}$ *-radius of the function* $h_{1/2}$ *is the smallest positive root of the equation*

$$\frac{2(10ar+2br-a+2b+4c)\sqrt{r}\cosh\sqrt{r}+(4ar+11a+14b+4c)r\sinh\sqrt{r}}{(4ar-5a+6b+4c)\sqrt{r}\cosh\sqrt{r}+(8ar+4br+3a-2b+4c)\sinh\sqrt{r}}=0.$$

Table 3: *Radii of uniformly convexity for* f_ν, g_ν *and* h_ν *for* $\nu = 1/2$

		$r^{ucv}(f_{1/2})$	$r^{ucv}(g_{1/2})$	$r^{ucv}(h_{1/2})$
$b=3$	$a=2$	0.1190	0.1643	0.0684
and	$a=3$	0.0921	0.1270	0.0407
$c=0$	$a=4$	0.0686	0.0945	0.0224
$a=-1$	$b=2$	0.1351	0.1865	0.0886
and	$b=3$	0.1539	0.2127	0.1154
$c=0$	$b=4$	0.1648	0.2278	0.1326
$a=1$	$c=2$	0.2349	0.3264	0.2796
and	$c=3$	0.2629	0.3660	0.3553
$b=2$	$c=4$	0.2845	0.3966	0.4208

Table 3 is related with radii of uniformly convexity for f_ν, g_ν and h_ν in special cases of a, b and c for $\nu = 1/2$ in the case (i) of Theorem 3, Theorem 4 and Theorem 5.

Table 4: *Radii of uniformly convexity for* f_ν, g_ν *and* h_ν *for* $\nu = 1/2$

		$r^{ucv}(f_{1/2})$	$r^{ucv}(g_{1/2})$	$r^{ucv}(h_{1/2})$
$b=-5$	$a=4$	0.4428	0.5591	0.6045
and	$a=5$	0.3311	0.4360	0.4059
$c=0$	$a=6$	0.2786	0.3726	0.3103
$a=-5$	$b=4$	0.3208	0.4251	0.3929
and	$b=5$	0.3902	0.5055	0.5228
$c=1$	$b=6$	0.4867	0.6072	0.6956
$a=-2$	$c=4$	0.3566	0.4745	0.4949
and	$c=5$	0.4236	0.5553	0.6499
$b=-1$	$c=6$	0.4947	0.6371	0.8173

Table 4 also is related with radii of uniformly convexity for f_ν, g_ν and h_ν in special cases of a, b and c for $\nu = 1/2$ in the case (ii) of Theorem 3, Theorem 4 and Theorem 5.

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