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SHARP ESTIMATE OF THE REMAINDER OF SOME ALTERNATING SERIES

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Abstract. For any two real numbers $\alpha > 0$ and $\beta > -\alpha$, we show that the best constants a and b (the smallest a and the largest b) such that the inequalities

$$\frac{1}{2\alpha n + a} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{\alpha k + \beta} \right| < \frac{1}{2\alpha n + b}$$

hold for every $n \geqslant 1$ are $a = \left(\frac{1}{\alpha + \beta} - S(\alpha, \beta) \right)^{-1} - 2\alpha$ and $b = \alpha + 2\beta$, where $S(\alpha, \beta) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\alpha n + \beta}$. In particular, we recover the main result of [6] and answer a question, stated in [6], about the Gregory-Leibniz series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$. More precisely, we show that the best constants c and d such that the inequalities

$$\frac{1}{4n + c} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \right| < \frac{1}{4n + d}$$

hold for every $n \geqslant 1$ are $c = \frac{4}{4-\pi} - 4$ and $d = 0$.

1. Introduction

Let $f : [1, \infty) \longrightarrow (0, \infty)$ be a function, satisfying the following properties:

(i) $f(n+1) < f(n)$, for all $n \in \mathbb{N}$.

(ii) $\lim\limits_{n \longrightarrow \infty} f(n) = 0$.

Throughout this paper, we denote by $g(n) := \frac{1}{f(n)}$.

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Consider the Leibniz series $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$. We denote by

$$R_n = \sum_{k=1}^{\infty}(-1)^{n+k-1}f(n+k),$$

$$\Delta f(n) := f(n+1) - f(n).$$

We assume $\Delta f(n) < \Delta f(n+1)$, then according to [2, Theorem 1.2], the following inequalities hold:

$$\frac{f(n+1)}{2} < |R_n| < \frac{f(n)}{2}.$$

The above inequalities can be rewritten as follows:

$$\frac{1}{2g(n+1)} < |R_n| < \frac{1}{2g(n)}.$$

A natural question is :which are the best constants ρ and σ (the largest ρ and the largest σ) such that the inequalities

$$\frac{1}{2g(n+1)-\rho} < |R_n| < \frac{1}{2g(n)+\sigma} \tag{1}$$

hold, for every $n \geqslant 1$?

Similar questions have been stated (cf. [6]) for the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ and the Gregory-Leibniz series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$.

Indeed, in [5], the author proved that the inequalities:

$$\frac{1}{2n+a} < \left|\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k}\right| < \frac{1}{2n+b} \tag{2}$$

and

$$\frac{1}{4n+c} < \left|\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\right| < \frac{1}{4n+d} \tag{3}$$

hold for every $n \geqslant 1$, where $a = 2\sqrt{7}-4$, $b = 1$, $c = 2\sqrt{19}-8$ and $d = 0$. In addition, in [6], the authors proved that $a = \frac{1}{1-\log 2} - 2$ and $b = 1$ are the best constants in Inequalities (2). The authors have also asked about the best constants c and d involved in (3).

More generally, let α, β be given real numbers such that $\alpha > 0$ and $\beta > -\alpha$. The aim of this paper is to find the best constants ρ, σ involved in (1), for $g(n) = \alpha n + \beta$. As a consequence, we answer the question about the Gregory-Leibniz series stated in [6].

We also express the n-th remainder $|R_n|$ of alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\alpha n+\beta}$ in terms of hypergeometric functions. This enables us establishing an estimation of $|R_n|$.

2. The best constants ρ,σ such that $\dfrac{1}{2g(n+1)-\rho} < |R_n| < \dfrac{1}{2g(n)+\sigma}$

Let $\sum_{n=1}^{\infty}(-1)^{n-1}f(n)$ be a Leibniz series such that $\Delta f(n) < \Delta f(n+1)$, for all integers $n \geqslant 1$. Consider the sequences (x_n) and (y_n) defined by

$$|R_n| = \frac{1}{2g(n)+x_n} = \frac{1}{2g(n+1)-y_n}. \tag{4}$$

Then the best constants ρ,σ (the greatest ρ and greatest σ) such that

$$\frac{1}{2g(n+1)-\rho} < |R_n| < \frac{1}{2g(n)+\sigma}, \tag{5}$$

for all integers $n \geqslant 1$ are given by

$$\rho = \inf\{y_n : n \geqslant 1\} \text{ and } \sigma = \inf\{x_n : n \geqslant 1\}.$$

Hence in order to determine ρ and σ, it suffices to study the monotonicity of the sequences (x_n) and (y_n).

THEOREM 1. *The best constants ρ,σ of the Leibniz series $S(\alpha,\beta) := \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{\alpha n+\beta}$ are given by $\rho = 2(2\alpha+\beta) - \left(\dfrac{1}{\alpha+\beta} - S(\alpha,\beta)\right)^{-1}$ and $\sigma = \alpha$,*

The proof is based on the following lemma.

LEMMA 1. *For $g(n) = \alpha n+\beta$, the sequences (x_n) and (y_n) defined in Equality 4 satisfy the following properties.*

1. *(x_n) is decreasing to α.*
2. *(y_n) is increasing to α.*

Proof. The monotonicity of the sequences (x_n) and (y_n) are related to the sequence (θ_n) defined in [4] implicitly by the relation $|R_n| = \dfrac{f(n+\theta_n)}{2}$.

The inverse function $f^{-1} : \left(0, \dfrac{1}{\alpha+\beta}\right] \longrightarrow [1,+\infty)$ is given by $f^{-1}(x) = \dfrac{1}{x\alpha} - \dfrac{\beta}{\alpha}$. Hence the function $\psi : [2,\infty) \longrightarrow \mathbb{R}$ defined in [4, Theorem 7] by

$$\psi(x) = f^{-1}\left(\frac{f(x-1)+f(x)}{2}\right)$$

can be simplified into

$$\psi(x) = \frac{(x+r-1)(x+r)}{x+r-\frac{1}{2}} - r = x+r-\frac{1}{2} - \frac{1}{4\left(x+r-\frac{1}{2}\right)} - r,$$

where $r = \beta/\alpha$. As $\psi''(x) = \dfrac{-1}{2\left(x+r-\frac{1}{2}\right)^3}$, $r > -1$ and $x \geqslant 2$, we deduce that ψ is concave, and consequently the sequence $(\theta_n, n \geqslant 1)$ is decreasing by [4, Theorem 7].

On the other hand, as $\lim_{t\to\infty} \dfrac{f'(t+1)}{f'(t)} = 1$, we deduce according to [4, Theorem 4] that $\lim_{n\to\infty} \theta_n = 1/2$. Now, as $x_n = 2\alpha\theta_n$ and $y_n = 2\alpha(1-\theta_n)$, we conclude that $(x_n, n \geqslant 1)$ is decreasing to α and $(y_n, n \geqslant 1)$ is increasing to α. $\square$

As a result, we have $\sigma = \alpha$ and $\rho = y_1 = 2(2\alpha+\beta) - \left(\dfrac{1}{\alpha+\beta} - S(\alpha,\beta)\right)^{-1}$. This enables us to extend the main result in [6].

COROLLARY 1. *Let $\alpha > 0$ and β be real numbers such that $\alpha+\beta > 0$. Then the best constants a and b (the smallest a and the largest b) such that the inequalities*

$$\frac{1}{2\alpha n + a} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{\alpha k + \beta} \right| < \frac{1}{2\alpha n + b} \tag{6}$$

hold for every $n \geqslant 1$ are $a = \left(\dfrac{1}{\alpha+\beta} - S(\alpha,\beta)\right)^{-1} - 2\alpha$ and $b = \alpha + 2\beta$.

REMARK 1. ([6, Theorem (2)]) The best constants a and b such that the inequalities

$$\frac{1}{2n + a} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k} \right| < \frac{1}{2n + b} \tag{7}$$

hold for every $n \geqslant 1$ are $a = \dfrac{1}{1-\log 2} - 2$ and $b = 1$.

Corollary 1 provides an answer of a question stated in [6] concerning the Gregory-Leibniz series.

COROLLARY 2. *The best constants c and d such that the inequalities*

$$\frac{1}{4n + c} < \left| \sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \right| < \frac{1}{4n + d} \tag{8}$$

hold for every $n \geqslant 1$ are $c = \dfrac{4}{4-\pi} - 4$ and $d = 0$.

3. Integral representation of $|R_n|$

The goal of this section is to give an integral representation of $|R_n|$, using the Gauss hypergeometric function ${}_2F_1$ defined as

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \tag{9}$$

where $(a)_n$ is the Pochhammer's symbol defined by $(a)_n := a(a+1)\dots(a+n-1)$, for any $n \geqslant 1$ and $(a)_0 = 1$.

The Euler integral representation of ${}_2F_1$ is formulated as follows (cf. [1, Theorem 2.2. page 651]: If $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, then

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)}\Gamma(c-b)\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,dt \tag{10} $$

in the x-plane cut along the real axis from 1 to ∞ (with $\arg t = \arg(1-t) = 0$, and $(1-xt)^{-a}$ has its principal value).

THEOREM 2. *Let α,β be real numbers such that $\alpha > 0$ and $\alpha+\beta > 0$ and $R_n = \sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{\alpha k+\beta}$. Then, we have*

1. $|R_n| = \dfrac{1}{\alpha(n+1)+\beta}\,{}_2F_1\left(1, n+\dfrac{\beta}{\alpha}+1; n+\dfrac{\beta}{\alpha}+2; -1\right)$,

2. $|R_n| = \displaystyle\int_0^1 \frac{t^{\alpha(n+1)+\beta-1}}{1+t^\alpha}dt$,

3. $S(\alpha,\beta) = \dfrac{1}{\alpha+\beta} - \displaystyle\int_0^1 \frac{t^{2\alpha+\beta-1}}{1+t^\alpha}dt$.

Proof.

1. The series $|R_n|$ is given by

$$ \begin{aligned} |R_n| &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\alpha(n+k)+\beta} \\ &= \frac{1}{\alpha(n+1)+\beta}\sum_{k=1}^{\infty}\frac{\alpha(n+1)+\beta}{\alpha(n+k)+\beta}(-1)^{k-1} \\ &= \frac{1}{\alpha(n+1)+\beta}\sum_{k=1}^{\infty}\frac{n+\frac{\beta}{\alpha}+1}{n+\frac{\beta}{\alpha}+k}(-1)^{k-1} \\ &= \frac{1}{\alpha(n+1)+\beta}\sum_{k=0}^{\infty}\frac{(1)_k\left(n+\frac{\beta}{\alpha}+1\right)_k}{\left(n+\frac{\beta}{\alpha}+2\right)_k}\frac{(-1)^k}{k!} \\ &= \frac{1}{\alpha(n+1)+\beta}\,{}_2F_1\left(1, n+\frac{\beta}{\alpha}+1; n+\frac{\beta}{\alpha}+2; -1\right). \end{aligned} $$

2. By Euler integral representation, if $x>0$ and $c>b>1$, we have

$$ {}_2F_1\left(1,b;c;-x\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{1+xt}dt, $$

so that

$$ \begin{aligned} {}_2F_1\left(1,n+\frac{\beta}{\alpha}+1;n+\frac{\beta}{\alpha}+2;-1\right) &= \frac{\Gamma\left(n+\frac{\beta}{\alpha}+2\right)}{\Gamma\left(n+\frac{\beta}{\alpha}+1\right)\Gamma(1)}\int_0^1\frac{t^{n+\frac{\beta}{\alpha}}}{1+t}dt \\ &= \left(n+\frac{\beta}{\alpha}+1\right)\int_0^1\frac{t^{n+\frac{\beta}{\alpha}}}{1+t}dt. \end{aligned} $$

Consequently,

$$ |R_n|=\frac{1}{\alpha}\int_0^1\frac{t^{n+\frac{\beta}{\alpha}}}{1+t}dt=|R_n|=\int_0^1\frac{s^{\alpha(n+1)+\beta-1}}{1+s^{\alpha}}ds. $$

3. It is enough to remark that

$$ S(\alpha,\beta)=\frac{1}{\alpha+\beta}-|R_1|=\frac{1}{\alpha+\beta}-\int_0^1\frac{t^{2\alpha+\beta-1}}{1+t^{\alpha}}dt. \qquad \square $$

As an application of Theorem 2-(3), we provide some examples of evaluation of the sum $\sum_{n=0}^{\infty}\frac{(-1)^n}{an+1}$.

EXAMPLE 1. Let $a>1$ be a real number, then

$$ S(a,1-a)=\sum_{n=0}^{\infty}\frac{(-1)^n}{an+1}=\int_0^1\frac{dt}{1+t^a}. $$

- For $a=1$, we get $S(1,0)=\log 2$.
- For $a=2$, we get $S(2,-1)=\frac{\pi}{4}$.
- For $a=3$, we decompose $\frac{1}{1+t^3}$ into the sum of partial fractions as follows

$$ \frac{1}{1+t^3}=-\frac{t-2}{3(t^2-t+1)}+\frac{1}{3(t+1)}, $$

and we obtain

$$ S(3,-2)=\frac{\sqrt{3}\pi+3\log 2}{9}. $$

In general, if $a \geqslant 3$ is an integer, then we need to decompose $\dfrac{1}{1+t^a}$ into partial fractions and then derive the sum $S(a,1-a)$ (for instance, $S(4,-3) = \dfrac{\pi + 2\log(1+\sqrt{2})}{4\sqrt{2}}$).

The following result gives an estimate of the remainder of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\alpha n+\beta}$.

THEOREM 3. *Let α,β be real numbers such that $\alpha > 0$, $\alpha+\beta > 0$. We let $f(n) = \dfrac{1}{\alpha n+\beta}$, $g(n) = \dfrac{1}{f(n)}$ and $R_n = \displaystyle\sum_{k=1}^{\infty}(-1)^{n+k-1} f(n+k)$, then we have the following inequalities*

$$\frac{1}{2g(n+1)-\rho_n} < |R_n| < \frac{1}{2g(n)+\sigma_n}, \tag{11}$$

for all $n \in \mathbb{N}$, where

$$\rho_n = \alpha\left(1 - \left(n + \frac{\beta}{\alpha} + 1\right)^{-1}\right),$$
$$\sigma_n = \alpha\left(1 + 3\left[\left(n+\frac{\beta}{\alpha}+1\right)^2 + 3\left(n+\frac{\beta}{\alpha}+1\right) + 3\right]^{-1}\right).$$

Proof. Using Theorem 2, we have

$$|R_n| = \frac{1}{\alpha(n+1)+\beta}\, {}_2F_1\left(1, n+\frac{\beta}{\alpha}+1; n+\frac{\beta}{\alpha}+2; -1\right).$$

Now, following [3, Page 338, Inequalities (4)], for $x > 0$ and $c > b > 1$, we have

$$\frac{c}{c+bx} < {}_2F_1(1,b;c;-x) < \frac{c(c+1)+(c-b)x}{c(c+1)+c(b+1)x}.$$

Therefore we obtain:

$$\left(\frac{1}{\alpha b}\right)\frac{b+1}{2b+1} < |R_n| < \left(\frac{1}{\alpha b}\right)\frac{(b+1)(b+2)+1}{(b+1)(b+2)+(b+1)^2},$$

where $b = n + \dfrac{\beta}{\alpha} + 1$. The lower and upper bounds may be written as follows

$$\left(\frac{1}{\alpha b}\right)\frac{b+1}{2b+1} = \frac{1}{2g(n+1)-\rho_n},$$
$$\left(\frac{1}{\alpha b}\right)\frac{(b+1)(b+2)+1}{(b+1)(b+2)+(b+1)^2} = \frac{1}{2g(n)+\sigma_n}. \quad \square$$

4. Further estimates of $|R_n|$

As in the previous sections, we let $f:[1,\infty)\longrightarrow(0,\infty)$ be a function, satisfying the following properties:

(i) $f(n+1)<f(n)$, for all $n\in\mathbb{N}$.

(ii) $\lim\limits_{n\longrightarrow\infty} f(n)=0$.

(iii) $\Delta f(n)<\Delta f(n+1)$, where $\Delta f(n):=f(n+1)-f(n)$.

We denote by $g(n):=\dfrac{1}{f(n)}$ and $R_n=\sum\limits_{k\geqslant 1}(-1)^{n+k-1}f(n+k)$.

We consider the two quantities

$$t_n=\sqrt{(\Delta g(n))^2+g(n+1)^2}-g(n)$$

and

$$\lambda_n=g(n+2)-\sqrt{(\Delta g(n+1))^2+g(n+1)^2},$$

then using the recursive relation $|R_{n+1}|+|R_n|=\frac{1}{g(n+1)}$, we have the following result.

PROPOSITION 4. Let n be a positive integer.

1. The following statements are equivalent:

   (i) $x_{n+1}<x_n$;

   (ii) $x_n>t_n$;

   (iii) $x_{n+1}<t_n$;

   (iv) $|R_n|<\dfrac{1}{2g(n)+t_n}$;

   (v) $|R_{n+1}|>\dfrac{1}{2g(n+1)+t_n}$.

2. The following statements are equivalent:

   (i) $y_{n+1}>y_n$;

   (ii) $y_n<\lambda_n$;

   (iii) $y_{n+1}<\lambda_n$;

   (iv) $|R_n|<\dfrac{1}{2g(n+1)-\lambda_n}$;

   (v) $|R_{n+1}|>\dfrac{1}{2g(n+2)-\lambda_n}$.

REMARK 2. Similar arguments may be used to show the equivalence between the reversed inequalities in the previous proposition.

For the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\alpha n+\beta}$, we have

$$t_n = \sqrt{\alpha^2 + (\alpha(n+1)+\beta)^2} - \alpha n - \beta$$

and

$$\lambda_n = \alpha(n+2)+\beta - \sqrt{\alpha^2 + (\alpha(n+1)+\beta)^2}.$$

Combining Lemma 1 and Proposition 4 yields the following result.

THEOREM 5. *The following inequalities hold.*

$$\frac{1}{\sqrt{\alpha^2+(\alpha n+\beta)^2}+\alpha(n+1)+\beta} < |R_n| < \frac{1}{\sqrt{\alpha^2+(\alpha(n+1)+\beta)^2}+\alpha n+\beta},$$

for all $n \geqslant 2$.

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