Mathematical
Inequalities
& Applications
Volume 26, Number 1 (2023), 109–129
doi:10.7153/mia-2023-26-09

THE GENERALIZED GAO'S CONSTANT OF ABSOLUTE NORMALIZED NORMS IN $\mathbb{R}^2$

ZHAN-FEI ZUO*, YI-MIN HUANG AND JING WANG

(*Communicated by S. Varošanec*)

Abstract. In this paper, we give the method to calculate the generalized Gao's constant under the absolute normalized norms in $\mathbb{R}^2$. Using this method, we can compute the exact values of the generalized Gao's constant of some concrete Banach spaces easily, such as Banach lattice, Lorentz sequence spaces etc.

1. Introduction

Let X be a real normed space with the unit ball B_X and the unit sphere S_X. Recently, the geometric constants have received widespread attention, for the reason that it not only essentially reflects the geometric properties of a space X, but also enables us to study the space quantitatively. Among which, the Baronti constant $A_2(X)$, the Gao constant $C'_{\mathrm{NJ}}(X)$ and the generalized von Neumann-Jordan constant $C_{\mathrm{NJ}}^{(p)}(X)$ have been treated by a lot of mathematicians (see [1–4, 7, 8, 19, 22]), they play an important role in the geometric theory of Banach spaces, which were defined as follows:

$$A_2(X) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} : x,y \in S_X\right\}.$$

$$C'_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2+\|x-y\|^2}{4} : x,y \in S_X\right\}.$$

$$C_{\mathrm{NJ}}^{(p)}(X) = \sup\left\{\frac{\|x+y\|^p+\|x-y\|^p}{2^{p-1}(\|x\|^p+\|y\|^p)} : \|x\|+\|y\| \neq 0\right\}.$$

They gave the specific descriptions of the geometric properties, such as uniformly non-square, normal structure etc in the context of fixed point property (see [7, 11, 13, 23, 24]).

Mathematics subject classification (2020): Primary 46B20; Secondary 46B25.
Keywords and phrases: The generalized Gao's constant, absolute normalized norms, convex function.
The work was sponsored by the Natural Science Foundation of Chongqing (CSTB2022NSCQ-MSX0290), the Scientific and Technological Research Program of Chongqing Municipal Education Commission (Grant No. KJZD-M202001201), China.

* Corresponding author.

Strongly motivated by the constant $A_2(X)$, $C'_{\mathrm{NJ}}(X)$ and $C^{(p)}_{\mathrm{NJ}}(X)$, Yang and Wang introduced the generalized Gao's constant $\tilde{C}^{(p)}_{\mathrm{NJ}}(X)$ in [20] as follows:

$$\tilde{C}^{(p)}_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^p + \|x-y\|^p}{2^p} : x,y \in S_X \right\} \ (1 \leqslant p < +\infty).$$

From the definition of the generalized Gao's constant, it is obvious that $\tilde{C}^{(1)}_{\mathrm{NJ}}(X) = A_2(X)$ and $\tilde{C}^{(2)}_{\mathrm{NJ}}(X) = C'_{\mathrm{NJ}}(X)$. Now, let us collect some properties of the constant $\tilde{C}^{(p)}_{\mathrm{NJ}}(X)$ in [20] as follows:

(i) Let X be a Banach space, then $\tilde{C}^{(p)}_{\mathrm{NJ}}(X) \leqslant C^{(p)}_{\mathrm{NJ}}(X) \leqslant 2^{2-p}[1+(2^{\frac{1}{q}}(\tilde{C}^{(p)}_{\mathrm{NJ}}(X))^{\frac{1}{p}} - 1)^q]^{p-1}$.

(ii) The Banach space X is uniformly non-square if and only if $\tilde{C}^{(p)}_{\mathrm{NJ}}(X) < 2$ for some $1 \leqslant p < +\infty$.

(iii) Let $X = \ell_{p,+\infty}$ with $p \geqslant 2$, then $\tilde{C}^{(p)}_{\mathrm{NJ}}(X) \leqslant \frac{3}{2} - (\frac{2^{\frac{1}{p}}-1}{2})^p$.

It is readily seen that the constant $\tilde{C}^{(p)}_{\mathrm{NJ}}(X)$ play a significant role in the geometry theory of Banach space, such as the relation between the generalized Gao's constant and some well known constants via several inequalities, equivalent conditions of uniformly non-square which are described by the generalized Gao's constant, the estimations of the constant on some specific space. Therefore, the calculation of the generalized Gao's constant $\tilde{C}^{(p)}_{\mathrm{NJ}}(X)$ for some concrete spaces is very important in studying geometric properties of the Banach space. However, some problems in the existing literature need solving: for instance, how to compute the values of the constant $\tilde{C}^{(p)}_{\mathrm{NJ}}(X)$ for the absolute normalized norms of some concrete Banach spaces?

In this paper, we are interested in determining the values of the generalized Gao's constant $\tilde{C}^{(p)}_{\mathrm{NJ}}(X)$ for the absolute normalized norms in $\mathbb{R}^2$. As an application, we get the exact values of the generalized Gao's constant $\tilde{C}^{(p)}_{\mathrm{NJ}}(X)$ for some concrete Banach spaces, such as ℓ_p^2 space, Banach lattice space X^p, $X_{p,q,\lambda}$ space, $Z_{\lambda,p,q,s}$ space, Lorentz sequence spaces $d^{(2)}(\omega,q)$ etc.

2. Preliminaries

To obtain the main results, we firstly recall some definitions and notions.

DEFINITION 2.1. If $a < b$ are real numbers, then any number $m(a,b)$ is called a mean of numbers a and b if it satisfies

$$a \leqslant m(a,b) \leqslant b.$$

One of the most known means is the weighted mean of order s, which is defined as

$$m^{[s]}(a,b;\omega,1-\omega)=\begin{cases}(\omega a^s+(1-\omega)b^s)^{\frac{1}{s}}, & s\neq 0,+\infty,-\infty,\\ a^{\omega}b^{1-\omega}, & s=0,\\ \max\{a,b\}, & s=+\infty,\\ \min\{a,b\}, & s=-\infty,\end{cases}$$

where a,b are positive real numbers and $\omega\in(0,1)$.

The norm on $\mathbb{R}^2$ is called absolute, if for all $(z,w)\in\mathbb{R}^2$, such that

$$\|(z,w)\|=\|(|z|,|w|)\|.$$

A norm $\|\cdot\|$ is called normalized if

$$\|(1,0)\|=\|(0,1)\|=1.$$

Let N_α denote the family of all absolute normalized norms on $\mathbb{R}^2$, and Ψ denote the family of all convex functions on $[0,1]$ such that

$$\psi(0)=\psi(1)=1 \text{ and } \max\{1-t,t\}\leqslant\psi(t)\leqslant 1.$$

PROPOSITION 2.2. ([5]) *If $\|.\|\in N_\alpha$, then $\psi(t)=\|(1-t,t)\|\in\Psi$. On the other hand, if $\psi(t)\in\Psi$, then*

$$\|(z,\omega)\|_\psi:=\begin{cases}(|z|+|\omega|)\psi\left(\frac{|\omega|}{|z|+|\omega|}\right), & (z,\omega)\neq(0,0),\\ 0, & (z,\omega)=(0,0),\end{cases}$$

is a norm $\|.\|_\psi\in N_\alpha$.

The typical example is the ℓ_p norm as follows:

$$\|(x,y)\|_p=\begin{cases}(|x|^p+|y|^p)^{\frac{1}{p}}, & 1\leqslant p<\infty,\\ \max\{|x|,|y|\}, & p=\infty.\end{cases}$$

The corresponding convex function $\psi_p(t)$ is defined as

$$\psi_p(t)=\begin{cases}\{(1-t)^p+t^p\}^{\frac{1}{p}}, & 1\leqslant p<\infty,\\ \max\{1-t,t\}, & p=\infty.\end{cases}$$

It is well known that $\|.\|_\infty\leqslant\|.\|_\psi\leqslant\|.\|_1$ for any $\|.\|_\psi\in N_\alpha$. Moreover, by taking different convex function $\psi(t)$, Proposition 2.2 also enables us to obtain many non-ℓ_p norms. In particular, $X=\mathbb{R}^2$ with an absolute normalized norm $\|.\|_X$ and with a function $\psi_X\in\Psi$, X^p denotes the space with the norm

$$\|x\|=\||x|^p\|_X^{\frac{1}{p}}.$$

It is proved that if X is a Banach lattice, then X^p space is a Banach lattice for any $p \in (1,+\infty)$, some more results about X^p space can be found in [15, 17]. The following lemma help us utilize our results.

LEMMA 2.3. ([18]) *Let $\psi(t) \geqslant \phi(t) \geqslant 0$ in $[a,b]$, if the function $\psi(t)-\phi(t)$ attain the maximum at $t=c \in [a,b]$ and the function $\phi(t)$ attain the minimum at $t=c$, then the function $\frac{\psi(t)}{\phi(t)}$ attains its maximum at $t=c$.*

3. Main results

Firstly, we can easily get the equivalent definitions of the generalized Gao's constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(X)$ from the Proposition 4.3 in [3].

PROPOSITION 3.1. *Let X be a non-trivial Banach space, for $1 \leqslant p < +\infty$, then*

$$\begin{aligned}\tilde{C}_{\mathrm{NJ}}^{(p)}(X) &= \sup\left\{\frac{\|x+y\|^p+\|x-y\|^p}{2^p} : x,y \in B_X\right\},\\ &= \sup\left\{\frac{\|x+y\|^p+\|x-y\|^p}{2^p \max(\|x\|^p,\|y\|^p)}, \|x\|+\|y\| \neq 0\right\}.\end{aligned}$$

PROPOSITION 3.2. *Let X be a non-trivial Banach space, for $1 \leqslant p < +\infty$, then*

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(X) = \sup\{\tilde{C}_{\mathrm{NJ}}^{(p)}(Y) : Y \in P(X)\},$$

where $P(X)$ is the set of all two-dimensional subspaces of X.

Proof. Firstly, it is obvious that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(X) \geqslant \sup\{\tilde{C}_{\mathrm{NJ}}^{(p)}(Y) : Y \in P(X)\}.$$

Secondly, for any $\varepsilon > 0$, there exist x_0 and y_0 in S_X such that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(X) < \frac{\|x+y\|^p+\|x-y\|^p}{2^p} + \varepsilon.$$

Let Y_0 be a two-dimensional subspace that contains x_0 and y_0, then

$$\frac{\|x+y\|^p+\|x-y\|^p}{2^p} \leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}(Y_0) \leqslant \sup\{\tilde{C}_{\mathrm{NJ}}^{(p)}(Y) : Y \in P(X)\},$$

thus, we obtain

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(X) < \sup\{\tilde{C}_{\mathrm{NJ}}^{(p)}(Y) : Y \in P(X)\} + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(X) \leqslant \sup\{\tilde{C}_{\mathrm{NJ}}^{(p)}(Y) : Y \in P(X)\}. \quad \square$$

THEOREM 3.3. *Let X and Y be isomorphic Banach spaces, then for Banach-Mazur distance $d(X,Y)$,*

$$\frac{\tilde{C}_{\mathrm{NJ}}^{(p)}(X)}{d(X,Y)^p} \leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}(Y) \leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}(X)d(X,Y)^p.$$

In particular, $\tilde{C}_{\mathrm{NJ}}^{(p)}(X) = \tilde{C}_{\mathrm{NJ}}^{(p)}(Y)$, if X and Y are isometric.

Proof. Suppose that $x,y \in S_X$, by the definition of Banach-Mazur distance, for each $\varepsilon > 0$, there exists an operator T from X onto Y such that

$$\|T\|\|T^{-1}\| \leqslant d(X,Y)(1+\varepsilon).$$

Consider

$$x_1 = \frac{Tx}{\|T\|} \in B_Y \ \text{ and } \ y_1 = \frac{Ty}{\|T\|} \in B_Y.$$

By the definition of $\tilde{C}_{\mathrm{NJ}}^{(p)}(X)$, we obtain

$$\begin{aligned}
\frac{\|x+y\|^p + \|x-y\|^p}{2^p} &= \frac{\|T\|^p(\|T^{-1}(x_1+y_1)\|^p + \|T^{-1}(x_1-y_1)\|^p)}{2^p}, \\
&\leqslant d(X,Y)^p(1+\varepsilon)^p\left(\frac{\|x_1+y_1\|^p + \|x_1-y_1\|^p}{2^p}\right), \\
&\leqslant d(X,Y)^p(1+\varepsilon)^p\tilde{C}_{\mathrm{NJ}}^{(p)}(Y),
\end{aligned}$$

which implies that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(X) \leqslant d(X,Y)^p(1+\varepsilon)^p\tilde{C}_{\mathrm{NJ}}^{(p)}(Y) \leqslant d(X,Y)^p\tilde{C}_{\mathrm{NJ}}^{(p)}(Y).$$

The last inequality is true for every $\varepsilon > 0$, so we obtain the left-hand side of our assertion, the right-hand side of the assertion follows by simply interchanging X and Y. □

In the following text, we will use notation X_ψ for the space X with norm $\|.\|_\psi$ and write $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi)$ instead of $\tilde{C}_{\mathrm{NJ}}^{(p)}(X_\psi)$.

COROLLARY 3.4. *Let $\|.\|$ and $|.|$ be two equivalent norms in X such that*

$$\alpha|.| \leqslant \|.\| \leqslant \beta|.| \quad (0 < \alpha < \beta),$$

then

$$\frac{\alpha^p\tilde{C}_{\mathrm{NJ}}^{(p)}(|.|)}{\beta^p} \leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|) \leqslant \frac{\beta^p\tilde{C}_{\mathrm{NJ}}^{(p)}(|.|)}{\alpha^p}.$$

Moreover, if $\|x\| = a|x|$, then $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|) = \tilde{C}_{\mathrm{NJ}}^{(p)}(|.|)$.

Proof. This follows from Theorem 3.3 and the fact that $d(X,Y) \leqslant \frac{\beta}{\alpha}$. □

Now, let us denote

$$M_1 = \max_{0\leqslant t\leqslant 1} \frac{\phi(t)}{\psi(t)} \text{ and } M_2 = \max_{0\leqslant t\leqslant 1} \frac{\psi(t)}{\phi(t)}.$$

THEOREM 3.5. *Let $\psi(t), \phi(t) \in \Psi$ and $\psi(t) \leqslant \phi(t)$ for all $t \in [0,1]$, if the function $\frac{\phi(t)}{\psi(t)}$ attains its maximum at $t = \frac{1}{2}$ and $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi) = \frac{1}{2^{p-1}\phi^p(\frac{1}{2})}$, then*

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) = \frac{1}{2^{p-1}\psi^p\left(\frac{1}{2}\right)}.$$

Proof. By the condition of $\psi(t) \leqslant \phi(t)$ and the definition of M_1, we have

$$\frac{1}{M_1}\|.\|_\phi \leqslant \|.\|_\psi \leqslant \|.\|_\phi.$$

By taking $\alpha = \frac{1}{M_1}$ and $\beta = 1$ in Corollary 3.4, which implies that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) \leqslant M_1^p \tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi).$$

It is noted that the function $\frac{\phi(t)}{\psi(t)}$ attains its maximum at $t = \frac{1}{2}$, i.e., $M_1 = \frac{\phi(\frac{1}{2})}{\psi(\frac{1}{2})}$ and $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi) = \frac{1}{2^{p-1}\phi^p(\frac{1}{2})}$, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) \leqslant M_1^p \tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi) = \frac{1}{2^{p-1}\psi^p(\frac{1}{2})}. \tag{3.1}$$

Let us put $x_1 = (\frac{1}{2\psi(\frac{1}{2})}, \frac{1}{2\psi(\frac{1}{2})})$, $y_1 = (\frac{1}{2\psi(\frac{1}{2})}, -\frac{1}{2\psi(\frac{1}{2})})$, then

$$\|x_1\|_\psi = \|y_1\|_\psi = 1,$$

$$\|x_1 + y_1\|_\psi = \|x_1 - y_1\|_\psi = \frac{1}{\psi(\frac{1}{2})},$$

$$\frac{\|x_1 + y_1\|^p + \|x_1 - y_1\|^p}{2^p} = \frac{1}{2^{p-1}\psi^p(\frac{1}{2})}. \tag{3.2}$$

From (3.1), (3.2), we obtain

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) = M_1^p \tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi) = \frac{1}{2^{p-1}\psi^p(\frac{1}{2})}.$$

We get the desired result. □

THEOREM 3.6. *Let* $\psi(t) \in \Psi$ *and* $\psi(t) \leqslant \psi_p(t)$ $(2 \leqslant p < \infty)$ *for all* $t \in [0,1]$, *then*

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) = M_1^p.$$

Proof. Let $x, y \in S_X$, by the condition of $\psi(t) \leqslant \psi_p(t)$ and Clarkson inequality in [6], we have

$$\begin{aligned}
\|x+y\|_\psi^p + \|x-y\|_\psi^p &\leqslant \|x+y\|_p^p + \|x-y\|_p^p \\
&\leqslant 2^{p-1}(\|x\|_p^p + \|y\|_p^p) \\
&\leqslant 2^{p-1}M_1^p(\|x\|_\psi^p + \|y\|_\psi^p) \\
&= 2^p M_1^p.
\end{aligned}$$

The definition of $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi)$ implies that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) \leqslant M_1^p. \tag{3.3}$$

On the other hand, note that the function $\frac{\psi_p(t)}{\psi(t)}$ attains its maximum at t_1, i.e. $M_1 = \frac{\psi_p(t_1)}{\psi(t_1)}$. Let us put $x_2 = (1-t_1, t_1)$, $y_2 = (1-t_1, -t_1)$, then

$$\|x_2\| = \|y_2\| = \psi(t_1).$$

$$\begin{aligned}
\|x_2+y_2\|_\psi^p + \|x_2-y_2\|_\psi^p &= 2^p[(1-t_1)^p + (t_1)^p] \\
&= 2^p \psi_p^p(t_1) \\
&= 2^p M_1^p \psi^p(t_1).
\end{aligned}$$

Therefore, we can get

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) \geqslant \frac{\|x_2+y_2\|_\psi^p + \|x_2-y_2\|_\psi^p}{2^p \max(\|x_2\|_\psi^p, \|y_2\|_\psi^p)} = M_1^p. \tag{3.4}$$

From inequalities (3.3) and (3.4), we infer that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) = M_1^p.$$

We end the proof. □

COROLLARY 3.7. *Let* X^p $(2 \leqslant p < \infty)$ *be a two-dimensional Banach lattice space, if the corresponding function* ψ_X *attains its minimum at the point* $t = \frac{1}{2}$, *then*

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{X^p}) = \frac{1}{2^{p-1}\psi_{X^p}^p(\frac{1}{2})}.$$

Proof. It is clear that $\|x\| = \||x|^p\|_X^{\frac{1}{p}} \in \mathbb{N}_\alpha$ from the norm of the space X^p, and its corresponding convex function is

$$\psi_{X^p}(t) = \|(1-t,t)\|_{X^p} = [(1-t)^p + t^p]^{\frac{1}{p}} \psi_X^{\frac{1}{p}}\left(\frac{t^p}{(1-t)^p + t^p}\right).$$

Since $\psi_X(t) \leqslant 1$, then $\psi_{X^p}(t) \leqslant \psi_p(t)$ $(2 \leqslant p < \infty)$ for all $t \in [0,1]$, it is easy to check that the function

$$\frac{\psi_p(t)}{\psi_{X^p}(t)} = \psi_X^{\frac{-1}{p}}\left(\frac{t^p}{(1-t)^p + t^p}\right).$$

For arbitrary $t \in [0,1]$, the variable $s = \frac{t^p}{(1-t)^p+t^p}$ is also belongs to $[0,1]$. Since the function $\psi_X(t)$ attains its minimum at the point $t = \frac{1}{2}$, then $\psi_X\left(\frac{t^p}{(1-t)^p+t^p}\right)$ attains its minimum at $t = \frac{1}{2}$, this implies that the function $\psi_X^{\frac{-1}{p}}\left(\frac{t^p}{(1-t)^p+t^p}\right)$ attains its maximum at $\frac{1}{2}$. By Theorem 3.6, we can get that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{X^p}) = \frac{1}{2^{p-1}\psi_{X^p}^p(\frac{1}{2})}.$$

We complete the proof. $\square$

THEOREM 3.8. *Let $\psi(t), \phi(t) \in \Psi$ and $\psi(t) \geqslant \phi(t)$ for all $t \in [0,1]$, if the function $\frac{\psi(t)}{\phi(t)}$ attains its maximum at $t = \frac{1}{2}$ and $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi) = 2\phi^p(\frac{1}{2})$, then*

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) = 2\psi^p\left(\frac{1}{2}\right).$$

Proof. From the condition of $\psi(t) \geqslant \phi(t)$ and the definition of M_2, we have

$$\|.\|_\phi \leqslant \|.\|_\psi \leqslant M_2\|.\|_\phi.$$

Taking $\alpha = 1$ and $\beta = M_2$ in Corollary 3.4, we get the following inequality

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) \leqslant M_2^p \tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi).$$

Since $M_2 = \frac{\psi(\frac{1}{2})}{\phi(\frac{1}{2})}$ and $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi) = 2\phi^p(\frac{1}{2})$, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi) \leqslant M_2^p \tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi) = 2\psi^p\left(\frac{1}{2}\right). \tag{3.5}$$

On the other hand, let us put $x_3 = (1,0)$, $y_3 = (0,1)$, then

$$\|x_3\| = \|y_3\| = 1.$$

$$\|x_3 + y_3\|_\psi = \|x_3 - y_3\|_\psi = 2\psi\left(\frac{1}{2}\right).$$

$$\frac{\|x_3+y_3\|_\psi^p+\|x_3-y_3\|_\psi^p}{2^p}=\frac{2^{p+1}\psi^p\left(\frac{1}{2}\right)}{2^p}=2\psi^p\left(\frac{1}{2}\right)\leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi). \tag{3.6}$$

By the inequalities (3.5) and (3.6), we can get that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi)=M_2^p\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\phi)=2\psi^p\left(\frac{1}{2}\right).$$

We end the proof. □

THEOREM 3.9. *Let* $\psi(t)\in\Psi$ *and* $\psi(t)\geqslant\psi_p(t)$ $(1<p\leqslant 2)$ *for all* $t\in[0,1]$, *if the function* $\frac{\psi(t)}{\psi_p(t)}$ *attains its maximum at* $t=\frac{1}{2}$, *then*

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi)=2^{2-p}M_2^p.$$

Proof. Let $x,y\in S_X$, by the condition of $\psi(t)\geqslant\psi_p(t)$ and the Clarkson inequality in [6],

$$\begin{aligned}\|x+y\|_\psi^p+\|x-y\|_\psi^p &\leqslant M_2^p(\|x+y\|_p^p+\|x-y\|_p^p)\\ &\leqslant 2M_2^p(\|x\|_p^p+\|y\|_p^p)\\ &\leqslant 2M_2^p(\|x\|_\psi^p+\|y\|_\psi^p).\end{aligned}$$

Which implies that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi)\leqslant 2^{2-p}M_2^p. \tag{3.7}$$

On the other hand, since $M_2=\frac{\psi(\frac{1}{2})}{\psi_p(\frac{1}{2})}$, let us put $x_4=(\frac{1}{2},0), y_4=(0,\frac{1}{2})$, then

$$\|x_4\|_\psi^p=\|y_4\|_\psi^p=\left(\frac{1}{2}\right)^p.$$

$$\|x_4+y_4\|_\psi=\|x_4-y_4\|_\psi=\psi\left(\frac{1}{2}\right).$$

$$\frac{\|x_4+y_4\|_\psi^p+\|x_4-y_4\|_\psi^p}{2^p\max(\|x_4\|_\psi^p,\|y_4\|_\psi^p)}=2\psi^p\left(\frac{1}{2}\right)=2^{2-p}\frac{\psi^p(\frac{1}{2})}{\psi_p^p(\frac{1}{2})}=2^{2-p}M_2^p\leqslant\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi). \tag{3.8}$$

From (3.7) and (3.8), we infer that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi)=2^{2-p}M_2^p.$$

The proof is completed. □

In the following, let us state the conclusion related to the general mean $m(t)$.

COROLLARY 3.10. *Let* $\psi(t)\leqslant\phi(t)\in\Psi$ *for all* $t\in[0,1]$ *and* $m(t):=m(\psi(t),\phi(t))$ *is the mean convex function of the functions* $\psi(t)$ *and* $\phi(t)$.

(i) If $\frac{\phi(t)}{m(t)}$ *attains its maximum at* $t=\frac{1}{2}$ *and* $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\phi})=\frac{1}{2^{p-1}\phi^p(\frac{1}{2})}$*, then*

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{m})=\frac{1}{2^{p-1}m^p\left(\frac{1}{2}\right)}.$$

(ii) If $\frac{m(t)}{\psi(t)}$ *attains its maximum at* $t=\frac{1}{2}$ *and* $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi})=2\psi^p(\frac{1}{2})$*, then*

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{m})=2m^p\left(\frac{1}{2}\right).$$

Proof. The general mean $m(t)$ has the property

$$\psi(t)\leqslant m(t)\leqslant \phi(t)$$

for all $t\in[0,1]$. Since $\psi(t),\varphi(t)\in\Psi$ and the assumption of the function $m(t)$ is convex, it is easy to check that $m(t)\in\Psi$. We can get the results from the Theorem 3.5 and Theorem 3.8, respectively. □

For the general case $\psi(t)\in\Psi$, we give the lower bound and upper bound of the generalized von Neumann-Jordan type constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi})$.

THEOREM 3.11. *Let* $\psi(t)\in\Psi$ *for all* $t\in[0,1]$*,* $M_1=\max\limits_{0\leqslant t\leqslant 1}\frac{\psi_p(t)}{\psi(t)}$*,* $M_2=\max\limits_{0\leqslant t\leqslant 1}\frac{\psi(t)}{\psi_p(t)}$*.*

(i) If $1<p\leqslant 2$*, then*

$$2^{2-p}M_2^p\leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}\left(\|\cdot\|_{\psi}\right)\leqslant 2^{2-p}M_1^pM_2^p.$$

(ii) If $2\leqslant p<\infty$*, then*

$$M_1^p\leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}\left(\|\cdot\|_{\psi}\right)\leqslant M_1^pM_2^p.$$

Proof. (i) If $1\leqslant p\leqslant 2$, it is easy to get the left inequality from the (3.8),

$$2^{2-p}M_2^p\leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}\left(\|\cdot\|_{\psi}\right).$$

Let $x,y\in S_X$, by the definition of M_1, M_2 and the Clarkson inequality, we obtain

$$\begin{aligned}\|x+y\|_{\psi}^p+\|x-y\|_{\psi}^p&\leqslant M_2^p(\|x+y\|_p^p+\|x-y\|_p^p)\\&\leqslant 2M_2^p(\|x\|_p^p+\|y\|_p^p)\\&\leqslant 2M_1^pM_2^p(\|x\|_{\psi}^p+\|y\|_{\psi}^p).\end{aligned}$$

The definition of $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi})$ implies that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi})\leqslant 2^{2-p}M_1^pM_2^p.$$

(ii) If $2 \leqslant p \leqslant \infty$, the left inequality is obvious from the inequality (3.4),

$$M_1^p \leqslant \tilde{C}_{\mathrm{NJ}}^{(p)}\left(\|\cdot\|_\psi\right).$$

Let $x, y \in S_X$, from the Clarkson inequality, we get

$$\begin{aligned}
\|x+y\|_\psi^p + \|x-y\|_\psi^p &\leqslant M_2^p(\|x+y\|_p^p + \|x-y\|_p^p) \\
&\leqslant 2^{p-1}M_2^p(\|x\|_p^p + \|y\|_p^p) \\
&\leqslant 2^{p-1}M_1^p M_2^p(\|x\|_\psi^p + \|y\|_\psi^p) \\
&= 2^p M_1^p M_2^p.
\end{aligned}$$

The definition of $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi)$ implies that the right inequality as follows:

$$\tilde{C}_{\mathrm{NJ}}^{(p)}\left(\|\cdot\|_\psi\right) \leqslant M_1^p M_2^p. \quad \square$$

In fact, from Theorem 3.6 and Theorem 3.9, the generalized Gao's constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\psi)$ coincides with the lower bound. In the following, we can only get some conditions under which the constants $A_2(\|.\|_\psi)$ and $C_{\mathrm{NJ}}^{'}(\|.\|_\psi)$ coincides with the upper bound in the case of $\psi(t) = \psi(1-t) \in \Psi$.

THEOREM 3.12. *Let $\psi(t) \in \Psi$ and $\psi(t) = \psi(1-t)$ for all $t \in [0,1]$. If there exist unique points $t_1, t_2 \in [0, \frac{1}{2}]$ such that*

$$M_1 = \frac{\psi_2(t_1)}{\psi(t_1)}, \quad M_2 = \frac{\psi(t_2)}{\psi_2(t_2)} \quad \text{and} \quad (1-t_1)(1-t_2) = \frac{1}{2},$$

then

$$A_2(\|.\|_\psi) = \sqrt{2}M_1M_2, \quad C_{\mathrm{NJ}}^{'}(\|.\|_\psi) = M_1^2M_2^2.$$

Proof. On the one hand, taking $\alpha = \frac{1}{M_1}$ and $\beta = M_2$ in Corollary 3.4, then

$$A_2(\|.\|_\psi) \leqslant A_2(\|.\|_2)M_1M_2, \quad C_{\mathrm{NJ}}^{'}(\|.\|_\psi) \leqslant C_{\mathrm{NJ}}^{'}(\|.\|_2)M_1^2M_2^2.$$

Since $A_2(\|.\|_2) = \sqrt{2}$ and $C_{\mathrm{NJ}}^{'}(\|.\|_2) = 1$, this implies that

$$A_2(\|.\|_\psi) \leqslant \sqrt{2}M_1M_2. \tag{3.9}$$

$$C_{\mathrm{NJ}}^{'}(\|.\|_\psi) \leqslant M_1^2M_2^2. \tag{3.10}$$

On the other hand, note that $(1-t_1)(1-t_2) = \frac{1}{2}$. Put $x = \frac{1}{\psi(t_1)}(1-t_1, t_1)$, $y = \frac{1}{\psi(t_1)}(t_1, t_1 - 1)$, we get $x+y = \frac{1}{\psi(t_1)}(1, 2t_1 - 1)$, $x - y = \frac{1}{\psi(t_1)}(1-2t_1, 1)$ and

$$\|x\|_\psi = 1, \quad \|y\|_\psi = 1,$$

$$\|x+y\|_\psi = \frac{(2-2t_1)}{\psi(t_1)}\psi\left(\frac{1-2t_1}{2-2t_1}\right) = \frac{\psi(t_2)}{\psi(t_1)(1-t_2)} = \frac{M_2\psi_2(t_2)}{\psi(t_1)(1-t_2)},$$

$$\|x-y\|_\psi = \frac{(2-2t_1)}{\psi(t_1)}\psi\left(\frac{1}{2-2t_1}\right) = \frac{\psi(1-t_2)}{\psi(t_1)(1-t_2)} = \frac{M_2\psi_2(t_2)}{\psi(t_1)(1-t_2)}.$$

It is well known that

$$\sqrt{2}(1-t)\psi_2\left(\frac{1}{2-2t}\right) = \psi_2(t).$$

Consequently, we obtain

$$A_2(\|.\|_\psi) \geqslant \frac{\|x+y\|_\psi + \|x-y\|_\psi}{2} = \sqrt{2}M_1M_2. \tag{3.11}$$

$$C'_{\mathrm{NJ}}(\|.\|_\psi) \geqslant \frac{\|x+y\|_\psi^2 + \|x-y\|_\psi^2}{4} = M_1^2M_2^2. \tag{3.12}$$

From the inequalities (3.9)–(3.12), we infer that

$$A_2(\|.\|_\psi) = \sqrt{2}M_1M_2, \quad C'_{\mathrm{NJ}}(\|.\|_\psi) = M_1^2M_2^2. \quad \square$$

4. Some Examples

In this section, we will calculate the exactly values of $\tilde{C}_{\mathrm{NJ}}^{(p)}(X)$ for some examples. These results which not only give the exact value of the generalized von Neumann-Jordan type constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(X)$, but also give some new supplement results about the constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(X)$ for some concrete Banach spaces.

EXAMPLE 4.1. If X is the $\ell_p^2\,(1 \leqslant p \leqslant \infty)$ space, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(X) = \begin{cases} 2^{2-p}, & 1 < p \leqslant 2, \\ 1, & 2 < p < \infty. \end{cases}$$

In particular, $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_1) = \tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\infty) = 2$.

Proof. On the one hand, let $1 < p \leqslant 2$ and $x, y \in S_X$, we can get that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_p) \leqslant 2^{2-p}, \tag{4.1}$$

from the Clarkson inequality

$$(\|x+y\|_p^p + \|x-y\|_p^p) \leqslant 2(\|x\|_p^p + \|y\|_p^p).$$

On the other hand, put $x = (1,0)$, $y = (0,1)$, then

$$\frac{\|x+y\|_p^p + \|x-y\|_p^p}{2^p \max(\|x\|_p^p, \|y\|_p^p)} = 2^{2-p}. \tag{4.2}$$

The definition of $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_p)$ from (4.1)–(4.2) implies that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_p) = 2^{2-p}. \tag{4.3}$$

Let $2 < p < \infty$ and $x, y \in S_X$, from the Clarkson inequality

$$(\|x+y\|_p^p + \|x-y\|_p^p) \leqslant 2^{p-1}(\|x\|_p^p + \|y\|_p^p).$$

then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_p) \leqslant 1. \tag{4.4}$$

On the other hand, put $x = (\frac{1}{2}, \frac{1}{2}), y = (\frac{1}{2}, -\frac{1}{2})$, then

$$\frac{\|x+y\|_p^p + \|x-y\|_p^p}{2^p \max(\|x\|_p^p, \|y\|_p^p)} = 1. \tag{4.5}$$

The definition of $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_p)$ from (4.4)–(4.5) implies that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_p) = 1. \tag{4.6}$$

Since $\|.\|_p \leqslant \|.\|_1 (1 < p \leqslant 2)$, it is well known that $\frac{\psi_1(t)}{\psi_p(t)} = \frac{1}{[(1-t)^p + t^p]^{\frac{1}{p}}}$ attains the maximum at $t = \frac{1}{2}$, then

$$M_2 = \max_{0 \leqslant t \leqslant 1} \frac{\psi_1(t)}{\psi_p(t)} = 2^{1-\frac{1}{p}}.$$

By Theorem 3.9, we obtain

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\infty) = 2^{2-p} M_2^p = 2.$$

Since $\|.\|_\infty \leqslant \|.\|_p \ (2 \leqslant p \leqslant \infty)$ and

$$\psi_\infty(t) = \begin{cases} 1-t, & 0 \leqslant t \leqslant \frac{1}{2}, \\ t, & \frac{1}{2} < t < 1. \end{cases}$$

(i) Let $0 \leqslant t \leqslant \frac{1}{2}$, $\frac{\psi_p(t)}{\psi_\infty(t)} = \frac{((1-t)^p + t^p)^{\frac{1}{p}}}{1-t} = g(t)$, then $g'(t) > 0$ and $M_1 = g(\frac{1}{2}) = 2^{\frac{1}{p}}$.

(ii) Let $\frac{1}{2} \leqslant t \leqslant 1$, $\frac{\psi_p(t)}{\psi_\infty(t)} = \frac{((1-t)^p + t^p)^{\frac{1}{p}}}{t} = h(t)$, then $h'(t) < 0$ and $M_1 = h(\frac{1}{2}) = 2^{\frac{1}{p}}$.

Therefore, $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_\infty) = M_1^p = 2$ by Theorem 3.6. □

EXAMPLE 4.2. Let $X = \mathbb{R}^2$, the convex function $\psi_X(t)$ is defined on $[0,1]$ as $\psi_X(t) = (1 - t + t^2)^{\frac{1}{2}}$, the corresponding norm is $\|(x,y)\| = (|x|^2 + |x||y| + |y|^2)^{\frac{1}{2}}$, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi_{X^p}}) = \frac{1}{2^{p-1}\psi_{X^p}^p\left(\frac{1}{2}\right)} = \frac{2\sqrt{3}}{3}.$$

Proof. It is obvious that $\|(x,y)\|$ is an absolute normalized norm on $\mathbb{R}^2$. By a standard discussion, it is easy to check that the corresponding function $\psi_X(t)=\sqrt{1-t+t^2}$ attains its minimum at the point $\frac{1}{2}$. For $p\geqslant 2$, the corresponding space X^p has the norm

$$\|(x,y)\|=((|x|^{2p}+|x|^p|y|^p+|y|^{2p})^{\frac{1}{2p}}.$$

And the corresponding convex function is

$$\psi_{X^p}(t)=\|(1-t,t)\|_{X^p}=[(1-t)^p+t^p]^{\frac{1}{p}}\psi_X^{\frac{1}{p}}\left(\frac{t^p}{(1-t)^p+t^p}\right)\leqslant\psi_p(t).$$

By Corollary 3.7, we have that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi_{X^p}})=\frac{1}{2^{p-1}\psi_{X^p}^p\left(\frac{1}{2}\right)}=\frac{2\sqrt{3}}{3}.\quad\square$$

REMARK 4.3. Since the generalized Gao's constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(X)$ has two-dimensional character and the concept of an absolute normalized norm concerns spaces with bases, therefore we can firstly consider the examples are norms in $\mathbb{R}^2$, such as the two-dimension space ℓ_p^2 and X^p in Example 4.1 and Example 4.2, This method can be helpful for us to deal with the values of $\tilde{C}_{\mathrm{NJ}}^{(p)}(X)$ for the general spaces X from the Proposition 3.2.

EXAMPLE 4.4. Let $X_{p,q,\lambda}$ be the space $\mathbb{R}^2$ with the norm

$$\|.\|_{p,q,\lambda}=\max\{\|.\|_p,\lambda\|.\|_q\},$$

where $1\leqslant q\leqslant p\leqslant\infty$ and $\lambda\in[2^{\frac{1}{p}-\frac{1}{q}},1]$, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{p,q,\lambda})=\begin{cases}2\lambda^p2^{\frac{p}{q}-p}, & \text{if } 1\leqslant q<p\leqslant 2,\\ \frac{2^{1-\frac{p}{q}}}{\lambda^p}, & \text{if } 2\leqslant q<p\leqslant\infty.\end{cases}$$

Proof. It is easy to check that $\|.\|_{p,q,\lambda}=\max\{\|.\|_p,\lambda\|.\|_q\}\in\mathbb{N}_\alpha$ and its corresponding convex function is

$$\psi(t)=\|(1-t,t)\|_{p,q,\lambda}=\max\{\psi_p(t),\lambda\psi_q(t)\}.$$

In fact, $\psi(t)$ is symmetric with respect to $t=\frac{1}{2}$, we can only consider the function $\psi(t)$ on the interval $[0,\frac{1}{2}]$, which is expanded to the whole interval $[0,1]$. Let $t_0\in[0,\frac{1}{2}]$ be a point such that $\psi_p(t_0)=\lambda\psi_q(t_0)$, then

$$\psi(t)=\begin{cases}\psi_p(t), & t\in[0,t_0],\\ \lambda\psi_q(t), & t\in[t_0,\frac{1}{2}].\end{cases}$$

(i) Suppose that $1 \leqslant q < p \leqslant 2$, it is obvious that $\psi(t) \geqslant \psi_p(t)$ and the function

$$\frac{\psi(t)}{\psi_p(t)} = \begin{cases} 1, & t \in [0,t_0] \cup [1-t_0,1], \\ \frac{\lambda \psi_q(t)}{\psi_p(t)}, & t \in [t_0, 1-t_0] \end{cases}$$

attains the maximum at $t = \frac{1}{2}$. By Theorem 3.9, we obtain

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{p,q,\lambda}) = 2^{2-p} M_2^p = 2\lambda^p 2^{\frac{p}{q}-p}.$$

(ii) Suppose that $2 \leqslant q < p \leqslant \infty$, since $\psi_p(t) \leqslant \psi_q(t)$ and $\lambda \psi_q(t) \leqslant \psi_q(t)$, then $\psi(t) \leqslant \psi_q(t)$, it is easy to check that the function

$$\frac{\psi_q(t)}{\psi(t)} = \begin{cases} \frac{\psi_q(t)}{\psi_p(t)}, & t \in [0,t_0] \cup [1-t_0,1], \\ \frac{1}{\lambda}, & t \in [t_0, 1-t_0] \end{cases}$$

attains its maximum at $t = \frac{1}{2}$. By Theorem 3.6, we get

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{p,q,\lambda}) = M_1^p = \frac{2^{1-\frac{p}{q}}}{\lambda^p}. \quad \square$$

EXAMPLE 4.5. Let $1 \leqslant p < q \leqslant \infty$, $1 \leqslant s < \infty$ and $\lambda > 0$. The Banach space $Z_{\lambda,p,q,s}$ and its corresponding function $\psi_{\lambda,p,q,s}(t)$ is defined on $[0,1]$ as

$$\psi_{\lambda,p,q,s}(t) = (1+\lambda)^{-\frac{1}{s}} (\psi_p^s(t) + \lambda \psi_q^s(t))^{\frac{1}{s}}.$$

i.e. $\psi_{\lambda,p,q,s}(t)$ is a weighted mean of order s of functions ψ_p and ψ_q with weights $\frac{1}{1+\lambda}$ and $\frac{\lambda}{1+\lambda}$. The corresponding norm is

$$\|\cdot\|_{\lambda,p,q,s} = (1+\lambda)^{-\frac{1}{s}} (\|\cdot\|_p^s + \lambda \|\cdot\|_q^s)^{\frac{1}{s}},$$

then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\lambda,p,q,s}) = \begin{cases} 2(1+\lambda)^{\frac{-p}{s}} (2^{\frac{s}{p}} + \lambda 2^{\frac{s}{q}})^{\frac{p}{s}}, & \text{if } 1 \leqslant p < q \leqslant 2, \\ 2(1+\lambda)^{\frac{p}{s}} (2^{\frac{s}{q}} + \lambda 2^{\frac{s}{q}})^{\frac{-p}{s}}, & \text{if } 2 \leqslant p < q \leqslant \infty. \end{cases}$$

Proof. Since $\psi_{\lambda,p,q,s}(t)$ is the weighted mean of order s of the functions $\psi_p(t)$ and $\psi_q(t)$, then

$$\psi_q(t) \leqslant \psi_{\lambda,p,q,s}(t) \leqslant \psi_p(t).$$

(i) Let $1 \leqslant p < q \leqslant 2$, since $\psi_{\lambda,p,q,s}(t) \geqslant \psi_q(t)$ and the function $\frac{\psi_{\lambda,p,q,s}(t)}{\psi_q(t)}$ attains the maximum at $t = \frac{1}{2}$ by the simple calculations. Take $\psi(t) = \psi_q(t)$ and $\phi(t) = \psi_p(t)$ in Corollary 3.10 (i), then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|\cdot\|_{\lambda,p,q,s}) = 2\psi_{\lambda,p,q,s}^p\left(\frac{1}{2}\right) = 2(1+\lambda)^{\frac{-p}{s}} (2^{\frac{s}{p}} + \lambda 2^{\frac{s}{q}})^{\frac{p}{s}}.$$

(ii) Let $2 \leqslant p < q \leqslant \infty$, since $\psi_{\lambda,p,q,s}(t) \leqslant \psi_p(t)$ and $\frac{\psi_p(t)}{\psi_{\lambda,p,q,s}(t)}$ attains its maximum at $t = \frac{1}{2}$. Similarly, take $\psi(t) = \psi_q(t)$ and $\phi(t) = \psi_p(t)$ in Corollary 3.10 (ii), then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\lambda,p,q,s}) = \frac{1}{2^{p-1}\psi_{\lambda,p,q,s}^{p}\left(\frac{1}{2}\right)} = 2(1+\lambda)^{\frac{p}{s}}(2^{\frac{s}{q}} + \lambda 2^{\frac{s}{q}})^{\frac{-p}{s}}. \quad \square$$

REMARK 4.6. (i) In fact, take $q = 2$ or $p = 2, q = 1$ in Example 4.4, some classical constants such as Baronti constant $A_2(X)$ and the Gao constant $C_{\mathrm{NJ}}^{'}(X)$ have been calculated for these concrete Banach spaces in [1, 16]. Now, Example 4.4 calculates the values of the constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{p,q,\lambda})$ for the general case $1 \leqslant q \leqslant p \leqslant \infty$ and $\lambda \in [2^{\frac{1}{p}-\frac{1}{q}}, 1]$.

(ii) In particular, the concrete Banach space $Z_{\lambda,2,\infty,s}$ in Example 4.5 has been studied in some papers (see [21, 22]). However, the exact value of $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\lambda,p,q,s})$ for the general case remain undiscovered. Example 4.5 give the exact value of the generalized von Neumann-Jordan type constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\lambda,p,q,s})$ for the general case $1 \leqslant p < q \leqslant \infty$, $1 \leqslant s < \infty$ and $\lambda > 0$.

EXAMPLE 4.7. Let $2 \leqslant p < \infty$ and V_p be the space $\mathbb{R}^2$ endowed with the norm

$$\|(x_1,x_2)\|_{V_p} = \max\left\{\left(\left|\frac{x_1}{2}\right|^p + |x_2|^p\right)^{\frac{1}{p}}, \left(|x_1|^p + \left|\frac{x_2}{2}\right|^p\right)^{\frac{1}{p}}\right\},$$

then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(V_p) = M_1^p = \frac{2^{p+1}}{1+2^p}.$$

Proof. The norm $\|(x_1,x_2)\|_{V_p}$ is absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is

$$\psi_{V_p}(t) = \begin{cases} ((1-t)^p + (\frac{t}{2})^p)^{\frac{1}{p}}, & 0 \leqslant t \leqslant \frac{1}{2}, \\ ((\frac{1-t}{2})^p + t^p)^{\frac{1}{p}}, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

From the form of function $\psi_{V_p}(t)$, it is easy to check that $\psi_{V_p}(t) \leqslant \psi_p(t)$, $\frac{\psi_p(t)}{\psi_{V_p}(t)}$ is symmetric with respect to $t = \frac{1}{2}$. Thus, it suffices to consider $\frac{\psi_p^p(t)}{\psi_{V_p}^p(t)}$ for $t \in [0, \frac{1}{2}]$, the function $\psi_p^p(t) - \psi_{V_p}^p(t) = (1 - \frac{1}{2^p})t^p$ attains the maximum at $t = \frac{1}{2}$ and the function $\psi_{V_p}(t)$ attains its minimum at $t = \frac{1}{2}$. By Lemma 2.3, we get the function $\frac{\psi_p(t)}{\psi_{V_p}(t)}$ attains its maximum at $t = \frac{1}{2}$. From Theorem 3.6, we get that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(V_p) = M_1^p = \frac{2^{p+1}}{1+2^p}. \quad \square$$

EXAMPLE 4.8. Let $0 < \omega < 1$ and $2 \leqslant q < \infty$. The two-dimensional Lorentz sequence space $d^{(2)}(\omega,q)$ is $\mathbb{R}^2$ with the norm

$$\|(x,y)\|_{\omega,q} = ((x^*)^q + \omega(y^*)^q)^{\frac{1}{q}},$$

where (x^*,y^*) is the rearrangement of $(|x|,|y|)$ satisfying $x^* \geqslant y^*$, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\omega,q}) = 2\left(\frac{1}{1+\omega}\right)^{\frac{p}{q}}.$$

Proof. The norm $\|(x,y)\|_{\omega,q}$ is an absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is

$$\psi_{\omega,q}(t) = \begin{cases} ((1-t)^q + \omega t^q)^{\frac{1}{q}}, & 0 \leqslant t \leqslant \frac{1}{2}, \\ (t^q + \omega(1-t)^q)^{\frac{1}{q}}, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is easy to check that $\psi_{\omega,q}(t) \leqslant \psi_q(t)$. Since $0 < \omega < 1$, $\frac{\psi_q(t)}{\psi_{\omega,q}(t)}$ is symmetric with respect to $t = \frac{1}{2}$, it suffices to consider $\frac{\psi_q(t)}{\psi_{\omega,q}(t)}$ for $t \in [0,\frac{1}{2}]$. For any $t \in [0,\frac{1}{2}]$, put $f(t) = \frac{\psi_q(t)^q}{\psi_{\omega,q}(t)^q}$, then

$$f'(t) = \frac{q(1-\omega)[t(1-t)]^{q-1}}{[(1-t)^q + \omega t^q]^2},$$

therefore, $f'(t) \geqslant 0$ for $0 \leqslant t \leqslant \frac{1}{2}$, this implies that the function $f(t)$ is increased for $0 \leqslant t \leqslant \frac{1}{2}$. Therefore, the function $\frac{\psi_q(t)}{\psi_{\omega,q}(t)}$ attains its maximum at $t = \frac{1}{2}$. By Theorem 3.6, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\omega,q}) = M_1^p = 2\left(\frac{1}{1+\omega}\right)^{\frac{p}{q}}. \quad \square$$

EXAMPLE 4.9. Let $2 \leqslant p < \infty$ and Y_p be the space $\mathbb{R}^2$ endowed with the norm

$$\|(x_1,x_2)\|_{Y_p} = \max\left\{\left(|x_1|^p + 2|x_2|^p\right)^{\frac{1}{p}}, \left(2|x_1|^p + |x_2|^p\right)^{\frac{1}{p}}\right\},$$

then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(Y_p) = M_1^p = \frac{4}{3}.$$

Proof. The norm $\|(x_1,x_2)\|_{Y_p}$ is absolute norm on $\mathbb{R}^2$, However, $\|(1,0)\|_{Y_p} = \|(0,1)\|_{Y_p} = 2^{\frac{1}{p}}$, so this norm is not normalized. Let $\|.\| = 2^{-\frac{1}{p}}\|.\|_{Y_p}$, it is easy to check that $\|.\|$ is absolute normalized norm on $\mathbb{R}^2$, and the corresponding convex function is given by the formula

$$\psi(t) = \begin{cases} \left(\frac{2(1-t)^p + t^p}{2}\right)^{\frac{1}{p}}, & 0 \leqslant t \leqslant \frac{1}{2}, \\ \left(\frac{(1-t)^p + 2t^p}{2}\right)^{\frac{1}{p}}, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is easy to check that $\psi(t) \leqslant \psi_p(t)$, $\frac{\psi_p(t)}{\psi(t)}$ is symmetric with respect to $t = \frac{1}{2}$. Thus, it suffices to consider $\frac{\psi_p^p(t)}{\psi^p(t)}$ for $t \in [0, \frac{1}{2}]$. For any $t \in [0, \frac{1}{2}]$, the function $\psi_p^p(t) - \psi^p(t) = \frac{t^p}{2}$ attains the maximum at $t = \frac{1}{2}$ and $\psi(t)$ attains its minimum at $t = \frac{1}{2}$, therefore the function $\frac{\psi_p(t)}{\psi(t)}$ attains its maximum at $t = \frac{1}{2}$ by Lemma 2.3. From Theorem 3.6, we get that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(Y_p) = M_1^p = \frac{4}{3}. \quad \square$$

REMARK 4.10. (i) Taking $q = 2$ and $\omega = 2^{\frac{2}{p}-1} \in (0,1)$ $(2 \leqslant p < \infty)$ in Example 4.8, we obtain the Lorentz sequence space $\ell_{p,2}$ which were studied in [10, 16] and the following formulas were established

$$A_2(\ell_{p,2}) = \frac{2}{(1+2^{\frac{2}{p}-1})^{\frac{1}{2}}}, \quad C_{\mathrm{NJ}}'(\ell_{p,2}) = \frac{2}{1+2^{\frac{2}{p}-1}}.$$

Now, we get the exact value of the generalized von Neumann-Jordan type constant $\tilde{C}_{\mathrm{NJ}}^{(p)}(d^{(2)}(\omega,q))$ for the general case $0 < \omega < 1$ and $2 \leqslant q < \infty$ in Example 4.8.

(ii) The Banach spaces V_2, Y_2 have been studied widely in [1, 13], some classical constants were calculated for these spaces. Now, the the values of $\tilde{C}_{\mathrm{NJ}}^{(p)}(X)$ are calculated for the general Banach spaces V_p, Y_p in Example 4.7 and Example 4.9 by Theorem 3.6.

In the above Examples, the maximum value M_1 always attains at $t = \frac{1}{2}$. However, we give some examples to show that M_1 does not attain at $t = \frac{1}{2}$.

EXAMPLE 4.11. For each $0 \leqslant \alpha \leqslant \frac{1}{2} \leqslant \beta \leqslant 1$, $X = (R^2, \|.\|_{\psi_{\alpha,\beta}})$ is the Banach space and its corresponding function is

$$\psi_{\alpha,\beta}(t) = \begin{cases} 1-t, & \text{if } 0 \leqslant t \leqslant \alpha, \\ \frac{(\alpha+\beta-1)t+\beta-2\alpha\beta}{\beta-\alpha}, & \text{if } \alpha \leqslant t \leqslant \beta, \\ t, & \text{if } \beta \leqslant t \leqslant 1. \end{cases}$$

If the function $\psi_{\alpha,\beta}(t) \leqslant \psi_p(t)$ $(2 \leqslant p \leqslant +\infty)$, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi_{\alpha,\beta}}) = \begin{cases} \frac{\beta^p+(1-\beta)^p}{\beta^p}, & \alpha+\beta \leqslant 1, \\ \frac{\alpha^p+(1-\alpha)^p}{(1-\alpha)^p}, & \alpha+\beta > 1. \end{cases}$$

Proof. Let us define the $f(t) = \frac{\psi_p(t)}{\psi_{\alpha,\beta}(t)}$, taking derivative of the function $f(t)$, by the similar discussion in Example 4.8 show that

$$M_1 = \begin{cases} \frac{\psi_p(\beta)}{\psi_{\alpha,\beta}(\beta)} = \frac{(\beta^p+(1-\beta)^p)^{\frac{1}{p}}}{\beta}, & \alpha+\beta \leqslant 1, \\ \frac{\psi_p(\alpha)}{\psi_{\alpha,\beta}(\alpha)} = \frac{(\alpha^p+(1-\alpha)^p)^{\frac{1}{p}}}{1-\alpha}, & \alpha+\beta > 1. \end{cases}$$

From Theorem 3.6, we get that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi_{\alpha,\beta}}) = M_1^p = \begin{cases} \frac{\beta^p+(1-\beta)^p}{\beta^p}, & \alpha+\beta \leqslant 1, \\ \frac{\alpha^p+(1-\alpha)^p}{(1-\alpha)^p}, & \alpha+\beta > 1. \end{cases} \qquad \square$$

EXAMPLE 4.12. Let $2 \leqslant p < \infty$ and $\frac{1}{2} < \beta \leqslant 2^{\frac{1}{p}-1}$, the corresponding convex function is given by $\psi_\beta(t) = \max\{1-t, t, \beta\}$, then

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi_\beta}) = M_1^p = \frac{\beta^p + (1-\beta)^p}{\beta^p}.$$

Proof. If $\frac{1}{2} < \beta \leqslant 2^{\frac{1}{p}-1}$, then $\psi_\beta(t) \leqslant \psi_p(t)$, it is easy to check that by the simple calculation

$$M_1 = \frac{\psi_p(\beta)}{\psi_\beta(\beta)} = \frac{\left((1-\beta)^p + \beta^p\right)^{\frac{1}{p}}}{\beta}.$$

From Theorem 3.6, we have that

$$\tilde{C}_{\mathrm{NJ}}^{(p)}(\|.\|_{\psi_\beta}) = M_1^p = \frac{\beta^p + (1-\beta)^p}{\beta^p}. \qquad \square$$

At last, we will present a practical example which satisfies the conditions of Theorem 3.12, thus the exact value of the von Neumann-Jordan type constant $A_2(X)$ and $C'_{\mathrm{NJ}}(X)$ coincide with their upper bound.

EXAMPLE 4.13. Let $\sqrt{3}-1 < c \leqslant 1$, the corresponding convex function is given by

$$\psi_c(t) = \max\left\{1-ct, 1-c+ct, 1-\frac{c^2}{2}\right\} \text{ for } 0 \leqslant t \leqslant 1,$$

then

$$A_2(X) = \frac{2(c^2-2c+2)}{\sqrt{2-c^2}} \text{ and } C'_{\mathrm{NJ}}(\|.\|_{\psi_c}) = \frac{2(c^2-2c+2)^2}{(2-c^2)^2}.$$

Proof. It is easy to check that $\psi_c(t) \in \Psi$ and $\psi(t) = \psi(1-t)$ for all $t \in [0,1]$. If $\sqrt{3}-1 < c \leqslant 1$, easy calculation shows that

$$M_1 = \frac{\psi_2(t_1)}{\psi_c(t_1)} = \sqrt{\frac{2(c^2-2c+2)}{(2-c^2)}}, \qquad M_2 = \frac{\psi_c(t_2)}{\psi_2(t_2)} = \sqrt{c^2-2c+2},$$

where $t_1 = \frac{c}{2}$, $t_2 = \frac{1-c}{2-c}$, which satisfy the condition $(1-t_1)(1-t_2) = \frac{1}{2}$ in Theorem 3.12, then

$$A_2(\|.\|_{\psi_c}) = \sqrt{2}M_1M_2 = \frac{2(c^2-2c+2)}{\sqrt{2-c^2}}.$$

$$C'_{\mathrm{NJ}}(\|.\|_{\psi_c}) = M_1^2M_2^2 = \frac{2(c^2-2c+2)^2}{(2-c^2)^2}. \quad \square$$

REFERENCES

[1] J. ALONSO AND E. LLORENS-FUSTER, *Geometric mean and triangles inscribed in a semicircle in Banach spaces*, J. Math. Anal. Appl., 340 (2008), 1271–1283.

[2] J. ALONSO, P. MARTÍN AND P. PAPINI, *Wheeling around von Neumann-Jordan constant in Banach Spaces*, Studia Math., 188 (2008), 135–50.

[3] M. BARONTI, E. CASINI, P. PAPINI, *Triangles inscribed in a semicircle, in Minkowski plane, and in normed spaces*, J. Math. Anal. Appl., 252 (2000), 121–146.

[4] M. BARONTI, E. CASINI, AND P. L. PAPINI, *Average distances and the geometry of Banach spaces*, Nonlinear Anal. Theory Methods Appl., 42 (2000), 533–541.

[5] F. BONSALL AND J. DUNCAN, *Numerical Ranges II.*, London Mathematical Society Lecture Notes Series, 10, Cambridge Univ. Press, New York, 1973.

[6] J. CLARKSON, *Uniformly convex spaces*, Trans. Amer. Math. Soc. 40 (1936), 396–414.

[7] Y. CUI, W. HUANG, H. HUDZIK, *Generalized von Neumann-Jordan constant and its relationship to the fixed point property*, Fixed point theory and applications, 40 (2015), 1–11.

[8] J. GAO, *A pythagorean approach in Banach spaces*, J. Inequal. Appl., 2006, Art. ID 94982.

[9] T. IKEDA, M. KATO, *Notes on von Neumann-Jordan and James constants for absolute norms on $\mathbb{R}^2$*, Mediterr. J. Math., 11 (2014), 633–642.

[10] M. KATO, L. MALIGRANDA, *On James and von Neumann-Jordan constants of Lorentz sequence spaces*, J. Math. Anal. Appl., 258 (2001), 457–465.

[11] M. KATO, L. MALIGRANDA AND Y. TAKAHASHI, *On James and Jordan-von Neumann constants and normal structure coefficient of Banach spaces*, Studia Math., 144 (2001), 275–295.

[12] N. KOMURO, K. SAITO AND K. MITANI, *Extremal structure of the set of absolute norms on R^2 and the von Neumann-Jordan constant*, J. Math. Anal. Appl., 370 (2010), 101–106.

[13] E. LLORENS-FUSTER, E. MAZCUÑÁN-NAVARRO, S. REICH, *The Ptolemy and Zbăganu constants of normed spaces*, Nonlinear Analysis., 72 (2010), 3984–3993.

[14] K. MITANI AND K. SAITO, *The James constant of absolute norms on R^2*, J. Nonlinear Convex Anal., 4 (2003), 399–410.

[15] L. NIKOLOVA, L. PERSSON, *Some properties of X^p spaces*, in: J. Musielak, H. Hudzik, R. Urbanski (eds.), Function spaces, Teubner-Texte zur Matematik, 120 (1991), 174–185.

[16] L. NIKOLOVA, S. VAROŠANEC, *Geometric constants for Banach spaces with absolute normalized norms which corresponding functions are comparable*, Journal of Nonlinear and Convex Analysis, 14 (2013), 529–549.

[17] L. PERSSON, *Some elementary inequalities in connection with X^p spaces*, in: B. Sendov (ed.) Constructive theory of functions, Publishing House of the Bulgarian Academy of Sciences, (1988), 367–376.

[18] K. SAITO, M. KATO, Y. TAKAHASHI, *Von Neumann-Jordan constant of absolute normalized norms on C^2*, J. Math. Anal. Appl., 244 (2000), 515–532.

[19] Y. TAKAHASHI, *Some geometric constants of Banach spaces–a unified approach*, in: Banach and Function Spaces II, Proc. International Symposium (ISBFS 2006) held in Kitakyushu (Sept. 14–17, 2006), Edited by Mikio Kato and Lech Maligranda, Yokohama Publ., Yokohama, (2008), 191–220.
[20] C. YANG AND F. WANG, *An extension of a simply inequality between von Neumann-Jordan and James Constants in Banach Spaces*, Acta Mathematica Sinica, 33 (2017), 1287–1396.
[21] C. YANG, *A note on Jordan-Von Neumann constant for $Z_{p,q}$ space*, Journal of Mathematical Inequalities, 2 (2015), 499–504.
[22] C. YANG, H. LI, *An inequality between Jordan-von Neumann constant and James constant*, Appl. Math. Lett., 23 (2010), 277–281.
[23] Z. ZUO, *Uniform normal structure of Banach spaces and their duals via geometrical coefficients*, Advances in Mathematics(China), 50 (2021), 593–602.
[24] Z. ZUO, *The normal structure and parametrized von Neumann-Jordan type constant*, Acta Mathematica Sinica, 63 (2020), 655–660.

(Received July 23, 2022)

Zhan-Fei Zuo
School of Mathematics and Statistics
Chongqing Three Gorges University
Wanzhou 404100, China
e-mail: zuozhanfei@163.com

Yi-Min Huang
School of Environmental and Chemical Engineering
Chongqing Three Gorges University
Wanzhou 404100, China
e-mail: huangyimin@sanxiau.edu.cn

Jing Wang
School of Mathematics and Statistics
Chongqing Three Gorges University
Wanzhou 404100, China