Mathematical
Inequalities
& Applications
Volume 26, Number 1 (2023), 161–182 doi:10.7153/mia-2023-26-13

INTEGRAL EQUATIONS ON COMPACT MANIFOLD WITH BOUNDARY

YAZHOU HAN

(Communicated by J. Jakšetić)

Abstract. Let (M^n, g, Σ) be a smooth compact Riemannian manifold with boundary and $n \geqslant 3$. This paper is devoted to studying a class of integral system

$$\begin{cases} g^{p_\alpha - 1}(x) = \int_\Sigma K(x,y) f(y) dS_y, & x \in M^n, \\ f^{\tilde{p}_\alpha - 1}(y) = \int_{M^n} K(x,y) g(x) dV_x, & y \in \Sigma, \end{cases}$$

where $\alpha \in (1,n)$, $p_\alpha = \frac{2n}{n+\alpha}$, $\tilde{p}_\alpha = \frac{2(n-1)}{n+\alpha-2}$, $(f,g) \in L^{\tilde{p}_\alpha}(\Sigma) \times L^{p_\alpha}(M^n)$ and the kernel function $K(x,y) \in C^\infty(\overline{M^n} \times \overline{M^n} \setminus \{(x,x)\})$ satisfies $K(x,y) \sim |x-y|_g^{\alpha-n}$ as $|x-y|_g \to 0$. Since the system is the Euler-Lagrange equations of extremal problem

$$N_K(\alpha, M) = \sup\left\{ \left| \int_{M^n} \int_\Sigma g(x) K(x,y) f(y) dS_y dV_x \right| : \|f\|_{L^{\tilde{p}_\alpha}(\Sigma)} = \|g\|_{L^{p_\alpha}(M^n)} = 1 \right\},$$

we will study the existence of the system by concentration-compactness principle. Firstly, we get $N_K(\alpha, M) \geqslant C_e(n, \alpha, \tilde{p}_\alpha)$, where $C_e(n, \alpha, \tilde{p}_\alpha)$ is the best constant of Hardy-Littlewood-Sobolev inequalities on the upper half space established by Dou and Zhu [6] and equals to $N_K(\alpha, M)$ when $(M^n, g, \Sigma) = (B_1(0), |\cdot|, \partial B_1(0))$ and $K(x,y) = |x-y|^{\alpha-n}$. Secondly, if $N_K(\alpha, M) > C_e(n, \alpha, \tilde{p}_\alpha)$, we prove that $N_K(\alpha, M)$ is attained. Namely, under the criterion $N_K(\alpha, M) > C_e(n, \alpha, \tilde{p}_\alpha)$, we get the existence of the system. Lastly, a concrete example satisfying the criterion is given. The example is closely related to the conformal problems studied by Escobar [9, 10].

1. Introduction

In [15, 14], a class of conformal integral equations

$$u^{\frac{Q-\alpha}{Q+\alpha}}(x) = \int_M [G_x(y)]^{\frac{Q-\alpha}{Q-2}} u(y) dV_y \tag{1.1}$$

were studied. In [15], M is a compact Riemannian manifold without boundary, Q is the geometric dimension and $G_x(y)$ is the Green's function with pole at x for the conformal Laplacian operator $-\Delta_g + \frac{n-2}{4(n-1)} R_g$. So, (1.1) is equivalent to Yamabe equation when $\alpha = 2$. While in [14], M is a compact CR manifold without boundary, Q is the homogenous dimension and $G_x(y)$ is the Green's function with pole at x for the CR conformal Laplacian operator $-b_n \Delta_b^\theta + R^\theta$. Therefore, if $\alpha = 2$, (1.1) is equivalent to CR Yamabe equation studied by Jerison and Lee [20].

Mathematics subject classification (2020): 45G15, 35A23, 45E10.
Keywords and phrases: Integral equations, concentration-compactness principle, compact Riemannian manifold with boundary, Hardy-Littlewood-Sobolev inequality.

Apart from their own research significance, as stated in [28], (1.1) is closed related to the curvature problems such as Yamabe problem, Q-curvature problem, etc.. This paper is mainly devoted to studying a class of integral equations with general kernel $K(x,y)$ on compact Riemannian manifold with boundary by concentration-compactness principle.

Let (M^n,g) be a smooth Riemannian manifold with boundary $\Sigma=\partial M^n$, where g is the Riemannian metric on M^n and $n\geqslant 3$. Denote by $|x-y|_g$ the geodesic distance from x to y on $\overline{M^n}$ under metric g. We will concern with a class of integral system

$$\begin{cases} g^{p_\alpha-1}(x)=\int_\Sigma K(x,y)f(y)dS_y, & x\in M^n,\\ f^{\tilde{p}_\alpha-1}(y)=\int_{M^n}K(x,y)g(x)dV_x, & y\in\Sigma,\end{cases} \tag{1.2}$$

where $p_\alpha:=\frac{2n}{n+\alpha}$, $\tilde{p}_\alpha:=\frac{2(n-1)}{n+\alpha-2}$, $(f,g)\in L^{\tilde{p}_\alpha}(\Sigma)\times L^{p_\alpha}(M^n)$ and the kernel function $K(x,y)\in C^\infty(\overline{M^n}\times\overline{M^n}\backslash\{(x,x)\})$ satisfies

$$(\mathbf{K}) \qquad K(x,y)\sim|x-y|_g^{\alpha-n} \text{ as } |x-y|_g\to 0.$$

It is easy to see that the system (1.2) is the Euler-Lagrange equations of extremal problem

$$\begin{aligned} N_K(\alpha,M)&=\sup\left\{\left|\int_{M^n}\int_\Sigma g(x)K(x,y)f(y)dS_ydV_x\right| : \|f\|_{L^{\tilde{p}_\alpha}(\Sigma)}=\|g\|_{L^{p_\alpha}(M^n)}=1\right\}\\ &=\sup\left\{\|E_{\alpha,K}f\|_{L^{q_\alpha}(M^n)} : \|f\|_{L^{\tilde{p}_\alpha}(\Sigma)}=1\right\}, \end{aligned} \tag{1.3}$$

where $E_{\alpha,K}f(x)=\int_\Sigma K(x,y)f(y)dS_y$ and $q_\alpha=\frac{2n}{n-\alpha}$ is the conjugate number of p_α. Particularly, when $K(x,y)=|x-y|_g^{\alpha-n}$, denote $E_{\alpha,K}f(x)$ by $E_\alpha f(x)$.

Let $\mathbb{R}^n_+:=\{x=(x',x_n)\in\mathbb{R}^n\,|\,x_n>0\}$, $n\geqslant 2$ be the upper half space and $|\cdot|$ be the common Euclidean distance function. On $(\mathbb{R}^n_+,|\cdot|)$, Dou and Zhu [6] proved that, for any $1<\alpha<n$, $1<p,q<+\infty$ satisfying $\frac{1}{q}=\frac{n-1}{n}(\frac{1}{p}-\frac{\alpha-1}{n-1})$,

$$C_e(n,\alpha,p)=\sup\{\|E_\alpha f\|_{L^q(\mathbb{R}^n_+)} : \|f\|_{L^p(\partial\mathbb{R}^n_+)}=1\} \tag{1.4}$$

can be achieved. If $(f,g)\in L^{\tilde{p}_\alpha}(\partial\mathbb{R}^n_+)\times L^{p_\alpha}(\mathbb{R}^n_+)$ is any solution of (1.2) with $K(x,y)=|x-y|^{\alpha-n}$, they classified them and computed the best constant $C_e(n,\alpha,\tilde{p}_\alpha)$.

Under the Möbius transformation $\mathscr{T}: x=(x',x_n)\in\mathbb{R}^n_+\to x\in B_1(0)$ defined as

$$\mathscr{T}(x)=\frac{4(x',x_n+2)}{|(x',x_n+2)|^2}-e_n, \quad e_n=(0,\cdots,0,1), \tag{1.5}$$

it holds that, if $K(x,y)=|x-y|^{\alpha-n}$, system (1.2) on $(\mathbb{R}^n_+,|\cdot|)$ is equivalent to the system on $(B_1(0),|\cdot|)$. So, if $(M^n,g)=(B_1(0),|\cdot|)$ and $K(x,y)=|x-y|^{\alpha-n}$ with $1<\alpha<n$, Dou and Zhu [6] also classified the solutions $(f,g)\in L^{\tilde{p}_\alpha}(\partial B_1(0))\times L^{p_\alpha}(B_1(0))$ of (1.2).

When $(M^n,g)=(\Omega,|\cdot|)$ and $K(x,y)=|x-y|^{2-n}$, where $\Omega\subset\mathbb{R}^n (n\geqslant 3)$ is a bounded smooth domain, Gluck and Zhu [13] studied the following extremal problem

$$\mathscr{E}_2(\Omega)=\sup\{\|E_2 f\|_{L^{\frac{2n}{n-2}}(\Omega)} : \|f\|_{L^{\frac{2(n-1)}{n}}(\partial\Omega)}=1\}. \tag{1.6}$$

They proved

$$\mathscr{E}_2(\Omega) \geqslant \mathscr{E}_2(B_1(0)) = C_e\left(n, 2, \frac{2(n-1)}{n}\right) = n^{\frac{n-2}{2(n-1)}} \omega_n^{1-\frac{1}{n}-\frac{1}{2(n-1)}},$$

and gave a criterion of the existence of extremal function of (1.6). Namely, if $\mathscr{E}_2(\Omega) > \mathscr{E}_2(B_1(0))$, the best constant $\mathscr{E}_2(\Omega)$ can be achieved by some extremal function $f \in L^{\frac{2(n-1)}{n}}(\partial\Omega)$. Moreover, for the annular domain $A_r = B_1(0) \setminus B_r(0)$, they proved that the criterion $\mathscr{E}_2(A_r) > \mathscr{E}_2(B_1(0))$ holds if r is sufficiently small. So, for any bounded smooth domain Ω, if $\mathscr{E}_2(\Omega) > \mathscr{E}_2(B_1(0))$, there exists function pair $(f,g) \in L^{\frac{2(n-1)}{n}}(\partial\Omega) \times L^{\frac{2n}{n+2}}(\Omega)$ satisfying (1.2) with $(\tilde{p}_\alpha, p_\alpha) = (\frac{2(n-1)}{n}, \frac{2n}{n+2})$.

If (M^n, g, Σ) is compact and the first eigenvalue $\lambda_1(L_g)$ of $L_g = -\Delta_g + \frac{n-2}{4(n-1)}R_g$ is positive, we know that there exists the Green function $G(x,y)$ with pole at y, which satisfies

$$\begin{cases} -\Delta_g G(x,y) + \frac{n-2}{4(n-1)} R_g G(x,y) = \delta_y(x), & \text{on } M^n, \\ \frac{\partial_g G(x,y)}{\partial \eta} + h_g G(x,y) = 0, & \text{on } \Sigma, \end{cases} \tag{1.7}$$

where R_g is the scalar curvature, η is the outward normal with respect to metric g, and h_g is the mean curvature of Σ. Escobar [9, 10] showed $G(x,y) \sim |x-y|_g^{2-n}$ as $|x-y|_g \to 0$. We can prove routinely that (1.2) is conformal if $K(x,y) = G(x,y)^{\frac{n-\alpha}{n-2}}$. A natural question is what about the existence of the solutions for (1.2) with kernel function $G(x,y)^{\frac{n-\alpha}{n-2}}$.

In the sequel, assume always that (M^n, g, Σ) is compact. The main goal is to study extremal problem (1.3) with general kernel $K(x,y)$ satisfying (**K**) and to get the existence of extremal functions.

Main results state as follows.

THEOREM 1.1. *Assume $1 < \alpha < n$, $1 < p, t < \infty$ with*

$$\frac{n-1}{n} \cdot \frac{1}{p} + \frac{1}{t} + \frac{n-\alpha+1}{n} = 2, \tag{1.8}$$

*and $K(x,y) \in C^\infty(\overline{M^n} \times \overline{M^n} \setminus \{(x,x)\})$ satisfying (**K**). Then there is a positive constant $C = C(n, \alpha, p, M)$, such that Hardy-Littlewood-Sobolev (HLS) inequality*

$$\left| \int_{M^n} \int_\Sigma g(x) K(x,y) f(y) dS_y dV_x \right| \leqslant C \|f\|_{L^p(\Sigma)} \|g\|_{L^t(M^n)} \tag{1.9}$$

holds for all $f \in L^p(\Sigma)$ and $g \in L^t(M^n)$. Define

$$N_K(p, \alpha, M) = \sup_{\|f\|_{L^p(\Sigma)} = \|g\|_{L^t(M^n)} = 1} \left| \int_{M^n} \int_\Sigma g(x) K(x,y) f(y) dS_y dV_x \right|. \tag{1.10}$$

Then, $N_K(p, \alpha, M) \geqslant C_e(n, \alpha, p)$ and $N_K(p, \alpha, M)$ can be attained if $N_K(p, \alpha, M) > C_e(n, \alpha, p)$. Moreover, if $N_K(\alpha, M) = N_K(\tilde{p}_\alpha, \alpha, M) > C_e(n, \alpha, \tilde{p}_\alpha)$, extremal pair of (1.3) satisfies system (1.2).

Since $G(x,y)^{\frac{n-\alpha}{n-2}} \sim |x-y|_g^{\alpha-n}$ as $|x-y|_g \to 0$, then as [15], it is sufficient to discuss the following HLS inequality: for any $f \in L^p(\Sigma)$ and $g \in L^t(M^n)$,

$$\Big|\int_{M^n}\int_{\Sigma} g(x)|x-y|_g^{\alpha-n} f(y) dS_y dV_x\Big| \leqslant C\|f\|_{L^p(\Sigma)}\|g\|_{L^t(M^n)}, \tag{1.11}$$

which is equivalent to

$$||E_\alpha f||_{L^q(M^n)} \leqslant C||f||_{L^p(\Sigma)}, \tag{1.12}$$

where q satisfies

$$\frac{1}{q} = \frac{n-1}{n}\Big(\frac{1}{p} - \frac{\alpha-1}{n-1}\Big), \tag{1.13}$$

and $\frac{1}{q} < \frac{n-1}{n}\frac{1}{p} < \frac{1}{p}$. i.e., $1 < p < q$. So, I will get the following result.

PROPOSITION 1.2. *Assume that* $1 < \alpha < n$, $1 < p < \frac{n-1}{\alpha-1}$ *and* q *is given by* (1.13). *Then there is a positive constant* $C = C(n,\alpha,p,M)$ *such that* (1.12) *holds for any* $f \in L^p(\Sigma)$. *Moreover, for* $1 \leqslant r < q$, *operator* $E_\alpha : L^p(M^n) \hookrightarrow L^r(\Sigma)$ *is compact.*

Define

$$N_{p,\alpha,M} = \sup\big\{||E_\alpha f||_{L^q(M^n)} : ||f||_{L^p(\Sigma)} = 1\big\}. \tag{1.14}$$

We have the following criterion for the existence of the extremal functions.

THEOREM 1.3. $N_{p,\alpha,M} \geqslant C_e(n,\alpha,p)$. *Moreover, if* $N_{p,\alpha,M} > C_e(n,\alpha,p)$, *then* $N_{p,\alpha,M}$ *can be attained.*

REMARK 1.4. If $(M^n,g,\Sigma) = (\Omega,|\cdot|,\partial\Omega)$ (where $\Omega \subset \mathbb{R}^n$ is bounded and smooth), $\alpha = 2$, $p = \frac{2(n-1)}{n}$ and $t = \frac{2n}{n+2}$, the result of Theorem 1.3 is the main result of Gluck and Zhu [13].

A more general example for Theorem 1.1 is given as follows. Assume that (M^n,g) is a locally conformally flat compact Riemannian manifold with umbilic boundary Σ. If $\lambda_1(L_g) > 0$, Escobar [9, 10] showed that the Green function $G(x,y)$ for conformal Laplacian L_g has the following asymptotic expansion

$$G(x,y) = |x-y|_g^{2-n} + A(y) + \alpha(x,y) \tag{1.15}$$

for $|x-y|_g$ small, where $A(y)$ is nonnegative, $\alpha(x,y)$ is a smooth harmonic function near $y \in \Sigma$ and $\alpha(y,y) = 0$. Moreover, $A = 0$ if and only if M^n is conformally diffeomorphic to $B_1(0)$.

Based on the above positive mass type result, we have

THEOREM 1.5. *Let* M^n *be a locally conformally flat compact Riemannian manifold with umbilic boundary* Σ *and* $K(x,y) = G(x,y)^{\frac{n-\alpha}{n-2}}$. *If* $\lambda_1(L_g) > 0$ *and* M^n *is not conformally diffeomorphic to* $B_1(0)$, *then*

$$N_K(\tilde{p}_\alpha,\alpha,M) > C_e(n,\alpha,\tilde{p}_\alpha),$$

which implies the existence of solutions to (1.2) *with* $K(x,y) = G(x,y)^{\frac{n-\alpha}{n-2}}$ *by Theorem* 1.1.

REMARK 1.6. When M^n is a bounded domain $\Omega \subset \mathbb{R}^n$ with umbilic boundary and isn't conformally diffeomorphic to $B_1(0)$, it holds $N_K\left(\frac{2(n-1)}{n}, 2, \Omega\right) > C_e\left(n, 2, \frac{2(n-1)}{n}\right)$ with $K(x,y) = G(x,y)$. For example, for any $r \in (0,1)$, annular domain $A_r = B_1^n(0) \setminus B_r(0)$ satisfies the foregoing conditions. The result is not same as the example given by Gluck and Zhu in [13].

Ngô [25] showed that the range $(1,n)$ of parameters α of (1.4) is not necessary. And an optimal HLS inequality with necessary range of α, which generalized the classical HLS inequality and the HLS inequality (1.4), was established by a new method in [26]. Namely, their proofs do not make use the layer cake representation technique nor the Marcinkiewicz interpolation inequality.

In Lemma 2.3, we will establish a subcritical type inequality with $0 < \alpha < n$. So, applying the Marcinkiewicz interpolation theorem, we can give a new proof for the HLS inequality on the upper half space.

PROPOSITION 1.7. *Let* $0 < \alpha < n$ *such that* $\tilde{p}_\alpha < q_\alpha$. *Then there exists a positive constant* $C := C(n,p,\alpha)$ *such that*

$$\left\| \int_{\partial\mathbb{R}^n_+} \frac{f(y)dy}{|x-y|^{n-\alpha}} \right\|_{L^{q_\alpha}(\mathbb{R}^n_+)} \leqslant C\|f\|_{L^{\tilde{p}_\alpha}(\partial\mathbb{R}^n_+)} \tag{1.16}$$

holds for any $f \in L^{\tilde{p}_\alpha}(\partial\mathbb{R}^n_+)$.

The paper is organized as follows. In Section 2, we first establish a subcritical type inequality (see Lemma 2.3) and a ε-level inequality (see Lemma 2.6). Then, complete the proof of Proposition 1.2. In Section 3, we present the concentration-compactness principle. Section 4 is devoted to the proof of Theorem 1.3. Therefore, we can get our main result – Theorem 1.1. Finally, conformal HLS inequality on the upper half space (Proposition 1.7) are established in Section 5.

2. HLS inequality

This section is devoted to establishing a rough HLS inequality on a smooth compact Riemannian manifold (M^n, g) with boundary $\Sigma = \partial M^n$.

For convenience, we introduce firstly some notations. For $\delta > 0$ small enough, write $M_\delta = \{x \in M^n : dist(x,\Sigma) \leqslant \delta\}$ is a tubular neighborhood of Σ and $\pi : M_\delta \to \Sigma$ denotes the nearest point projection. For $\xi \in \Sigma$, choose a normal coordinate for Σ at ξ, namely $\tau_1, \ldots, \tau_{n-1}$. Let $C_\delta = \{x \in M_\delta : dist(\pi(x), \xi) \leqslant \delta\}$. For δ small, we have a coordinate near ξ for M^n as

$$\phi : C_\delta \to \overline{B_\delta^{n-1}(0)} \times [0,\delta] : x \mapsto (\tau(\pi(x)), t(x)),$$

where $t(x) := dist(x,\Sigma)$. It is usually called the Fermi coordinate at ξ. We will identify C_δ with $\overline{B_\delta^{n-1}(0)} \times [0,\delta]$ through ϕ.

Note that $g = g_{ij}dx_i \otimes dx_j + dx_n \otimes dx_n$. Under the Fermi coordinate (see e.g., [17]), we know that for any $\varepsilon > 0$ and $\xi \in \Sigma$, there exists $\delta > 0$ such that

$$(1-\varepsilon)I \leqslant g \leqslant (1+\varepsilon)I \quad \text{in } C_\delta. \tag{2.1}$$

2.1. Subcritical HLS inequality

Firstly, we recall the following Hausdorff-Young type inequality established in [17].

LEMMA 2.1. (Lemma 2.3 in [17]) *Assume X and Y be measure spaces, and $1 \leqslant p, q_0, q_1, r \leqslant \infty$, $p \leqslant r$, $q_0 \leqslant r$ and $\frac{1}{p} + \frac{1}{q_1} = \frac{q_0}{q_1 r} + 1$. Define $(Kf)(x) = \int_Y K(x,y) f(y)dy$, where K is defined in $X \times Y$ and satisfies*

$$\Big(\int_X |K(x,y)|^{q_0} dx\Big)^{\frac{1}{q_0}} \leqslant A, \quad \Big(\int_Y |K(x,y)|^{q_1} dy\Big)^{\frac{1}{q_1}} \leqslant A.$$

then for a function f defined on Y,

$$\|Kf\|_{L^r(X)} \leqslant A\|f\|_{L^p(Y)}. \tag{2.2}$$

REMARK 2.2. By a careful study of the origin proof of Lemma 2.3 in [17], I find that the condition of $q_0, q_1 \geqslant 1$ can be weakened to $q_0, q_1 > 0$.

As an application of the above Hausdorff-Young type inequality and Remark 2.2, we have the following subcritical inequality.

LEMMA 2.3. *Let α, p, q, r satisfy $\alpha \in (0,n)$, $1 \leqslant p \leqslant r$, $q \geqslant 1$ and*

$$\frac{1}{r} > \frac{1}{q} = \frac{n-1}{n}\Big(\frac{1}{p} - \frac{\alpha-1}{n-1}\Big). \tag{2.3}$$

Then there is a positive constant $C(\alpha, p, M^n, g)$, such that

$$||E_\alpha f||_{L^r(M^n)} \leqslant C(n,\alpha,p,M^n,g)||f||_{L^p(\Sigma)} \tag{2.4}$$

holds for all $f \in L^p(\Sigma)$. Moreover, for $1 \leqslant r < q$, operator $E_\alpha : L^p(M^n) \hookrightarrow L^r(\Sigma)$ is compact.

Proof. Let $q_1 > 0$ and $q_0 = \frac{n}{n-1} q_1$ satisfy

$$\frac{1}{r} + \frac{q_1}{q_0} = \frac{q_1}{q_0}\Big(\frac{1}{p} + \frac{1}{q_1}\Big). \tag{2.5}$$

Then $q_0 \leqslant r$, $q_0 \in (0, \frac{n}{n-\alpha})$ and $q_1 \in (0, \frac{n-1}{n-\alpha})$.

For any $x \in M^n$ and $y \in \Sigma$, since $(n-\alpha)q_0 < (n-\alpha)\frac{n}{n-\alpha} = n$, we have

$$\int_{M^n} |x-y|_g^{(\alpha-n)q_0} dV(x) \leqslant C(n, q_0, \alpha, M^n).$$

Using $(n-\alpha)q_1 < (n-\alpha)\frac{n-1}{n-\alpha} = n-1$ and the fact

$$|\pi(x)-y|_g \leqslant |\pi(x)-x|_g + |x-y|_g \leqslant 2|x-y|_g,$$

we have

$$\int_{\Sigma} |x-y|_g^{(\alpha-n)q_1} dS_g(y) \leqslant C\int_{\Sigma} |\pi(x)-y|_g^{(\alpha-n)q_1} dS_g(y) \leqslant C(n, q_1, \alpha, M^n).$$

Hence, the inequality (2.4) follows from Lemma 2.1 and Remark 2.2.

Now, we show the embedding $E_\alpha : L^p(M^n) \hookrightarrow L^r(\Sigma)$ is compact. In fact, it is sufficient to prove the result for the case $p \leqslant r < q$.

Let $\{f_m(y)\} \subset L^p(\Sigma)$ be a bounded sequence. Then there exists a subsequence (still denoted by $\{f_m(y)\}$) and some function $f \in L^p(\Sigma)$ such that

$$f_m \rightharpoonup f \quad \text{weakly in } L^p(\Sigma).$$

For some constant $\rho > 0$ determined later, we write $E_\alpha f_m(x)$ as

$$\begin{aligned} E_\alpha f_m(x) &= \int_\Sigma \frac{f_m(y)}{|x-y|_g^{n-\alpha}} dS_g(y) \\ &= \int_{\Sigma\cap\{|x-y|_g>\rho\}} \frac{f_m(y)}{|x-y|_g^{n-\alpha}} dS_g(y) + \int_{\Sigma\cap\{|x-y|_g\leqslant\rho\}} \frac{f_m(y)}{|x-y|_g^{n-\alpha}} dS_g(y) \\ &=: E_\alpha^1 f_m(x) + E_\alpha^2 f_m(x). \end{aligned} \tag{2.6}$$

For a given point $x \in M^n$, since $|x-y|_g^{\alpha-n}\chi_{\{y:|x-y|_g\geqslant\rho\}} \in L^{p/(p-1)}(\Sigma)$, the weakly convergence implies that

$$E_\alpha^1 f_m \to E_\alpha^1 f \quad \text{pointwisely a.e. in } M^n,$$

and by Hölder inequality we deduce that

$$\begin{aligned} |E_\alpha^1 f_m| &\leqslant \int_{\Sigma\cap\{|x-y|_g>\rho\}} \frac{|f_m(y)|}{|x-y|_g^{n-\alpha}} dS_g(y) \\ &\leqslant \Big(\int_{\{|x-y|_g>\rho\}} |x-y|_g^{\frac{(\alpha-n)p}{p-1}} dS_g(y)\Big)^{\frac{p-1}{p}} \|f_m\|_{L^p(\Sigma)} \leqslant C(\rho). \end{aligned}$$

Hence, by dominated convergence theorem, we have

$$E_\alpha^1 f_m \to E_\alpha^1 f \quad \text{strongly in } L^r(M^n). \tag{2.7}$$

Since $q_0 \in (0, \frac{n}{n-\alpha})$ and $q_1 \in (0, \frac{n-1}{n-\alpha})$ in (2.5), we have

$$\Big(\int_{\{M^n\cap|x-y|_g\leqslant\rho\}} |x-y|_g^{(\alpha-n)q_0} dV(y)\Big)^{\frac{1}{q_0}} \leqslant C\rho^{\alpha-n+\frac{n}{q_0}}$$

and

$$\Big(\int_{\{\Sigma\cap|x-y|_g\leqslant\rho\}}|x-y|_g^{(\alpha-n)q_1}dS_g(y)\Big)^{\frac{1}{q_1}}\leqslant C\rho^{\alpha-n+\frac{n-1}{q_1}}.$$

Write $A=C\max\left\{\rho^{\alpha-n+\frac{n}{q_0}},\rho^{\alpha-n+\frac{n-1}{q_1}}\right\}$. Obviously, $A\to 0$ as $\rho\to 0$. From Lemma 2.1, it yields

$$\|E_\alpha^2(f_m-f)\|_{L^r(M^n)}\leqslant A\|f_m-f\|_{L^p(\Sigma)}. \tag{2.8}$$

Choosing ρ small firstly in (2.8) and then sending m to infinity in (2.7), we arrive at

$$E_\alpha f_m\to E_\alpha f \quad \text{strongly in } L^r(M^n).$$

Hence the embedding is compact. □

REMARK 2.4. Under the conditions of Lemma 2.3, we have $1\leqslant p\leqslant r<q$, $p<\frac{n-1}{\alpha-1}$ if $1<\alpha<n$ and $p<\frac{1}{1-\alpha}$ if $0<\alpha<1$.

Define the extremal problem for inequality (2.4) as

$$N_{p,r,\alpha,M}:=\sup\{\|E_\alpha f\|_{L^r(M^n)}:\ \|f\|_{L^p(\Sigma)}=1\}. \tag{2.9}$$

By the compactness of Lemma 2.3, we have

PROPOSITION 2.5. *$N_{p,r,\alpha,M}$ can be attained. Namely, there exists some nonnegative function $f\in L^p(\Sigma)$ such that $\|f\|_{L^p(\Sigma)}=1$ and $N_{p,r,\alpha,M}=\|E_\alpha f\|_{L^r(M^n)}$. Moreover, f satisfies the following Euler-Lagrange equation*

$$N^r_{p,r,\alpha,M}f^{p-1}(y)=\int_{M^n}\frac{(E_\alpha f)^{r-1}(x)}{|x-y|_g^{n-\alpha}}dV_x. \tag{2.10}$$

Moreover, $f\in L^\infty(\Sigma)$.

Proof. Since the proof is standard, we omit the details in here. □

2.2. Roughly HLS inequality

We firstly establish the following ε-version inequality.

LEMMA 2.6. *Let $1<\alpha<n$, and $1<p<\frac{n-1}{\alpha-1}$ and q satisfy (1.13). Then, for any small $\varepsilon>0$, there is a constant $C(\varepsilon)>0$ such that*

$$||E_\alpha f||^p_{L^q(M^n)}\leqslant\big(C_e(n,\alpha,p)+\varepsilon\big)^p||f||^p_{L^p(\Sigma)}+C(\varepsilon)\|E_{\alpha+1}f\|^p_{L^q(M^n)} \tag{2.11}$$

holds for any $f\in L^p(\Sigma)$.

Proof. Without loss of generality, we assume $f \geqslant 0$.

Using compactness of Σ, (2.1) and the HLS inequality established by Dou and Zhu [6], for fixed $\varepsilon > 0$, we can choose $\delta > 0$ small and take $\{\eta_{i,\varepsilon}\}_{i=1}^{k}$ be a partition of the unit covering of M_δ, such that $0 \leqslant \eta_{i,\varepsilon} \leqslant 1$, $\operatorname{supp}\{\eta_{i,\varepsilon}\} \cap \Sigma \neq \emptyset$ for all $i = 1,2,\cdots,k$, $\sum_{i=1}^{k} \eta_{i,\varepsilon}^{p} = 1$, and for all $i = 1,2,\cdots,k$,

$$||E_\alpha(\eta_{i,\varepsilon} f)||_{L^q(\operatorname{supp}\{\eta_{i,\varepsilon}\})} \leqslant \big(C_e(n,\alpha,p)+\varepsilon\big)||\eta_{i,\varepsilon} f||_{L^p(\Sigma\cap\operatorname{supp}\{\eta_{i,\varepsilon}\})}. \tag{2.12}$$

Thus, similar to the computation of Proposition 2.5 of [15], we have

$$\begin{aligned}
&||E_\alpha f||^p_{L^q(M_\delta)} = ||(E_\alpha f)^p||_{L^{q/p}(M_\delta)}\\
=&||\sum_{i=1}^{k} \eta_{i,\varepsilon}^{p}(E_\alpha f)^p||_{L^{q/p}(M_\delta)} \leqslant \sum_{i=1}^{k} ||\eta_{i,\varepsilon}^{p}(E_\alpha f)^p||_{L^{q/p}(\operatorname{supp}\{\eta_{i,\varepsilon}\})}\\
=&\sum_{i=1}^{k} ||\eta_{i,\varepsilon} E_\alpha f||^p_{L^q(\operatorname{supp}\{\eta_{i,\varepsilon}\})}\\
\leqslant&\sum_{i=1}^{k}\Big(||E_\alpha(\eta_{i,\varepsilon} f)||_{L^q(\operatorname{supp}\{\eta_{i,\varepsilon}\})} + ||\eta_{i,\varepsilon} E_\alpha f - E_\alpha(\eta_{i,\varepsilon} f)||_{L^q(\operatorname{supp}\{\eta_{i,\varepsilon}\})}\Big)^p\\
\leqslant&\sum_{i=1}^{k}\big((C_e(n,\alpha,p)+\varepsilon)||\eta_{i,\varepsilon} f||_{L^p(\Sigma\cap\operatorname{supp}\{\eta_{i,\varepsilon}\})} + C(\varepsilon)\|E_{\alpha+1} f\|_{L^q(M^n)}\big)^p\\
\leqslant&\sum_{i=1}^{k}(C_e(n,\alpha,p)+\varepsilon)^p(1+\varepsilon)||\eta_{i,\varepsilon} f||^p_{L^p(\Sigma\cap\operatorname{supp}\{\eta_{i,\varepsilon}\})} + C(\varepsilon)\|E_{\alpha+1} f\|^p_{L^q(M^n)}\\
=&(C_e(n,\alpha,p)+\varepsilon)^p(1+\varepsilon)||f||^p_{L^p(\Sigma)} + C(\varepsilon)\|E_{\alpha+1} f\|^p_{L^q(M^n)}.
\end{aligned} \tag{2.13}$$

On the other hand, if $x \in M^n \setminus M_\delta$, then

$$E_\alpha f(x) \leqslant \frac{1}{\delta}\int_\Sigma \frac{|x-y|_g f(y)}{|x-y|_g^{n-\alpha}} dS_y = \frac{1}{\delta} E_{\alpha+1} f(x).$$

Thus, we arrive at

$$\begin{aligned}
||E_\alpha f||^p_{L^q(M^n)} &\leqslant \Big(||E_\alpha f||_{L^q(M^n\setminus M_\delta)} + ||E_\alpha f||_{L^q(M_\delta)}\Big)^p\\
&\leqslant \Big(\frac{1}{\delta}||E_{\alpha+1} f||_{L^q(M^n\setminus M_\delta)} + ||E_\alpha f||_{L^q(M_\delta)}\Big)^p,
\end{aligned}$$

Combining the above and (2.13) we get (2.11). □

PROPOSITION 2.7. *Let $1 < \alpha < n$, and $1 < p < \frac{n-1}{\alpha-1}$ and q satisfy (1.13). Then, there exists some positive constant $C = C(\alpha,p,M^n,g)$ such that, for any $f \in L^p(\Sigma)$,*

$$||E_\alpha f||_{L^q(M^n)} \leqslant C||f||_{L^p(\Sigma)}. \tag{2.14}$$

Proof. Combining Lemma 2.6 and Lemma 2.3, we complete the proof. □

REMARK 2.8. For any bounded sequence $\{f_m\} \subset L^p(\Sigma)$, there exists a subsequence (still denoted by $\{f_m\}$) and some function $f \in L^p(\Sigma)$ such that

$$\begin{aligned} f_m \rightharpoonup f &\quad \text{weakly in} \quad L^p(\Sigma), \\ E_\alpha f_m \rightharpoonup E_\alpha f &\quad \text{weakly in} \quad L^q(M^n), \\ E_\alpha f_m \to E_\alpha f &\quad \text{strongly in} \quad L^r(M^n) \end{aligned}$$

for all $r \in [1,q)$. Furthermore, $E_\alpha f_m \to E_\alpha f$ pointwisely a.e. in M^n.

3. Concentration-compactness principle

LEMMA 3.1. *Assume that $\{f_m\} \subset L^p(\Sigma)$ are a bounded nonnegative sequence and there exists some function $f \in L^p(\Sigma)$ such that*

$$f_m \rightharpoonup f \quad \textit{weakly in} \quad L^p(\Sigma).$$

After passing to a subsequence, assume that $|E_\alpha f_m|^q dV(x)$, $|f_m|^p dS_y$ converge weakly in the sense of measure to some bounded nonnegative measures ν, μ on M^n. Then,

i). *There exist some countable set J, a family $\{P_j : j \in J\}$ of distinct points in Σ, and a family $\{\nu_j : j \in J\}$ of nonnegative numbers such that*

$$\nu = |E_\alpha f|^q dV(x) + \sum_{j\in J} \nu_j \delta_{P_j}, \tag{3.1}$$

where δ_{P_j} are the Dirac-mass of mass 1 concentrated at $P_j \in \Sigma$;

ii). *In addition,*

$$\mu \geqslant |f|^p dS_y + \sum_{j\in J} \mu_j \delta_{P_j} \tag{3.2}$$

for some family $\{\mu_j > 0 : j \in J\}$, where μ_j satisfy

$$\nu_j^{1/q} \leqslant C_e(n,\alpha,p)\mu_j^{1/p} \quad \textit{for all} \quad j \in J. \tag{3.3}$$

In particular, $\sum_{j\in J} \nu_j^{p/q} < +\infty$.

Proof. We firstly show $i)$. Let $\{f_m\} \subset L^p(\Sigma)$ be a bounded nonnegative sequence. It follows from Remark 2.8 that

$$E_\alpha f_m \rightharpoonup E_\alpha f \quad \text{weakly in} \quad L^q(M^n), \tag{3.4}$$

$$E_\alpha f_m \to E_\alpha f \quad \text{strongly in} \quad L^r(M^n), \tag{3.5}$$

$$E_\alpha f_m \to E_\alpha f \quad \text{pointwisely a.e. in} \quad M^n, \tag{3.6}$$

where $r \in [1,q)$. Then, Brézis-Lieb Lemma leads that

$$\begin{aligned} 0 &= \lim_{m\to+\infty} \int_{M^n} (|E_\alpha f_m|^q - |E_\alpha(f_m - f)|^q - |E_\alpha f|^q)\, dV(x) \\ &= \int_{M^n} d\nu - \int_{M^n} |E_\alpha f|^q dV(x) - \lim_{m\to+\infty} \int_{M^n} |E_\alpha(f_m - f)|^q dV(x). \end{aligned}$$

So, it is sufficient to discuss the case $f \equiv 0$.

Let $\varphi(x) \in C_0^\infty(M^n \backslash \Sigma)$. Since $dist(supp\{\varphi\}, \Sigma) > 0$ and (3.6), we have

$$\left|\varphi(x) \int_\Sigma \frac{f_m(y)}{|x-y|_g^{n-\alpha}} dS_y\right| \leqslant C \int_\Sigma |f_m| dS_y \leqslant C$$

and, as $m \to \infty$,

$$\varphi(x) E_\alpha f_m \to \varphi(x) E_\alpha f \quad \text{pointwisely a.e. in } M^n.$$

It follows from dominated convergence theorem that for any $\varphi(x) \in C_0^\infty(M^n \backslash \Sigma)$,

$$\lim_{m \to +\infty} \int_{M^n} |\varphi(x) E_\alpha f_m|^q dV(x) = \int_{M^n} |\varphi(x) E_\alpha f|^q dV(x).$$

Hence, $\nu|_{M^n \backslash \Sigma} = |E_\alpha f|^q dV(x)$.

Suppose that $\varphi(x) \in C_0^\infty(\overline{M^n})$ satisfies $\mathrm{supp}(\varphi) \cap \Sigma \neq \emptyset$ and $\mathrm{supp}(\varphi) \subset M_\delta$. By the classical argument of Lions (see [22, 23]), it is sufficient to prove that there exists some positive constant C such that

$$\Big(\int_{M^n} |\varphi|^q d\nu\Big)^{1/q} \leqslant C \Big(\int_\Sigma |\varphi|^p d\mu\Big)^{1/p}, \quad \forall \varphi \in C_0^\infty(M^n). \tag{3.7}$$

Indeed,

$$\begin{aligned}
&\Big(\int_{M_\delta} |\varphi \cdot E_\alpha f_m|^q dV(x)\Big)^{1/q} \\
\leqslant &\Big(\int_{M_\delta} |E_\alpha(\varphi f_m)|^q dV(x)\Big)^{1/q} + \Big(\int_{M_\delta} |\varphi \cdot E_\alpha f_m - E_\alpha(\varphi f_m)|^q dV(x)\Big)^{1/q} \\
\leqslant &C\Big(\int_\Sigma |\varphi f_m|^p dS_y\Big)^{1/p} + \Big(\int_{M_\delta} |\varphi \cdot E_\alpha f_m - E_\alpha(\varphi f_m)|^q dV(x)\Big)^{1/q}.
\end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}
&|\varphi \cdot E_\alpha f_m - E_\alpha(\varphi f_m)| \\
=&\left|\int_\Sigma (\varphi(x) - \varphi(y))|x-y|_g^{\alpha-n} f_m(y) dS_y\right| \\
\leqslant&\left|\int_{B_\delta^{n-1}(\pi(x))} (\varphi(x) - \varphi(y))|x-y|_g^{\alpha-n} f_m(y) dS_y\right| \\
&+\left|\int_{\Sigma \backslash B_\delta^{n-1}(\pi(x))} (\varphi(x) - \varphi(y))|x-y|_g^{\alpha-n} f_m(y) dS_y\right| \\
=:&J_1 + J_2.
\end{aligned}$$

Write $R(x,y) := (\varphi(x) - \varphi(y))|x-y|_g^{\alpha-n}$. It is easy to check that $R(x,y) \in L^r(M^n)$, for $r \leqslant +\infty$ if $\alpha + 1 - n \geqslant 0$ and $r < \frac{n-1}{n-\alpha-1}$ if $\alpha + 1 - n < 0$. Note that

$$J_1 \leqslant C\left|\int_{B_\delta^{n-1}(\pi(x))} |x-y|_g^{\alpha-n+1} f_m(y) dS_y\right|.$$

For $\alpha+1-n \geqslant 0$, by dominated convergence theorem we have

$$\lim_{m\to+\infty}\int_{M_\delta} J_1^q dV_x = 0. \tag{3.9}$$

For $\alpha+1-n<0$, by the HLS inequalities (1.12), we obtain

$$J_1 \in L^s(M^n),$$

where $\frac{1}{s}=\frac{n-1}{n}\left(\frac{1}{p}-\frac{\alpha}{n-1}\right)<\frac{1}{q}$. Furthermore, repeating the proof process of Lemma 2.3, we have (3.9) again. Since $R(x,y)$ is uniformly bounded for $y\in\Sigma\backslash B_\delta^{n-1}(\pi(x))$, by dominated convergence theorem, we arrive at

$$\begin{aligned}\int_{M_\delta} J_2^q dV(x) &= \int_{M_\delta} \Big|\int_{\Sigma\backslash B_\delta^{n-1}(\pi(x))} R(x,y) f_m(y) dS_y\Big|^q dV(x)\\ &\to \int_{M_\delta} \Big|\int_{\Sigma\backslash B_\delta^{n-1}(\pi(x))} R(x,y) f(y) dS_y\Big|^q dV(x) = 0.\end{aligned}$$

strongly in $L^q(M_\delta)$. Hence, we get

$$\lim_{m\to+\infty}\Big(\int_{M^n} |\varphi(x)E_\alpha f_m - E_\alpha(\varphi f_m)|^q dV_x\Big)^{1/q} = 0. \tag{3.10}$$

Letting $m\to+\infty$ in (3.8), and using (3.10) we deduce (3.7). Furthermore, by P. Lion's Lemma (see Lemma 1.2 in [23]), there exist some countable set J, a family $\{P_j : j\in J\}$ of distinct points in Σ such that

$$\lim_{m\to\infty} |E_\alpha(f_m-f)|^q dV(x) = \sum_{j\in J} \nu_j \delta_{P_j},$$

and

$$\nu = |E_\alpha f|^q dV(x) + \sum_{j\in J} \nu_j \delta_{P_j},$$

where $\nu_j = \nu(\{P_j\})$.

Next we show $ii)$. Since

$$f_m \rightharpoonup f \quad \text{weakly in} \quad L^p(\Sigma),$$

then, $\mu \geqslant |f|^p dS_y$. So, we just have to show that for each fixed $j\in J$,

$$\nu_j^{1/q} = \nu(\{P_j\})^{1/q} \leqslant C_e(n,\alpha,p)\mu(\{P_j\})^{1/p} = C_e(n,\alpha,p)\mu_j^{1/p}.$$

For point $P_j\in\Sigma$ and $\delta>0$ small enough, we can choose a neighbourhood $C_{\delta,P_j} := \{x\in M_\delta : d_\Sigma(\pi(x),P_j)\leqslant\delta\}\subset M_\delta$ so that

$$(1-\varepsilon)I \leqslant g(x) \leqslant (1+\varepsilon)I, \quad \forall x\in C_{\delta,P_j}.$$

Define $\varphi_\lambda(x) = \varphi(\frac{x}{\lambda})$, where $\varphi(x) \in C_0^\infty(\overline{\mathbb{R}^n_+})$ satisfies $0 \leqslant \varphi(x) \leqslant 1$, $\varphi(0) = 1$, $\operatorname{supp}\varphi \subset B_1^n(0) \cap \overline{\mathbb{R}^n_+}$ and $\lambda \in (0,\delta)$. Then,

$$\begin{aligned} E_\alpha((\varphi_\lambda \circ \phi) \cdot f_m) &= \int_\Sigma (\varphi_\lambda \circ \phi(y) f_m(y)|x-y|_g^{\alpha-n} dS_g(y) \\ &= \int_{B_\delta^n(0)\cap\partial\mathbb{R}^n_+} \varphi_\lambda(y)(f_m \circ \phi^{-1})(y)|x-y|_g^{\alpha-n}\sqrt{\det g(y)}dy \\ &\leqslant \frac{(1+\varepsilon)^{n/2}}{(1-\varepsilon)^{\frac{n-\alpha}{2}}} \int_{B_\delta^n(0)\cap\partial\mathbb{R}^n_+} \varphi_\lambda(y)(f_m \circ \phi^{-1})(y)|x-y|^{\alpha-n}dy \end{aligned}$$

and

$$\begin{aligned} &\Big(\int_{\phi^{-1}(B_\delta^n(0)\cap\mathbb{R}^n_+)} |E_\alpha((\varphi_\lambda \circ \phi)\cdot f_m)|^q dV_x\Big)^{1/q} \\ \leqslant\, &(1+\varepsilon)^{\frac{n}{2q}}\Big(\int_{B_\delta^n(0)} |E_\alpha((\varphi_\lambda \circ \phi)\cdot f_m)|^q dx\Big)^{1/q} \\ \leqslant\, &\frac{(1+\varepsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\varepsilon)^{\frac{n-\alpha}{2}}}\Big(\int_{B_\delta^n(0)}\Big|\int_{B_\delta^n(0)\cap\partial\mathbb{R}^n_+} \varphi_\lambda(y)(f_m\circ\phi^{-1})(y)|x-y|^{\alpha-n}dy\Big|^q dx\Big)^{1/q} \\ \leqslant\, &\frac{(1+\varepsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\varepsilon)^{\frac{n-\alpha}{2}}} C_e(n,\alpha,p)\Big(\int_{B_\delta^n(0)\cap\partial\mathbb{R}^n_+} |\varphi_\lambda(y)(f_m\circ\phi^{-1})(y)|^p dy\Big)^{1/p} \\ \leqslant\, &\frac{(1+\varepsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\varepsilon)^{\frac{n}{2p}+\frac{n-\alpha}{2}}} C_e(n,\alpha,p)\Big(\int_{\phi^{-1}(B_\delta^n(0)\cap\partial\mathbb{R}^n_+)} |(\varphi_\lambda\circ\phi)\cdot f_m|^p dy\Big)^{1/p}. \end{aligned}$$

Combining the above we have

$$\begin{aligned} &\Big(\int_{M^n} |(\varphi_\lambda\circ\phi)\cdot E_\alpha f_m|^q dV(x)\Big)^{1/q} \\ \leqslant\, &\Big(\int_{\phi^{-1}(B_\delta^n(0)\cap\mathbb{R}^n_+)} |E_\alpha((\varphi_\lambda\circ\phi)\cdot f_m)|^q dV(x)\Big)^{1/q} \\ &\quad + \Big(\int_{\phi^{-1}(B_\delta^n(0)\cap\mathbb{R}^n_+)} |(\varphi_\lambda\circ\phi)\cdot E_\alpha f_m - E_\alpha((\varphi_\lambda\circ\phi)\cdot f_m)|^q dV(x)\Big)^{1/q} \\ \leqslant\, &\frac{(1+\varepsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\varepsilon)^{\frac{n}{2p}+\frac{n-\alpha}{2}}} C_e(n,\alpha,p)\Big(\int_{\phi^{-1}(B_\delta^n(0)\cap\partial\mathbb{R}^n_+)} |(\varphi_\lambda\circ\phi)\cdot f_m|^p dS_y\Big)^{1/p} + \mathbf{I}, \qquad (3.11) \end{aligned}$$

where

$$\mathbf{I} := \Big(\int_{\phi^{-1}(B_\delta^n(0)\cap\mathbb{R}^n_+)} |(\varphi_\lambda\circ\phi)\cdot E_\alpha f_m - E_\alpha((\varphi_\lambda\circ\phi)\cdot f_m)|^q dV(x)\Big)^{1/q}.$$

Arguing as (3.8), we have

$$\mathbf{I} \to \Big(\int_{\phi^{-1}(B_\delta^n(0)\cap\mathbb{R}^n_+)} |(\varphi_\lambda\circ\phi)\cdot E_\alpha f - E_\alpha((\varphi_\lambda\circ\phi)\cdot f)|^q dV(x)\Big)^{1/q}, \quad \text{as } m \to +\infty.$$

Hence, letting $m \to +\infty$ leads

$$
\begin{aligned}
&\Big(\int_{M^n} |\varphi_\lambda \circ \phi|^q dv\Big)^{1/q} \\
\leqslant &\frac{(1+\varepsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\varepsilon)^{\frac{n}{2p}+\frac{n-\alpha}{2}}} C_e(n,\alpha,p)\Big(\int_\Sigma |(\varphi_\lambda \circ \phi)|^p d\mu\Big)^{1/p} \\
&+\left(\int_{\phi^{-1}(B^n_\delta(0)\cap\mathbb{R}^n_+)} |(\varphi_\lambda \circ \phi)\cdot E_\alpha f - E_\alpha((\varphi_\lambda\circ\phi)\cdot f)|^q dV(x)\right)^{1/q}. \qquad (3.12)
\end{aligned}
$$

Since

$$
\int_{\phi^{-1}(B^n_\delta(0)\cap\mathbb{R}^n_+)} |(\varphi_\lambda \circ \phi)\cdot E_\alpha f|^q dV(x) \to 0 \quad \text{as} \quad \lambda \to 0^+
$$

and

$$
\begin{aligned}
&\Big(\int_{\phi^{-1}(B^n_\delta(0)\cap\mathbb{R}^n_+)} |E_\alpha((\varphi_\lambda\circ\phi)\cdot f)|^q dV(x)\Big)^{1/q} \\
\leqslant &C\Big(\int_{\phi^{-1}(B^n_\delta(0)\cap\partial\mathbb{R}^n_+)} |(\varphi_\lambda\circ\phi)\cdot f|^p dS_y\Big)^{1/p} \to 0 \quad \text{as} \quad \lambda \to 0^+,
\end{aligned}
$$

the proof is complete by letting $\lambda \to 0^+$ and $\varepsilon \to 0^+$. □

4. A criterion for the existence of extremal problems

In this section, we discuss the existence of extremal problems to (1.12) (Theorem 1.3) and give an example for which the criterion $N_K(\tilde{p}_\alpha,\alpha,M) > C_e(n,\alpha,\tilde{p}_\alpha)$ is satisfied (Theorem 1.5).

4.1. Criterion for the existence of (1.12)

Because of the definition of $C_e(n,\alpha,p)$ (see (1.4)), we know that $C_e(n,\alpha,p)$ can also be defined as

$$
C_e(n,\alpha,p) = \sup\Big\{\Big|\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} \frac{f(y)g(x)}{|x-y|^{n-\alpha}} dydx\Big| : \|f\|_{L^p(\partial\mathbb{R}^n_+)} = \|g\|_{L^t(\mathbb{R}^n_+)} = 1\Big\}, \quad (4.1)
$$

where $t = q' = \frac{q}{q-1}$. Dou and Zhu [6] proved the best constant $C_e(n,\alpha,p)$ can be attained by a pair of nonnegative functions $(f,g) \in L^p(\partial\mathbb{R}^n_+)\times L^t(\mathbb{R}^n_+)$. Hence, extremal pair satisfies the Euler-Lagrange equations

$$
\begin{cases}
C_e(n,\alpha,p) f^{p-1}(y) = \int_{\mathbb{R}^n_+} g(x)|x-y|^{\alpha-n}dx, \\
C_e(n,\alpha,p) g^{t-1}(x) = \int_{\partial\mathbb{R}^n_+} f(y)|x-y|^{\alpha-n}dy.
\end{cases} \quad (4.2)
$$

Moreover, by scaling, we know that function pairs

$$f_\lambda(y) = \lambda^{-\frac{n-1}{p}} f(y/\lambda), \quad g_\lambda(x) = \lambda^{-\frac{n}{t}} g(x/\lambda), \quad \forall \lambda > 0 \tag{4.3}$$

also satisfy (4.1) and (4.2).

Similar to (4.1), extremal problems to (1.14) is equivalent to the following form

$$N_{p,\alpha,M} = \sup_{\|f\|_{L^p(\Sigma)}>0, \|g\|_{L^t(M^n)}>0} \frac{\left|\int_{M^n}\int_\Sigma f(y)g(x)|x-y|_g^{\alpha-n} dV_x dS_y\right|}{\|f\|_{L^p(\Sigma)}\|g\|_{L^t(M^n)}}. \tag{4.4}$$

LEMMA 4.1. $N_{p,\alpha,M} \geqslant C_e(n,\alpha,p)$.

Proof. Let $\lambda > 0$ be small positive constant and $\delta > 0$ be some fixed constant selected later. Define

$$\tilde{f}(y) = \begin{cases} f_\lambda(y), & \text{in } B_\delta^{n-1}(0) \\ 0, & \text{in } \partial\mathbb{R}^n_+ \backslash B_\delta^{n-1}(0) \end{cases} \quad \text{and} \quad \tilde{g}(x) = \begin{cases} g_\lambda(x), & \text{in } C_\delta^n(0) \\ 0, & \text{in } \mathbb{R}^n_+ \setminus C_\delta^n(0) \end{cases}$$

where $f_\lambda(x)$, $g_\lambda(y)$ are given in (4.3) and $C_\delta^n(0) = B_\delta^{n-1}(0) \times [0,\delta]$. Then, $(\tilde{f},\tilde{g}) \in L^p(\partial\mathbb{R}^n_+) \times L^t(\mathbb{R}^n_+)$, and we have by (4.1)

$$\begin{aligned}
&\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} \tilde{g}(x)|x-y|^{\alpha-n}\tilde{f}(y)dydx \\
=&\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} g_\lambda(x)|x-y|^{\alpha-n}f_\lambda(y)dydx - \int_{\mathbb{R}^n_+}\int_{|y|>\delta} g_\lambda(x)|x-y|^{\alpha-n}f_\lambda(y)dydx \\
&-\int_{\mathbb{R}^n_+\backslash C_\delta^n(0)}\int_{\partial\mathbb{R}^n_+} g_\lambda(x)|x-y|^{\alpha-n}f_\lambda(y)dydx \\
&+\int_{\mathbb{R}^n_+\backslash C_\delta^n(0)}\int_{|y|>\delta} g_\lambda(x)|x-y|^{\alpha-n}f_\lambda(y)dydx \\
=&C_e(n,\alpha,p) - C_e(n,\alpha,p)\int_{|y|>\delta} f_\lambda^p(y)dy - C_e(n,\alpha,p)\int_{\mathbb{R}^n_+\backslash C_\delta^n(0)} g_\lambda^t(x)dx \\
&+\int_{\mathbb{R}^n_+\backslash C_\delta^n(0)}\int_{|y|>\delta} f_\lambda(y)g_\lambda(x)|x-y|^{\alpha-n}dxdy \\
:=&C_e(n,\alpha,p) - \mathbf{I} - \mathbf{II} + \mathbf{III}.
\end{aligned} \tag{4.5}$$

and then

$$\begin{aligned}
&\frac{\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} \tilde{f}(y)\tilde{g}(x)|x-y|^{\alpha-n}dydx}{\|\tilde{f}\|_{L^p(\partial\mathbb{R}^n_+)}\|\tilde{g}\|_{L^t(\mathbb{R}^n_+)}} \\
\geqslant&\frac{C_e(n,\alpha,p) - \mathbf{I} - \mathbf{II}}{\|f_\lambda\|_{L^p(\partial\mathbb{R}^n_+)}\|g_\lambda\|_{L^t(\mathbb{R}^n_+)}} = C_e(n,\alpha,p) - \mathbf{I} - \mathbf{II}.
\end{aligned} \tag{4.6}$$

For fixed $\delta > 0$, since $\|f\|_{L^p(\partial\mathbb{R}^n_+)} = \|g\|_{L^t(\mathbb{R}^n_+)} = 1$, we have for $\lambda \to 0^+$,

$$\mathbf{I} := C_e(n,\alpha,p)\int_{|y|>\delta} f^p_\lambda(y)dy = C_e(n,\alpha,p)\int_{|y|>\delta/\lambda} f^p(y)dy \to 0,$$
$$\mathbf{II} := C_e(n,\alpha,p)\int_{\mathbb{R}^n_+\setminus C^n_\delta(0)} g^t_\lambda(x)dx \to 0.$$

So, for $\lambda \to 0^+$,

$$\frac{\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+}\tilde{f}(y)\tilde{g}(x)|x-y|^{\alpha-n}dydx}{\|\tilde{f}\|_{L^p(\partial\mathbb{R}^n_+)}\|\tilde{g}\|_{L^t(\mathbb{R}^n_+)}} \geqslant C_e(n,\alpha,p) + o(1). \tag{4.7}$$

For any given point $P \in \Sigma$, choose a neighbourhood $\Omega_P \subset M^n$ so that for $\delta > 0$ small enough, in the Fermi coordinate around P, $C_\delta(P) := \phi^{-1}(C^n_\delta(0)) \subset \Omega_P$, and

$$(1-\varepsilon)I \leqslant g(x) \leqslant (1+\varepsilon)I, \quad \forall\, x \in C_\delta(P).$$

Thus,

$$(1-\varepsilon)^{\frac{1}{2}}|\phi(x)-\phi(y)| \leqslant |x-y|_g \leqslant (1+\varepsilon)^{\frac{1}{2}}|\phi(x)-\phi(y)|, \qquad \forall\, x,\ y \in C_\delta(P).$$

Defining

$$u(y) = \begin{cases} f_\lambda(\phi(y)), & \text{in} \quad C_\delta(P)\cap\Sigma, \\ 0, & \text{in} \quad \Sigma\setminus C_\delta(P) \end{cases} \quad \text{and} \quad v(x) = \begin{cases} g_\lambda(\phi(x)), & \text{in} \quad C_\delta(P), \\ 0, & \text{in} \quad M^n\setminus C_\delta(P), \end{cases}$$

we have

$$\int_{M^n} |v|^t dV(x) \leqslant (1+\varepsilon)^{\frac{n}{2}} \int_{C^n_\delta(0)} |g_\lambda(x)|^t dx, \tag{4.8}$$

$$\int_\Sigma |u|^p dS_y \leqslant (1+\varepsilon)^{\frac{n-1}{2}} \int_{B^{n-1}_\delta(0)} |f_\lambda(y)|^p dy, \tag{4.9}$$

$$\begin{aligned}
&\int_{M^n}\int_\Sigma u(y)v(x)|x-y|_g^{\alpha-n}dS_y dV(x) \\
&\quad = \int_{C^n_\delta(0)}\int_{B^{n-1}_\delta(0)} \frac{u(y)v(x)}{|x-y|_g^{n-\alpha}}\sqrt{\det g(y)}\sqrt{\det g(x)}dydx \\
&\quad \geqslant \int_{C^n_\delta(0)}\int_{B^{n-1}_\delta(0)} \frac{f_\lambda(y)g_\lambda(x)}{(1+\varepsilon)^{\frac{n-\alpha}{2}}|x-y|^{n-\alpha}}(1-\varepsilon)^{n-\frac{1}{2}}dydx \\
&\quad = \frac{(1-\varepsilon)^{n-1/2}}{(1+\varepsilon)^{\frac{n-\alpha}{2}}}\int_{C^n_\delta(0)}\int_{B^{n-1}_\delta(0)} \frac{f_\lambda(y)g_\lambda(x)}{|x-y|^{n-\alpha}}dydx.
\end{aligned} \tag{4.10}$$

It follows from (4.5)–(4.10) that

$$
\begin{aligned}
N_{p,\alpha,M} &\geqslant \frac{\int_{M^n}\int_{\Sigma} v(x)u(y)|x-y|_g^{\alpha-n}dS_y dV(x)}{\|u\|_{L^p(\Sigma)}\|v\|_{L^t(M^n)}}\\
&\geqslant \frac{\frac{(1-\varepsilon)^{n-1/2}}{(1+\varepsilon)^{\frac{n-\alpha}{2}}}\int_{C_\delta^n(0)}\int_{B_\delta^{n-1}(0)}\frac{f_\lambda(y)g_\lambda(x)}{|x-y|^{n-\alpha}}dydx}{(1+\varepsilon)^{\frac{n}{2t}+\frac{n-1}{2p}}\|f_\lambda\|_{L^p(B_\delta^{n-1}(0))}\|g_\lambda\|_{L^t(C_\delta^n(0))}}\\
&\geqslant \frac{(1-\varepsilon)^{n-1/2}}{(1+\varepsilon)^{\frac{n}{2t}+\frac{n-1}{2p}+\frac{n-\alpha}{2}}}\left(C_e(n,\alpha,p)-\mathbf{I}-\mathbf{II}\right).
\end{aligned}
$$

Sending ε and λ to 0, the estimate are obtained. □

Proof of Theorem 1.3. Letting p and q satisfy the conditions of Proposition 1.2 and taking $r>p$, then

$$
\frac{1}{q} > \frac{n-1}{n}\Big(\frac{1}{r}-\frac{\alpha-1}{n-1}\Big).
$$

Replacing r,p by q,r in Lemma 2.3 and Proposition 2.5, we obtain a function sequence $\{f_r\}$ which satisfy $\|f_r\|_{L^r(\Sigma)}=1$ and $N_{r,q,\alpha,M}=\|E_\alpha f_r\|_{L^q(M^n)}$. It is easy to prove that $\{f_r\}$ is uniformly bounded in $L^p(\Sigma)$ as $r\to p^+$ and

$$
N_{p,q,\alpha,M}=\lim_{r\to p^+}N_{r,q,\alpha,M}=\lim_{r\to p^+}\frac{\|E_\alpha f_r\|_{L^q(M^n)}}{\|f_r\|_{L^p(\Sigma)}}. \tag{4.11}
$$

Then, there exists a subsequence of $\{f_p\}$ (denoted as $\{f_m\}$) and some function $f\in L^p(\Sigma)$ such that

$$
f_m \rightharpoonup f \quad \text{weakly in} \quad L^p(\Sigma).
$$

From HLS inequalities (1.12), we know that

$$
\mu_m = |f_m|^p dS_y,\quad \nu_m=|E_\alpha f_m|^q dV(x) \tag{4.12}
$$

are two families of bounded measures. So, there exist two nonnegative bounded measures μ and ν on Σ and M^n such that

$$
\mu_m \rightharpoonup \mu, \text{ and } \nu_m \rightharpoonup \nu
$$

weakly in the sense of measure.

Applying Lemma 3.1 we have

$$
\nu = |E_\alpha f|^q dV(x)+\sum_{j\in J}\nu_j\delta_{P_j},\quad \mu\geqslant |f|^p dS_y+\sum_{j\in J}\mu_j\delta_{P_j}, \tag{4.13}
$$

and $\nu_j^{1/q}\leqslant C_e(n,\alpha,p)\mu_j^{1/p}$ for all $j\in J$. Since $\int_\Sigma d\mu=\lim_{m\to+\infty}\int_\Sigma|f_m|^p dS_y=1$, then $\int_\Sigma |f|^p dS_y\leqslant 1$ and $\mu_j\leqslant 1,\ j\in J$.

We claim that $\mu_j=0$, for $j\in J$, which implies that $\nu_j=0$, for $j\in J$.

In fact, otherwise, combining (4.13) and the fact $\frac{q}{p} > 1$, we have

$$\begin{aligned} N^q_{p,\alpha,M} &= \lim_{m\to+\infty}\int_{M^n}|E_\alpha f_m|^q dV_x = \int_{M^n} d\nu \\ &= \int_{M^n}|E_\alpha f|^q dV(x) + \sum_{j\in J}\nu_j \\ &\leqslant N^q_{p,\alpha,M}\|f\|^q_{L^p(\Sigma)} + \sum_{j\in J} C_e(n,\alpha,p)^q \mu_j^{q/p} \\ &< N^q_{p,\alpha,M}\left(\int_\Sigma |f|^p dV_x\right)^{q/p} + \sum_{j\in J} N^q_{p,\alpha,M}\mu_j^{q/p} \\ &\leqslant N^q_{p,\alpha,M}\left(\int_\Sigma |f|^p dS_y + \sum_{j\in J}\mu_j\right)^{q/p} \\ &\leqslant N^q_{p,\alpha,M}\left(\int_\Sigma d\mu\right)^{q/p} = N^q_{p,\alpha,M}, \end{aligned} \tag{4.14}$$

which yields a contradiction.

Repeating the process of (4.14), we have that

$$N^q_{p,\alpha,M} = \int_{M^n}|E_\alpha f|^q dV(x) \quad \text{and} \quad \int_\Sigma |f|^p dVS_y = 1,$$

i.e., f is a maximizer. □

4.2. An example for which $N_K(\tilde{p}_\alpha,\alpha,M) > C_e(n,\alpha,\tilde{p}_\alpha)$

Assume that (M^n,g) is locally conformally flat and its boundary Σ is umbilic. Now we will apply Theorem 1.1 to give the existence of the solutions for conformal integral equations (1.2) with $K(x,y) = G(x,y)^{\frac{n-\alpha}{n-2}}$.

Proof of Theorem 1.5. By [6], we know that the solutions of (4.2) with $p = \tilde{p}_\alpha$, $t = p_\alpha$ are smooth and there exists a positive constant μ such that

$$f^{\tilde{p}_\alpha-1}(y) = \left(\frac{\mu}{|y|}\right)^{n-\alpha} f^{\tilde{p}_\alpha-1}\left(\frac{\mu y}{|y|^2}\right), \quad g^{p_\alpha-1}(x) = \left(\frac{\mu}{|x|}\right)^{n-\alpha} g^{p_\alpha-1}\left(\frac{\mu x}{|x|^2}\right)$$

for any $\alpha\in(1,n)$, which implies that

$$\lim_{|y|\to+\infty}|y|^{n+\alpha-2}f(y) = \mu^{n+\alpha-2}f(0),$$
$$\lim_{|x|\to+\infty}|x|^{n+\alpha}g(x) = \mu^{n+\alpha}g(0).$$

Now, taking $\lambda\to 0^+$, we have

$$\int_{|y|>\delta} f_\lambda^{\tilde{p}_\alpha}(y)dy = \int_{|y|>\delta/\lambda} f^{\tilde{p}_\alpha}(y)dy \leqslant C(\delta/\lambda)^{-n+1},$$
$$\int_{\mathbb{R}^n_+\setminus C^n_\delta(0)} g_\lambda^{p_\alpha}(x)dx \leqslant C(\delta/\lambda)^{-n}$$

and

$$\begin{aligned}&\int_{C^n_\delta(0)}\int_{|y|\leqslant\delta}|x-y|^{\alpha-2}f_\lambda(y)g_\lambda(x)dydx\\ =&\lambda^{n-2}\int_{C^n_{\delta/\lambda}(0)}\int_{|y|\leqslant\delta/\lambda}|x-y|^{\alpha-2}f(y)g(x)dydx\geqslant C\lambda^{n-2}.\end{aligned}$$

Repeating the proof of Proposition 2.9 of [15], we can prove the existence of (1.2). □

5. HLS inequality on the upper half space

We firstly establish the HLS inequality on the $(B_1(0),|\cdot|)$.

PROPOSITION 5.1. *Let α,p,q,r satisfy $\alpha\in(0,n)$, $1<p<q$ and $\frac{1}{q}=\frac{n-1}{n}\left(\frac{1}{p}-\frac{\alpha-1}{n-1}\right)$. Then there is a positive constant $C:=C(n,\alpha,p)$, such that*

$$\Big\|\int_{\partial B_1(0)}\frac{f(y)dy}{|x-y|^{n-\alpha}}\Big\|_{L^q(B_1(0))}\leqslant C\|f\|_{L^p(\partial B_1(0))} \tag{5.1}$$

holds for all $f\in L^p(\partial B_1(0))$.

Proof. If I can prove weak estimate, then the strong estimate can be deduced by Marcinkiewicz interpolation theorem. So, I only need to prove the following weak type estimate

$$\Big\|\int_{\partial B_1(0)}\frac{f(y)dy}{|x-y|^{n-\alpha}}\Big\|_{L^q_W(B_1(0))}\leqslant C\|f\|_{L^p(\partial B_1(0))}, \tag{5.2}$$

namely,

$$meas\left\{x:\Big|\int_{\partial B_1(0)}\frac{f(y)dy}{|x-y|^{n-\alpha}}\Big|>\lambda\right\}\leqslant\left(\frac{C\|f\|_{L^p(\partial B_1(0))}}{\lambda}\right)^q,\ \forall\lambda>0.$$

By homogeneity, we assume $\|f\|_{L^p(\partial B_1(0)}=1$ without loss of generality and will prove

$$meas\left\{x:\Big|\int_{\partial B_1(0)}\frac{f(y)dy}{|x-y|^{n-\alpha}}\Big|>\lambda\right\}\leqslant C\lambda^{-q},\ \forall\lambda>0. \tag{5.3}$$

For constant $\gamma>0$ determined later, define

$$E^1_{\alpha,\gamma}f(x):=\int_{\partial B_1(0),|x-y|\leqslant\gamma}\frac{f(y)dy}{|x-y|^{n-\alpha}}$$

and

$$E^2_{\alpha,\gamma}f(x):=\int_{\partial B_1(0),|x-y|>\gamma}\frac{f(y)dy}{|x-y|^{n-\alpha}}.$$

Then for any $\lambda > 0$,

$$meas\left\{x : \left|\int_{\partial B_1(0)} \frac{f(y)dy}{|x-y|^{n-\alpha}}\right| > 2\lambda\right\}$$
$$\leqslant meas\{x : |E^1_{\alpha,\gamma}f(x)| > \lambda\} + meas\{x : |E^2_{\alpha,\gamma}f(x)| > \lambda\}.$$

By Hölder's inequality,

$$|E^2_{\alpha,\gamma}f(x)| \leqslant \left(\int_{\partial B_1(0),|x-y|>\gamma} \frac{dy}{|x-y|^{(n-\alpha)p'}}\right)^{1/p'}$$
$$\leqslant C_1\gamma^{\frac{(\alpha-n)p'+(n-1)}{p'}} = C_1\gamma^{-n/q},$$

where $p' = \frac{p}{p-1}$. Choose $\gamma = (\lambda/C_1)^{-q/n}$ and then $meas\{x : |E^2_{\alpha,\gamma}f(x)| > \lambda\} = 0$.

Take $q_1 > 0$ and $q_0 = \frac{n}{n-1}q_1$ satisfying $\frac{1}{p} + \frac{q_1}{q_0} = \frac{q_1}{q_0}(\frac{1}{p} + \frac{1}{q_1})$. Then, $q_1 \in (0, \frac{n-1}{n-\alpha})$, $q_0 \in (0, \frac{n}{n-\alpha})$, $\frac{1}{p} + \frac{p-q_0}{pq_1} = 1$ and $q_0 < p$. By Hölder's inequality,

$$|E^1_{\alpha,\gamma}f(x)| \leqslant \int_{\partial B_1(0),|x-y|\leqslant\gamma} \frac{|f(y)|dy}{|x-y|^{(n-\alpha)(q_0/p+(p-q_0)/p)}}$$
$$\leqslant \left(\int_{\partial B_1(0),|x-y|\leqslant\gamma} \frac{|f(y)|^p dy}{|x-y|^{(n-\alpha)q_0}}\right)^{1/p}$$
$$\cdot \left(\int_{\partial B_1(0),|x-y|\leqslant\gamma} |x-y|^{-(n-\alpha)q_1} dy\right)^{\frac{p-q_0}{pq_1}}$$
$$\leqslant C_2\gamma^{[(\alpha-n)q_1+(n-1)]\frac{p-q_0}{pq_1}} \left(\int_{\partial B_1(0),|x-y|\leqslant\gamma} \frac{|f(y)|^p dy}{|x-y|^{(n-\alpha)q_0}}\right)^{1/p}.$$

Then,

$$\int_{B_1(0)} |E^1_{\alpha,\gamma}f(x)|^p dx$$
$$\leqslant C_2^p\gamma^{[(\alpha-n)q_1+(n-1)]\frac{p-q_0}{q_1}} \int_{B_1(0)} \int_{\partial B_1(0),|x-y|\leqslant\gamma} \frac{|f(y)|^p dydx}{|x-y|^{(n-\alpha)q_0}}$$
$$\leqslant C_3\gamma^{[(\alpha-n)q_1+(n-1)]\frac{p-q_0}{q_1}+[(\alpha-n)q_0+n]} = C_3\gamma^{p(\alpha-1)+1}$$

and

$$meas\{x : |E^1_{\alpha,\gamma}f(x)| > \lambda\} \leqslant \frac{\|E^1_{\alpha,\gamma}f(x)\|^p_{L^p(B_1(0))}}{\lambda^p} \leqslant \frac{C_3\gamma^{p(\alpha-1)+1}}{\lambda^p}$$
$$\leqslant C_4\lambda^{-\frac{q}{n}[p(\alpha-1)+1]-p} = C_4\lambda^{-q}.$$

So, I have

$$meas\left\{x : \left|\int_{\partial B_1(0)} \frac{f(y)dy}{|x-y|^{n-\alpha}}\right| > 2\lambda\right\} \leqslant C_5(2\lambda)^{-q}$$

and complete the proof. □

REMARK 5.2. Note that there are analytic differences between the case $\alpha \in (1,n)$ and the case $\alpha \in (0,1)$. In fact, If $\alpha \in (1,n)$, $\int_{\partial B_1(0)} |x-y|^{\alpha-n} dy$ is uniformly bounded for any $x \in B_1(0)$. While for the case $\alpha \in (0,1]$, $\int_{\partial B_1(0)} |x-y|^{\alpha-n} dy$ converges to ∞ as $dist(x, \partial B_1(0)) \to 0$. The fact will bring some new difficulties. I will study the inequality (5.1) with $\alpha \in (0,1)$ in the further work.

Proof of Proposition 1.7. By the Möbius transformation (1.5), I know that (1.16) is equivalent to (5.1) with $p = \tilde{p}_\alpha$, $q = q_\alpha$. For conciseness, I omit the detailed proof. □

Acknowledgements. The author would like to thank Professor Meijun Zhu and Professor Jingbo Dou for some helpful discussions and their valuable comments. The project is supported by the National Natural Science Foundation of China (Grant No. 12071269 and 11571268) and Natural Science Foundation of Zhejiang Province (Grant No. LY18A010013).

REFERENCES

[1] S. CHEN, *A new family of sharp conformally invariant integral inequalities*, Int. Math. Res. Not., 5 (2014), 1205–1220.

[2] L. CHEN, G. LU, C. TAO, *Reverse Stein-Weiss inequalities on the upper half space and the existence of their extremals*, Adv. Nonlinear Stud., 19 (2019), 475–494.

[3] L. CHEN, G. LU, C. TAO, *Existence of extremal functions for the Stein-Weiss inequalities on the Heisenberg group*, J. Funct. Anal., 277 (2019), 1112–1138.

[4] J. DOU, Q. GUO, M. ZHU, *Subcritical approach to sharp Hardy-Littlewood-Sobolev type inequalities on the upper half space*, Adv. Math., 312 (2017), 1–45. Corrigendum to "Subcritical approach to sharp Hardy-Littlewood-Sobolev type inequalities on the upper half space" [Adv. Math. 312 (2017) 1–45], Adv. Math., 317 (2017), 640–644.

[5] J. DOU, Q. GUO, M. ZHU, *Negative power nonlinear integral equations on bounded domains*, J. Diff. Eqs., 269 (2020), 10527–10557.

[6] J. DOU, M. ZHU, *Sharp Hardy-Littlewood-Sobolev inequality on the upper half space*, Int. Math. Res. Not., 3 (2015), 651–687.

[7] J. DOU, M. ZHU, *Reversed Hardy-Littewood-Sobolev inequality*, Int. Math. Res. Not., 19 (2015), 9696–9728.

[8] J. DOU, M. ZHU, *Nonlinear integral equations on bounded domains*, J. Funct. Anal., 277 (2019), 111–134.

[9] J. F. ESCOBAR, *The Yamabe problem on manifolds with boundary*, J. Diff. Geom., 35 (1992), 21–84.

[10] J. F. ESCOBAR, *Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary*, Ann. of Math. 136 (1992), 1–50.

[11] R. L. FRANK, E. H. LIEB, *Sharp constants in several inequalities on the Heisenberg group*, Ann. of Math., 176 (2012), 349–381.

[12] M. GLUCK, *Subcritical approach to conformally invariant extension operators on the upper half space*, J. Funct. Anal., 278 (1) (2020), 1–46.

[13] M. GLUCK, M. ZHU, *An extension operator on bounded domains and applications*, Calc. Var. PDE, 58 (2019), Article No. 79.

[14] Y. HAN, *Integral equations on compact CR manifolds*, Discrete and Continuous Dynamical Systems, 41 (5) (2021), 2187–2204.

[15] Y. HAN, M. ZHU, *Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds and applications*, J. Diff. Eqs., 260 (2016), 1–25.

[16] F. HANG, X. WANG, X. YAN, *Sharp integral inequalities for harmonic functions*, Comm. Pure Appl. Math., 61 (2008), 0054–0095.

[17] F. HANG, X. WANG, X. YAN, *An integral equation in conformal geometry*, Ann. Inst. H. Poincaré Analyse Non Linéaire, 26 (2009), 1–21.
[18] G. H. HARDY, J. E. LITTLEWOOD, *Some properties of fractional integrals (1)*, Math. Zeitschr., 27 (1928), 565–606.
[19] G. H. HARDY, J. E. LITTLEWOOD, *On certain inequalities connected with the calculus of variations*, J. London Math. Soc., 5 (1930), 34–39.
[20] D. JERISON, J. M. LEE, *The Yamabe problem on CR manifolds*, J. Diff. Geom. 25 (1987), 167–197.
[21] E. H. LIEB, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. 118 (1983), 349–374.
[22] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The limit case, Part 1*, Rev. Mat. Iberoamericana, 1 (1) (1985), 145–201.
[23] P. L. LIONS, *The concentration-compactness principle in the calculus of variations. The limit case, Part 2*, Rev. Mat. Iberoamericana, 1 (2) (1985), 45–121.
[24] Q. N. NGÔ, V. H. NGUYEN, *Sharp reversed Hardy-Littlewood-Sobolev inequality on the half space* $\mathbb{R}^n_+$, Int. Math. Res. Not., 20 (2017), 6187–6230.
[25] Q. N. NGÔ, *All conditions for Stein-Weiss inequalities are necessary*, arXiv: 2110.14220[math. FA].
[26] Q. N. NGÔ, Q. H. NGUYEN, V. H. NGUYEN, *An optimal Hardy-Littlewood-Sobolev inequality on* $\mathbb{R}^{n-k}\times\mathbb{R}^n$ *and its consequences*, arXiv: 2009.09868[math.FA].
[27] S. L. SOBOLEV, *On a theorem of functional analysis*, Mat. Sb. (N.S.), 4 (1938), 471–479. A. M. S. Transl. Ser., 2, 34 (1963), 39–68.
[28] M. ZHU, *Prescribing integral curvature equation*, Diff. Inte. Eqs., 29 (2016), 889–904.

(Received September 28, 2022)

Yazhou Han
Department of Mathematics
College of Science, China Jiliang University
Hangzhou, 310018, China
e-mail: yazhou.han@gmail.com