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ON A STEVIĆ–SHARMA TYPE OPERATOR FROM WEIGHTED–TYPE SPACES INTO BLOCH–TYPE SPACES

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Abstract. The boundedness, compactness and essential norm of a Stević-Sharma type operator from weighted-type spaces into Bloch-type spaces are investigated in this paper.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ the unit circle and $H(\mathbb{D})$ be the class of all functions analytic in $\mathbb{D}$. For $a \in \mathbb{D}$, let σ_a be the automorphism of $\mathbb{D}$ exchanging 0 for a. Then $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$.

We denote by $S(\mathbb{D})$ the set of all analytic self-maps of $\mathbb{D}$. Let $\varphi \in S(\mathbb{D})$. The composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ. See [2, 20, 36, 38] and the references therein for the study of various properties of composition operators.

For $n \in \mathbb{N}_0$, the nth differentiation operator D^n is defined by

$$D^n f = f^{(n)}, \quad f \in H(\mathbb{D}),$$

where $f^{(0)} = f$. If $n = 1$, it is the classical differentiation operator D and typically unbounded on many holomorphic function spaces.

Products of composition and differentiation operators between spaces of holomorphic functions have been studied for almost two decades. Some of the first results in the topic can be found, for example, in [4, 10, 18, 23].

Let $\psi \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. We denote the generalized weighted composition operator (also called weighted differentiation composition operator) by $D^n_{\psi,\varphi}$, i.e.,

$$D^n_{\psi,\varphi} f = \psi \cdot f^{(n)} \circ \varphi, \quad f \in H(\mathbb{D}).$$

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When $n=0$, $D^n_{\psi,\varphi}$ is the well-known weighted composition operator which we denote here by ψC_φ. The operator $D^n_{\psi,\varphi}$ seems studied for the first time by Zhu in [43]. For more results on the operator see, for example, [11, 22, 24, 25, 43, 44, 45, 46, 47]. For some n-dimensional generalizations of the operator see [26, 27, 29, 30].

Let $\mu>0$. The Bloch-type space, denoted by $\mathscr{B}^\mu$, is the space consisting of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{B}^\mu}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)^\mu|f'(z)|<\infty.$$

We write $\|f\|=\sup_{z\in\mathbb{D}}(1-|z|^2)^\mu|f'(z)|$, $\mathscr{B}^\mu$ is a Banach space with the above norm. When $\mu=1$, $\mathscr{B}^1=\mathscr{B}$ is the classical Bloch space. For more about the Bloch space, see [1, 9, 13, 14, 16, 17, 41, 42]. Let $H^\infty=H^\infty(\mathbb{D})$ denote the set of all bounded analytic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty=\sup_{z\in\mathbb{D}}|f(z)|$. Note that $H^\infty\subset\mathscr{B}$ and that $\|f\|_{\mathscr{B}}\leqslant\|f\|_\infty$ if $f\in H^\infty$. For $\varphi\in S(\mathbb{D})$, $\|\varphi\|_{\mathscr{B}}\leqslant\|\varphi\|_\infty\leqslant 1$. In [5], under the assumption that $\psi C_\varphi: H^\infty\to\mathscr{B}$ is bounded, Hu, Li and Wulan characterized the essential norm of $\psi C_\varphi: H^\infty\to\mathscr{B}$ and showed that

$$\|\psi C_\varphi\|_{e,H^\infty\to\mathscr{B}}\approx\max\{P,Q\}\approx\limsup_{n\to\infty}\|\psi\varphi^n\|_{\mathscr{B}},$$

where

$$P:=\limsup_{|\varphi(z)|\to1}(1-|z|^2)|\psi'(z)|,\quad Q:=\limsup_{|\varphi(z)|\to1}\frac{(1-|z|^2)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}.$$

We say that a function $v:\mathbb{D}\to\mathbb{R}_+$ is a weight, if v is a continuous, strictly positive and bounded function. The general weighted-type space, denoted by H^∞_v, is the space consisting of all $f\in H(\mathbb{D})$ such that

$$\|f\|_v=\sup_{z\in\mathbb{D}}v(z)|f(z)|<\infty.$$

H^∞_v is a Banach space under the norm $\|\cdot\|_v$. The weight v is called radial if $v(z)=v(|z|)$ for all $z\in\mathbb{D}$. The associated weight $\widetilde{v}$ of v is defined by

$$\widetilde{v}=(\sup\{|f(z)|: f\in H^\infty_v,\|f\|_v\leqslant1\})^{-1},\ \ z\in\mathbb{D}.$$

When $v=v_\alpha(z)=(1-|z|^2)^\alpha$ $(0<\alpha<\infty)$, it is easy to check that $\widetilde{v}_\alpha(z)=v_\alpha(z)$. In this case, we denote H^∞_v by H^∞_α, where,

$$H^\infty_\alpha=\{f\in H(\mathbb{D}):\|f\|_{v_\alpha}=\sup_{z\in\mathbb{D}}|f(z)|(1-|z|^2)^\alpha<\infty\}.$$

Composition operators, weighted composition operators and related concrete operators from or into weighted-type spaces and their generalizations have been studied a lot, see, for example, [6, 15, 19, 21, 22, 24, 25, 35, 39, 46].

Studying sums of generalized weighted composition operators have been proposed by Stević and Sharma. The first paper in the topic was [32]. Soon after that the Stević-Sharma type operators have attracted some attention (see, for example, [8, 12, 33, 40]).

For the case of holomorphic functions on the upper half-plane see [31]. In [34], Stević and his collaborators studied a more general operator. Soon after the publication of [34] Stević proposed to his collaborators studying the following general operator

$$T^{k,n}_{\vec{\psi},\varphi} f = \sum_{j=0}^{k} \psi_j \cdot f^{(n+j)} \circ \varphi = \sum_{j=0}^{k} D^{n+j}_{\psi_j,\varphi} f, \quad f \in H(\mathbb{D}),$$

where $n,k \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $\psi_j \in H(\mathbb{D})$, $j = 0,1,\ldots,k$, which generalizes previously studied operators. He also proposed studying several n-dimensional generalizations. One of them can be found in [28]. The case $n = 0$ has been recently studied in [37]. Here we also study the case $n = 0$, that is, the operator

$$T^{k}_{\vec{\psi},\varphi} f = \sum_{j=0}^{k} \psi_j \cdot f^{(j)} \circ \varphi = \sum_{j=0}^{k} D^{j}_{\psi_j,\varphi} f, \quad f \in H(\mathbb{D}),$$

where $k \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $\psi_j \in H(\mathbb{D})$, $j = 0,1,\ldots,k$.

The purpose of this paper is to characterize the boundedness and compactness of the operator $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$. Moreover, we also give some estimates for the essential norm of the operator $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$.

Recall that the essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators $\mathscr{K}$ mapping X into Y, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - \mathscr{K}\|_{X\to Y} : \mathscr{K} \text{ is compact}\},$$

where X,Y are Banach spaces and $\|\cdot\|_{X\to Y}$ is the operator norm.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Boundedness

In this section, we characterize the boundedness of the operator $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$. For this purpose, we need some lemmas as follows.

LEMMA 2.1. [42] *Assume that* $0 < \alpha < \infty$. *Let* n *be a nonnegative integer and* $f \in H^\infty_\alpha$. *Then there is a positive constant* C *independent of* f *such that*

$$|f^{(n)}(z)| \leqslant C \frac{\|f\|_{v_\alpha}}{(1-|z|^2)^{\alpha+n}}.$$

LEMMA 2.2. [15] *Let* v *and* w *be radial, non-increasing weights tending to zero at the boundary of* $\mathbb{D}$. *Then the weighted composition operator* $\psi C_\varphi : H^\infty_v \to H^\infty_w$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))} |\psi(z)| < \infty.$$

Moreover, the following holds

$$\|\psi C_\varphi\|_{H_v^\infty \to H_w^\infty} = \sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))}|\psi(z)|.$$

LEMMA 2.3. [6] *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then the weighted composition operator $\psi C_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if*

$$\sup_{n\geqslant 0} \frac{\|\psi\varphi^n\|_w}{\|\xi^n\|_v} < \infty,$$

with the norm comparable to the above supermum.

LEMMA 2.4. [7] *For $\alpha > 0$, we have $\lim_{n\to\infty} n^\alpha \|\xi^{n-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.*

Now we are in a position to state and prove our main results in this paper.

THEOREM 2.1. *Let $\min\{\alpha,\mu\} > 0$, $k \in \mathbb{N}_0$, $\varphi \in S(\mathbb{D})$ and $\psi_j \in H(\mathbb{D})$, $j = 0,1,\dots,k$. Then the following statements are equivalent:*

(a) The operator $T^k_{\vec{\psi},\varphi} : H_\alpha^\infty \to \mathscr{B}^\mu$ is bounded.

(b) $\sup_{a\in\mathbb{D}} \|T^k_{\vec{\psi},\varphi} f_{j,a}\|_{\mathscr{B}^\mu} < \infty$, for $j = 0,1,\dots,k+1$. Here

$$f_{j,a} = \frac{1-|a|^2}{(1-\bar{a}z)^{\alpha+1}} \sigma_a^j(z), \quad for \quad j = 0,1,\dots,k+1.$$

(c) (i) $M_0 = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\mu |\psi_0'(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty$;

$$(ii)\ \ M_j = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\mu |\psi_{j-1}(z)\varphi'(z) + \psi_j'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}} < \infty, \quad for \quad j = 1,2,\dots,k;$$

$$(iii)\ \ M_{k+1} = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\mu |\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}} < \infty.$$

(d) $\sup_{n\geqslant 1} n^\alpha \|\psi_0'\varphi^{n-1}\|_{v_\mu} < \infty$;

$$\sup_{n\geqslant 1} n^{\alpha+j} \|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_\mu} < \infty, \quad for \quad j = 1,2,\dots,k;$$

$$\sup_{n\geqslant 1} n^{\alpha+k+1} \|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu} < \infty.$$

Proof. $(a) \Rightarrow (b)$ Assume that $T^k_{\vec{\psi},\varphi} : H_\alpha^\infty \to \mathscr{B}^\mu$ is bounded. For each $a \in \mathbb{D}$, it is easy to check that $f_{j,a} \in H_\alpha^\infty$ for $j = 0,1,\dots,k+1$. Moreover $\|f_{j,a}\|_{v_\alpha} \leqslant 2^{\alpha+1}$ for $j = 0,1,\dots,k+1$. By the boundedness of $T^k_{\vec{\psi},\varphi} : H_\alpha^\infty \to \mathscr{B}^\mu$, we get

$$\sup_{a\in\mathbb{D}} \|T^k_{\vec{\psi},\varphi} f_{j,a}\|_{\mathscr{B}^\mu} \leqslant \|T^k_{\vec{\psi},\varphi}\| \sup_{a\in\mathbb{D}} \|f_{j,a}\|_{v_\alpha} \leqslant C\|T^k_{\vec{\psi},\varphi}\| < \infty,$$

for $j=0,1,\dots,k+1$, as desired.

$(b)\Rightarrow(c)$ Assume that (b) holds. From the assumption we see that

$$\sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi}f_{j,\varphi(a)}\|_{\mathscr{B}^\mu}<\infty, \tag{2.1}$$

for $j=0,1,\dots,k+1$. We shall prove the conditions (i)–(iii) hold. Fix $a\in\mathbb{D}$. First, we prove that the condition (iii) holds. It is easy to check that $f_{k+1,\varphi(a)}\in H^\infty_\alpha$ with $\|f_{k+1,\varphi(a)}\|_{v_\alpha}\leqslant 2^{\alpha+1}$, $f^{(i)}_{k+1,\varphi(a)}(\varphi(a))=0$ for $i=0,1,\dots,k$ and

$$\left|f^{(k+1)}_{k+1,\varphi(a)}(\varphi(a))\right|=\frac{(k+1)!}{(1-|\varphi(a)|^2)^\alpha(1-|\varphi(a)|^2)^{k+1}}=\frac{(k+1)!}{(1-|\varphi(a)|^2)^{\alpha+k+1}}.$$

Thus,

$$\begin{aligned}
\|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}&\geqslant\|T^k_{\vec{\psi},\varphi}f_{k+1,\varphi(a)}\|_{\mathscr{B}^\mu}\geqslant(1-|a|^2)^\mu|(T^k_{\vec{\psi},\varphi}f_{k+1,\varphi(a)})'(a)|\\
&=(1-|a|^2)^\mu\left|\sum_{j=0}^k\left(\psi_j'(a)f^{(j)}_{k+1,\varphi(a)}(\varphi(a))+\psi_j(a)\varphi'(a)f^{(j+1)}_{k+1,\varphi(a)}(\varphi(a))\right)\right|\\
&=(1-|a|^2)^\mu\Bigg|\psi_0'(a)f_{k+1,\varphi(a)}(\varphi(a))+\psi_k(a)\varphi'(a)f^{(k+1)}_{k+1,\varphi(a)}(\varphi(a))\\
&\quad+\sum_{j=1}^k\left(\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\right)f^{(j)}_{k+1,\varphi(a)}(\varphi(a))\Bigg|\\
&=(1-|a|^2)^\mu|\psi_k(a)\varphi'(a)||f^{(k+1)}_{k+1,\varphi(a)}(\varphi(a))|\\
&=\frac{(1-|a|^2)^\mu|\psi_k(a)\varphi'(a)|(k+1)!}{(1-|\varphi(a)|^2)^{\alpha+k+1}}.
\end{aligned}\tag{2.2}$$

Therefore, by (2.1) we have

$$\begin{aligned}
M_{k+1}&=\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}}\leqslant\frac{1}{(k+1)!}\sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi}f_{k+1,\varphi(a)}\|_{\mathscr{B}^\mu}\\
&<\infty,
\end{aligned}\tag{2.3}$$

and

$$M_{k+1}=\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}}\lesssim\frac{1}{(k+1)!}\|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}. \tag{2.4}$$

Next, we will prove that the condition (ii) holds. For $k\geqslant 1$ and $a\in\mathbb{D}$, it is easy to see that $f_{k,\varphi(a)}\in H^\infty_\alpha$ with $\|f_{k,\varphi(a)}\|_{v_\alpha}\leqslant 2^{\alpha+1}$. Moreover, $f^{(i)}_{k,\varphi(a)}(\varphi(a))=0$ for $i=0,1,\dots,k-1$ and

$$\left|f^{(k)}_{k,\varphi(a)}(\varphi(a))\right|=\frac{k!}{(1-|\varphi(a)|^2)^\alpha(1-|\varphi(a)|^2)^k}=\frac{k!}{(1-|\varphi(a)|^2)^{\alpha+k}}. \tag{2.5}$$

Using Lemma 2.1 and (2.5), we have

$$\begin{aligned}
&\|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu} \geqslant \|T^k_{\vec{\psi},\varphi}f_{k,\varphi(a)}\|_{\mathscr{B}^\mu} \geqslant (1-|a|^2)^\mu|(T^k_{\vec{\psi},\varphi}f_{k,\varphi(a)})'(a)|\\
\geqslant\ &(1-|a|^2)^\mu\big|\psi_k'(a)+\psi_{k-1}(a)\varphi'(a)\big|\big|f^{(k)}_{k,\varphi(a)}(\varphi(a))\big|\\
&-(1-|a|^2)^\mu\big|\psi_k(a)\varphi'(a)\big|\big|f^{(k+1)}_{k,\varphi(a)}(\varphi(a))\big|\\
\geqslant\ &\frac{(1-|a|^2)^\mu\big|\psi_k'(a)+\psi_{k-1}(a)\varphi'(a)\big|k!}{(1-|\varphi(a)|^2)^{\alpha+k}}\\
&-\frac{C\|f_{k,\varphi(a)}\|_{v_\alpha}(1-|a|^2)^\mu|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}}.
\end{aligned} \tag{2.6}$$

Thus, using (2.1), (2.3) and (2.6), we have

$$\begin{aligned}
M_k &= \sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu\big|\psi_k'(a)+\psi_{k-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^{\alpha+k}}\\
&\leqslant \frac{1}{k!}\left(\sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi}f_{k,\varphi(a)}\|_{\mathscr{B}^\mu}+C\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}}\right)\\
&\lesssim \sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi}f_{k,\varphi(a)}\|_{\mathscr{B}^\mu}+C\sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi}f_{k+1,\varphi(a)}\|_{\mathscr{B}^\mu}\\
&<\infty.
\end{aligned} \tag{2.7}$$

Using (2.4) and (2.6), we have

$$\begin{aligned}
M_k &= \sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu\big|\psi_k'(a)+\psi_{k-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^{\alpha+k}}\\
&\leqslant \frac{1}{k!}\left(\|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}+C\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu|\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}}\right)\\
&\lesssim \|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}.
\end{aligned} \tag{2.8}$$

This proves condition (ii) for $j=k$. Further, fix $1\leqslant j\leqslant k-1$ and assume that

$$M_i \lesssim \|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}, \tag{2.9}$$

for $i=j+1,\dots,k$. We will prove

$$M_j \lesssim \|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}.$$

It is easy to see that $f_{j,\varphi(a)}\in H^\infty_\alpha$ such that $\|f_{j,\varphi(a)}\|_{v_\alpha}\leqslant 2^{\alpha+1}$, $f^{(s)}_{j,\varphi(a)}(\varphi(a))=0$ for all $s<j$ and

$$\big|f^{(j)}_{j,\varphi(a)}(\varphi(a))\big| = \frac{j!}{(1-|\varphi(a)|^2)^\alpha(1-|\varphi(a)|^2)^j} = \frac{j!}{(1-|\varphi(a)|^2)^{\alpha+j}}. \tag{2.10}$$

Using Lemma 2.1 and (2.10), we have

$$\begin{aligned}
\|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu} &\geqslant \|T^k_{\vec{\psi},\varphi} f_{j,\varphi(a)}\|_{\mathscr{B}^\mu} \geqslant (1-|a|^2)^\mu |(T^k_{\vec{\psi},\varphi} f_{j,\varphi(a)})'(a)| \\
&\geqslant (1-|a|^2)^\mu \big|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big|\big|f^{(j)}_{j,\varphi(a)}(\varphi(a))\big| \\
&\quad -(1-|a|^2)^\mu \big|\psi_k(a)\varphi'(a)\big|\big|f^{(k+1)}_{j,\varphi(a)}(\varphi(a))\big| \\
&\quad -\sum_{i=j+1}^{k}(1-|a|^2)^\mu \big|\psi_i'(a)+\psi_{i-1}(a)\varphi'(a)\big|\big|f^{(i)}_{j,\varphi(a)}(\varphi(a))\big| \\
&\geqslant \frac{(1-|a|^2)^\mu \big|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big| j!}{(1-|\varphi(a)|^2)^{\alpha+j}} \\
&\quad -\sum_{i=j+1}^{k}\frac{C\|f_{j,\varphi(a)}\|_{v_\alpha}(1-|a|^2)^\mu \big|\psi_i'(a)+\psi_{i-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^{\alpha+i}} \\
&\quad -\frac{C\|f_{j,\varphi(a)}\|_{v_\alpha}(1-|a|^2)^\mu |\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}}. \qquad (2.11)
\end{aligned}$$

Thus, by (2.1), (2.3), (2.7) and (2.11), we obtain

$$\begin{aligned}
M_j &= \sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu \big|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^{\alpha+j}} \\
&\leqslant \frac{1}{j!}\Bigg(\sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi} f_{j,\varphi(a)}\|_{\mathscr{B}^\mu} + C\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu |\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}} \\
&\quad + C\sum_{i=j+1}^{k}\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu \big|\psi_i'(a)+\psi_{i-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^{\alpha+i}}\Bigg) \\
&\lesssim \sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi} f_{j,\varphi(a)}\|_{\mathscr{B}^\mu} + C\sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi} f_{k+1,\varphi(a)}\|_{\mathscr{B}^\mu} + C\sum_{i=j+1}^{k}\sup_{a\in\mathbb{D}}\|T^k_{\vec{\psi},\varphi} f_{i,\varphi(a)}\|_{\mathscr{B}^\mu} \\
&< \infty. \qquad (2.12)
\end{aligned}$$

By (2.4), (2.8) and (2.11), we get

$$\begin{aligned}
M_j &= \sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu \big|\psi_j'(a)+\psi_{j-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^{\alpha+j}} \\
&\leqslant \frac{1}{j!}\Bigg(\|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu} + C\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu |\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}} \\
&\quad + C\sum_{i=j+1}^{k}\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^\mu \big|\psi_i'(a)+\psi_{i-1}(a)\varphi'(a)\big|}{(1-|\varphi(a)|^2)^{\alpha+i}}\Bigg) \\
&\lesssim \|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}. \qquad (2.13)
\end{aligned}$$

This proves (ii).

Finally, we prove that (i) holds. It is easy to see that $\|f_{0,\varphi(a)}\|_{v_\alpha} \leqslant 2^{\alpha+1}$ for all $a \in \mathbb{D}$, and

$$\left|f_{0,\varphi(a)}(\varphi(a))\right| = \frac{1}{(1-|\varphi(a)|^2)^\alpha}. \tag{2.14}$$

Using Lemma 2.1 and (2.14), we have

$$\begin{aligned}
\|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha \to \mathscr{B}^\mu} &\geqslant \|T^k_{\vec{\psi},\varphi} f_{0,\varphi(a)}\|_{\mathscr{B}^\mu} \geqslant (1-|a|^2)^\mu |(T^k_{\vec{\psi},\varphi} f_{0,\varphi(a)})'(a)| \\
&= (1-|a|^2)^\mu \left| \sum_{j=0}^k \left(\psi_j'(a) f^{(j)}_{0,\varphi(a)}(\varphi(a)) + \psi_j(a)\varphi'(a) f^{(j+1)}_{0,\varphi(a)}(\varphi(a)) \right) \right| \\
&= (1-|a|^2)^\mu \Big| \psi_0'(a) f_{0,\varphi(a)}(\varphi(a)) + \psi_k(a)\varphi'(a) f^{(k+1)}_{0,\varphi(a)}(\varphi(a)) \\
&\quad + \sum_{j=1}^k \Big(\psi_j'(a) + \psi_{j-1}(a)\varphi'(a) \Big) f^{(j)}_{0,\varphi(a)}(\varphi(a)) \Big| \\
&\geqslant \frac{(1-|a|^2)^\mu |\psi_0'(a)|}{(1-|\varphi(a)|^2)^\alpha} - \frac{C\|f_{0,\varphi(a)}\|_{v_\alpha} (1-|a|^2)^\mu |\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}} \\
&\quad - \sum_{j=1}^k \frac{C\|f_{0,\varphi(a)}\|_{v_\alpha} (1-|a|^2)^\mu \left|\psi_j'(a) + \psi_{j-1}(a)\varphi'(a)\right|}{(1-|\varphi(a)|^2)^{\alpha+j}}.
\end{aligned} \tag{2.15}$$

Thus, using (2.1), (2.3), (2.12) and (2.15), we have

$$\begin{aligned}
M_0 &= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\mu |\psi_0'(a)|}{(1-|\varphi(a)|^2)^\alpha} \\
&\lesssim \sup_{a\in\mathbb{D}} \|T^k_{\vec{\psi},\varphi} f_{0,\varphi(a)}\|_{\mathscr{B}^\mu} + \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\mu |\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}} \\
&\quad + \sum_{j=1}^k \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\mu \left|\psi_j'(a) + \psi_{j-1}(a)\varphi'(a)\right|}{(1-|\varphi(a)|^2)^{\alpha+j}} \\
&\lesssim \sup_{a\in\mathbb{D}} \|T^k_{\vec{\psi},\varphi} f_{0,\varphi(a)}\|_{\mathscr{B}^\mu} + \sum_{j=1}^k \sup_{a\in\mathbb{D}} \|T^k_{\vec{\psi},\varphi} f_{j,\varphi(a)}\|_{\mathscr{B}^\mu} + \sup_{a\in\mathbb{D}} \|T^k_{\vec{\psi},\varphi} f_{k+1,\varphi(a)}\|_{\mathscr{B}^\mu} \\
&< \infty.
\end{aligned}$$

Using (2.4), (2.8), (2.13) and (2.15), we have

$$\begin{aligned}
M_0 &= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\mu |\psi_0'(a)|}{(1-|\varphi(a)|^2)^\alpha} \\
&\lesssim \|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha \to \mathscr{B}^\mu} + \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\mu |\psi_k(a)\varphi'(a)|}{(1-|\varphi(a)|^2)^{\alpha+k+1}} \\
&\quad + \sum_{j=1}^k \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^\mu \left|\psi_j'(a) + \psi_{j-1}(a)\varphi'(a)\right|}{(1-|\varphi(a)|^2)^{\alpha+j}} \\
&\lesssim \|T^k_{\vec{\psi},\varphi}\|_{H^\infty_\alpha \to \mathscr{B}^\mu}.
\end{aligned} \tag{2.16}$$

This proves (i). Thus (c) holds.

$(c) \Rightarrow (a)$ Suppose that conditions (i)–(iii) hold. Let $f \in H_\alpha^\infty$. By Lemma 2.1 we have

$$\begin{aligned}
\|T^k_{\vec{\psi},\varphi} f\|_{\mathscr{B}^\mu} &= |T^k_{\vec{\psi},\varphi} f(0)| + \|T^k_{\vec{\psi},\varphi} f\| = \Big|\sum_{j=0}^{k} \psi_j(0) f^{(j)}(\varphi(0))\Big| \\
&\quad + \sup_{z\in\mathbb{D}} (1-|z|^2)^\mu \Big|\sum_{j=0}^{k}\Big(\psi_j'(z) f^{(j)}(\varphi(z)) + \psi_j(z)\varphi'(z) f^{(j+1)}(\varphi(z))\Big)\Big| \\
&\leqslant \sum_{j=0}^{k} |\psi_j(0)||f^{(j)}(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu |\psi_0'(z)||f(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu \sum_{j=1}^{k} |\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)||f^{(j)}(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu |\psi_k(z)\varphi'(z)||f^{(k+1)}(\varphi(z))| \\
&\leqslant C\|f\|_{v_\alpha}\Bigg(\sum_{j=0}^{k}\frac{|\psi_j(0)|}{(1-|\varphi(0)|^2)^{\alpha+j}} + \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\mu|\psi_0'(z)|}{(1-|\varphi(z)|^2)^\alpha} \\
&\quad + \sum_{j=1}^{k}\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\mu|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}} \\
&\quad + \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\mu|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}\Bigg) \\
&\lesssim \|f\|_{v_\alpha}\left(C + \sum_{j=0}^{k+1} M_j\right) < \infty. \qquad (2.17)
\end{aligned}$$

This proves that the operator $T^k_{\vec{\psi},\varphi} : H_\alpha^\infty \to \mathscr{B}^\mu$ is bounded. Thus (a) holds.

$(d) \Leftrightarrow (a)$ We have proved that $T^k_{\vec{\psi},\varphi} : H_\alpha^\infty \to \mathscr{B}^\mu$ is bounded if and only if (c) holds. Thus we know that the operator $T^k_{\vec{\psi},\varphi} : H_\alpha^\infty \to \mathscr{B}^\mu$ is bounded if and only if

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\mu|\psi_0'(z)|}{(1-|\varphi(z)|^2)^\alpha} < \infty; \qquad (2.18)$$

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\mu|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}} < \infty, \quad \text{for} \quad j=1,2,\ldots,k; \qquad (2.19)$$

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\mu|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}} < \infty. \qquad (2.20)$$

By Lemma 2.2, it is easy to see that (2.18) is equivalent to the operator $\psi_0' C_\varphi : H_{v_\alpha}^\infty \to H_{v_\mu}^\infty$ is bounded. By Lemma 2.3, this is equivalent to

$$\sup_{n\geqslant 1}\frac{\|\psi_0'\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_\alpha}} < \infty. \qquad (2.21)$$

By Lemma 2.2, it is easy to see that (2.19) is equivalent to the operator $(\psi_{j-1}\varphi' + \psi_j')C_\varphi : H^\infty_{v_{\alpha+j}} \to H^\infty_{v_\mu}$ is bounded. By Lemma 2.3, this is equivalent to

$$\sup_{n\geqslant 1} \frac{\|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_{\alpha+j}}} < \infty, \quad \text{for} \quad j = 1,2,\dots,k. \tag{2.22}$$

By Lemma 2.2, we see that (2.20) is equivalent to the operator $\psi_k\varphi' C_\varphi : H^\infty_{v_{\alpha+k+1}} \to H^\infty_{v_\mu}$ is bounded. By Lemma 2.3, this is equivalent to

$$\sup_{n\geqslant 1} \frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_{\alpha+k+1}}} < \infty. \tag{2.23}$$

By Lemma 2.4, we see that $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$ is bounded if and only if

$$\sup_{n\geqslant 1} n^\alpha \|\psi_0'\varphi^{n-1}\|_{v_\mu} \approx \sup_{n\geqslant 1} \frac{n^\alpha \|\psi_0'\varphi^{n-1}\|_{v_\mu}}{n^\alpha \|\xi^{n-1}\|_{v_\alpha}} < \infty;$$

$$\sup_{n\geqslant 1} n^{\alpha+j} \|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_\mu} \approx \sup_{n\geqslant 1} \frac{n^{\alpha+j}\|(\psi_{j-1}\varphi' + \psi_j')\varphi^{n-1}\|_{v_\mu}}{n^{\alpha+j}\|\xi^{n-1}\|_{v_{\alpha+j}}} < \infty,$$
$$\text{for} \quad j = 1,2,\dots,k$$

and

$$\sup_{n\geqslant 1} n^{\alpha+k+1} \|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu} \approx \sup_{n\geqslant 1} \frac{n^{\alpha+k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu}}{n^{\alpha+k+1}\|\xi^{n-1}\|_{v_{\alpha+k+1}}} < \infty.$$

As desired. The proof is complete. □

3. Essential norm

In this section, we give some estimates for the essential norm of the operator $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$. For this purpose, we first state some lemmas which will be used in the proofs of the main results in this section.

LEMMA 3.1. [6] *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Suppose $\psi C_\varphi : H^\infty_v \to H^\infty_w$ is bounded. Then*

$$\|\psi C_\varphi\|_{e,H^\infty_v \to H^\infty_w} = \limsup_{n\to\infty} \frac{\|\psi\varphi^n\|_w}{\|\xi^n\|_v}.$$

LEMMA 3.2. [36] *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

(1) The point evaluation functionals on Y are continuous.

(2) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(3) $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

LEMMA 3.3. *Let $\min\{\alpha,\mu\} > 0$, $k \in \mathbb{N}_0$, $\psi_j \in H(\mathbb{D})$, $j = 0,1,\ldots,k$ and $\varphi \in S(\mathbb{D})$ with $\|\varphi\|_\infty < 1$ such that $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$ is bounded. Then $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$ is compact.*

Proof. Assume that the operator $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$ is bounded, from Theorem 2.1, we obtain

$$T_0 := \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu|\psi_0'(z)| < \infty,$$

$$T_j := \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)| < \infty, \quad \text{for} \quad j = 1,2,\ldots,k$$

and

$$T_{k+1} := \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu|\psi_k(z)\varphi'(z)| < \infty.$$

Let $\{f_n\}_{n\in\mathbb{N}}$ be a bounded sequence in H^∞_α such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Cauchy's estimates imply that $f_n^{(j)} \to 0$, $j = 0,1,\ldots,k+1$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. Thus, for the compact subset $K = \{\varphi(z) : |\varphi(z)| \leqslant \|\varphi\|_\infty\} \subseteq \mathbb{D}$, we have

$$\sup_{z\in\mathbb{D}}|f_n^{(j)}(\varphi(z))| \to 0 \text{ as } n \to \infty. \tag{3.1}$$

For $i = 0,1,\ldots,k+1$, since $\{f_n^{(i)}\}_{n\in\mathbb{N}}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, it can be seen that $|T^k_{\vec{\psi},\varphi}f_n(0)| \to 0$ as $n \to \infty$. Further, using (3.1) we have

$$\begin{aligned}
\|T^k_{\vec{\psi},\varphi}f_n\|_{\mathscr{B}^\mu} &= |T^k_{\vec{\psi},\varphi}f_n(0)| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu\left|\sum_{j=0}^k\left(\psi_j'(z)f_n^{(j)}(\varphi(z))+\psi_j(z)\varphi'(z)f_n^{(j+1)}(\varphi(z))\right)\right| \\
&\leqslant |T^k_{\vec{\psi},\varphi}f_n(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu|\psi_0'(z)||f_n(\varphi(z))| \\
&\quad + \sum_{j=1}^k\sup_{z\in\mathbb{D}}(1-|z|^2)^\mu|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)||f_n^{(j)}(\varphi(z))| \\
&\quad + \sup_{z\in\mathbb{D}}(1-|z|^2)^\mu|\psi_k(z)\varphi'(z)||f_n^{(k+1)}(\varphi(z))|
\end{aligned}$$

$$\leqslant |T^k_{\vec{\psi},\varphi} f_n(0)| + T_0 \sup_{z\in\mathbb{D}} |f_n(\varphi(z))| + \sum_{j=1}^{k} T_j \sup_{z\in\mathbb{D}} |f_n^{(j)}(\varphi(z))|$$
$$+T_{k+1} \sup_{z\in\mathbb{D}} |f_n^{(k+1)}(\varphi(z))| \to 0 \text{ as } n\to\infty.$$

This proves that $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$ is a compact operator. $\square$

Next, we state and prove the main results in this section. For the simplicity, we set

$$A_i = \lim_{|a|\to 1} \|T^k_{\vec{\psi},\varphi} f_{i,a}\|_{\mathscr{B}^\mu}, \quad \text{for} \quad i = 0,1,\dots,k+1;$$
$$B_0 = \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu |\psi_0'(z)|}{(1-|\varphi(z)|^2)^\alpha};$$
$$B_j = \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu |\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}}, \text{ for } \quad j=1,2,\dots,k;$$
$$B_{k+1} = \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu |\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}.$$

THEOREM 3.1. *Let* $\min\{\alpha,\mu\}>0$, $k\in\mathbb{N}_0$, $\varphi\in S(\mathbb{D})$ *and* $\psi_j \in H(\mathbb{D})$, $j = 0,1,\dots,k$. *If the operator* $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$ *is bounded, then*

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \approx \max_{0\leqslant i\leqslant k+1} A_i \approx \max_{0\leqslant i\leqslant k+1} B_i.$$

Proof. When $\|\varphi\|_\infty < 1$. It is easy to see that $T^k_{\vec{\psi},\varphi} : H^\infty_\alpha \to \mathscr{B}^\mu$ is compact by using Lemma 3.3. In this case, the asymptotic relations vacuously hold. Now we consider the case $\|\varphi\|_\infty = 1$. First we prove that

$$\max_{0\leqslant i\leqslant k+1} A_i \leqslant \|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu}.$$

Let $a\in\mathbb{D}\backslash\{0\}$. It is easy to see that $f_{i,a}$ $(i=0,1,\dots,k+1)\in H^\infty_\alpha$ and $f_{i,a}$ $(i = 0,1,\dots,k+1)$ converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $|a|\to 1$. Thus, for any compact operator $\mathscr{K} : H^\infty_\alpha \to \mathscr{B}^\mu$, by Lemma 3.2 we have $\lim_{|a|\to 1} \|\mathscr{K} f_{i,a}\|_{\mathscr{B}^\mu} = 0$, for $i=0,1,\dots,k+1$. Hence, for $i=0,1,\dots,k+1$,

$$\|T^k_{\vec{\psi},\varphi} - \mathscr{K}\|_{H^\infty_\alpha\to\mathscr{B}^\mu} \gtrsim \limsup_{|a|\to 1} \|(T^k_{\vec{\psi},\varphi} - \mathscr{K}) f_{i,a}\|_{\mathscr{B}^\mu}$$
$$\geqslant \limsup_{|a|\to 1} \|T^k_{\vec{\psi},\varphi} f_{i,a}\|_{\mathscr{B}^\mu} - \limsup_{|a|\to 1} \|\mathscr{K} f_{i,a}\|_{\mathscr{B}^\mu} = A_i.$$

Therefore, from the definition of the essential norm, we obtain

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} = \inf_{\mathscr{K}} \|T^k_{\vec{\psi},\varphi} - \mathscr{K}\|_{H^\infty_\alpha\to\mathscr{B}^\mu} \gtrsim \max_{0\leqslant i\leqslant k+1} A_i.$$

Next, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$ such that

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}=\lim_{n\to\infty}\frac{(1-|z_n|^2)^{\mu}|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+k+1}}. \tag{3.2}$$

For each n, define $f_{k+1,n}(z)=\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}}\sigma^{k+1}_{\varphi(z_n)}(z)$, $z\in\mathbb{D}$. It is easy to see that $f_{k+1,n}\in H^\infty_\alpha$ and $\|f_{k+1,n}\|_{v_\alpha}\leqslant 2^{\alpha+1}$, $f^{(i)}_{k+1,n}(\varphi(z_n))=0$ for $i=0,1,\dots,k$ and

$$\left|f^{(k+1)}_{k+1,n}(\varphi(z_n))\right|=\frac{(k+1)!}{(1-|\varphi(z_n)|^2)^{\alpha+k+1}}. \tag{3.3}$$

Clearly, $\{f_{k+1,n}\}_{n\in\mathbb{N}}$ is a bounded sequence in H^∞_α and converges to zero uniformly on each compact subset of $\mathbb{D}$. Hence, if $\mathscr{K}:H^\infty_\alpha\to\mathscr{B}^\mu$ is a compact operator, then by Lemma 3.2 we have $\lim_{n\to\infty}\|\mathscr{K}f_{k+1,n}\|_{\mathscr{B}^\mu}=0$. Further, we have

$$\|T^k_{\vec{\psi},\varphi}-\mathscr{K}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}\geqslant\limsup_{n\to\infty}\|(T^k_{\vec{\psi},\varphi}-\mathscr{K})f_{k+1,n}\|_{\mathscr{B}^\mu}\geqslant\limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{k+1,n}\|_{\mathscr{B}^\mu},$$

and hence

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu}=\inf_{\mathscr{K}}\|T^k_{\vec{\psi},\varphi}-\mathscr{K}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}\geqslant\limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{k+1,n}\|_{\mathscr{B}^\mu}. \tag{3.4}$$

Using (3.3), (3.4) we get

$$\begin{aligned}\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu}&\geqslant\limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{k+1,n}\|_{\mathscr{B}^\mu}\\&\geqslant\limsup_{n\to\infty}(1-|z_n|^2)^{\mu}\left|\sum_{j=0}^{k}\left(\psi_j'(z_n)f^{(j)}_{k+1,n}(\varphi(z_n))+\psi_j(z_n)\varphi'(z_n)f^{(j+1)}_{k+1,n}(\varphi(z_n))\right)\right|\\&=\limsup_{n\to\infty}\frac{(k+1)!(1-|z_n|^2)^{\mu}|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+k+1}}.\end{aligned} \tag{3.5}$$

Since $|\varphi(z_n)|\to 1$ as $n\to\infty$, it follows from (3.2) and (3.5) that

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu}\gtrsim\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}=B_{k+1}. \tag{3.6}$$

Next, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$, such that

$$\begin{aligned}&\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_{k-1}(z)\varphi'(z)+\psi_k'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k}}\\&=\lim_{n\to\infty}\frac{(1-|z_n|^2)^{\mu}|\psi_{k-1}(z_n)\varphi'(z_n)+\psi_k'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+k}}.\end{aligned} \tag{3.7}$$

For each n, define $f_{k,n}(z)=\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}}\sigma^{k}_{\varphi(z_n)}(z)$, $z\in\mathbb{D}$. It is easy to see that $f_{k,n}\in H^\infty_\alpha$ and $\|f_{k,n}\|_{v_\alpha}\leqslant 2^{\alpha+1}$, $f^{(i)}_{k,n}(\varphi(z_n))=0$ for $i=0,1,\dots,k-1$ and

$$|f^{(k)}_{k,n}(\varphi(z_n))|=\frac{k!}{(1-|\varphi(z_n)|^2)^{\alpha+k}}. \tag{3.8}$$

Clearly, $\{f_{k,n}\}_{n\in\mathbb{N}}$ is a bounded sequence in H_α^∞ and converges to zero uniformly on compact subsets of $\mathbb{D}$. Hence, if $\mathscr{K}: H_\alpha^\infty \to \mathscr{B}^\mu$ is a compact operator, then by Lemma 3.2 we have $\lim_{n\to\infty}\|\mathscr{K} f_{k,n}\|_{\mathscr{B}^\mu}=0$. Further, we have

$$\|T^k_{\vec{\psi},\varphi}-\mathscr{K}\|_{H_\alpha^\infty\to\mathscr{B}^\mu} \geqslant \limsup_{n\to\infty}\|(T^k_{\vec{\psi},\varphi}-\mathscr{K})f_{k,n}\|_{\mathscr{B}^\mu} = \limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{k,n}\|_{\mathscr{B}^\mu},$$

and hence

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H_\alpha^\infty\to\mathscr{B}^\mu} = \inf_{\mathscr{K}}\|T^k_{\vec{\psi},\varphi}-\mathscr{K}\|_{H_\alpha^\infty\to\mathscr{B}^\mu} \geqslant \limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{k,n}\|_{\mathscr{B}^\mu}. \tag{3.9}$$

Using Lemma 2.1 and (3.8), (3.9) implies that

$$\begin{aligned}
&\|T^k_{\vec{\psi},\varphi}\|_{e,H_\alpha^\infty\to\mathscr{B}^\mu} \geqslant \limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{k,n}\|_{\mathscr{B}^\mu}\\
&\geqslant \limsup_{n\to\infty}(1-|z_n|^2)^\mu\left|\sum_{j=0}^k\left(\psi_j'(z_n)f_{k,n}^{(j)}(\varphi(z_n))+\psi_j(z_n)\varphi'(z_n)f_{k,n}^{(j+1)}(\varphi(z_n))\right)\right|\\
&\geqslant \limsup_{n\to\infty}(1-|z_n|^2)^\mu\left|\psi_k'(z_n)+\psi_{k-1}(z_n)\varphi'(z_n)\right|\left|f_{k,n}^{(k)}(\varphi(z_n))\right|\\
&\quad-\limsup_{n\to\infty}(1-|z_n|^2)^\mu\left|\psi_k(z_n)\varphi'(z_n)\right|\left|f_{k,n}^{(k+1)}(\varphi(z_n))\right|\\
&\geqslant \limsup_{n\to\infty}\frac{k!(1-|z_n|^2)^\mu\left|\psi_k'(z_n)+\psi_{k-1}(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^{\alpha+k}}\\
&\quad-\limsup_{n\to\infty}\frac{C\|f_{k,n}\|_{v_\alpha}(1-|z_n|^2)^\mu|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+k+1}}.
\end{aligned}\tag{3.10}$$

Since $|\varphi(z_n)|\to 1$ as $n\to\infty$, it follows from (3.2), (3.7) and (3.10) that

$$\begin{aligned}
\|T^k_{\vec{\psi},\varphi}\|_{e,H_\alpha^\infty\to\mathscr{B}^\mu} &\geqslant \limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{k,n}\|_{\mathscr{B}^\mu}\\
&\geqslant \limsup_{n\to\infty}\frac{k!(1-|z_n|^2)^\mu\left|\psi_k'(z_n)+\psi_{k-1}(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^{\alpha+k}}\\
&\quad-C\|f_{k,n}\|_{v_\alpha}\limsup_{n\to\infty}\frac{(1-|z_n|^2)^\mu|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+k+1}}\\
&\geqslant \lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{k!(1-|z|^2)^\mu\left|\psi_k'(z)+\psi_{k-1}(z)\varphi'(z)\right|}{(1-|\varphi(z)|^2)^{\alpha+k}}\\
&\quad-C\|f_{k,n}\|_{v_\alpha}\lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^\mu|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}.
\end{aligned}$$

Thus, applying (3.6),

$$\lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^\mu\left|\psi_k'(z)+\psi_{k-1}(z)\varphi'(z)\right|}{(1-|\varphi(z)|^2)^{\alpha+k}} \lesssim (1+C)\|T^k_{\vec{\psi},\varphi}\|_{e,H_\alpha^\infty\to\mathscr{B}^\mu}.$$

Therefore

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \gtrsim \frac{1}{1+C}\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^\mu\left|\psi_k'(z)+\psi_{k-1}(z)\varphi'(z)\right|}{(1-|\varphi(z)|^2)^{\alpha+k}} \approx B_k. \quad (3.11)$$

Now we fix $1\leqslant j\leqslant k-1$ and assume that

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \gtrsim \lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^\mu\left|\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)\right|}{(1-|\varphi(z)|^2)^{\alpha+i}} = B_i, \quad (3.12)$$

for each $i=j+1,\dots,k$. Now we show that (3.12) holds for $i=j$. For that, let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$ such that

$$\begin{aligned}&\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^\mu|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}}\\ &=\lim_{n\to\infty}\frac{(1-|z_n|^2)^\mu|\psi_{j-1}(z_n)\varphi'(z_n)+\psi_j'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+j}}. \quad (3.13)\end{aligned}$$

For each n, define $f_{j,n}(z)=\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}}\sigma^j_{\varphi(z_n)}(z)$, $z\in\mathbb{D}$. It is easy to see that $f_{j,n}\in H^\infty_\alpha$ and $\|f_{j,n}\|_{v_\alpha}\leqslant 2^{\alpha+1}$, $f^{(i)}_{j,n}(\varphi(z_n))=0$ for $i=0,1,\dots,j-1$ and

$$\left|f^{(j)}_{j,n}(\varphi(z_n))\right| = \frac{j!}{(1-|\varphi(z_n)|^2)^{\alpha+j}}. \quad (3.14)$$

Clearly, $\{f_{j,n}\}_{n\in\mathbb{N}}$ is a bounded sequence in H^∞_α which converges to zero uniformly on compact subsets of $\mathbb{D}$. Similarly, we have

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} = \inf_{\mathscr{K}}\|T^k_{\vec{\psi},\varphi}-\mathscr{K}\|_{H^\infty_\alpha\to\mathscr{B}^\mu} \geqslant \limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{j,n}\|_{\mathscr{B}^\mu}. \quad (3.15)$$

Using Lemma 2.1, (3.14) and (3.15), similarly we get

$$\begin{aligned}\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \geqslant &\limsup_{n\to\infty}\frac{j!(1-|z_n|^2)^\mu\left|\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right||\varphi(z_n)^n|}{(1-|\varphi(z_n)|^2)^{\alpha+j}}\\ &-C_1\sum_{i=j+1}^{k}\limsup_{n\to\infty}\frac{\|f_{j,n}\|_{v_\alpha}(1-|z_n|^2)^\mu|\psi_i'(z_n)+\psi_{i-1}(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+i}}\\ &-C_2\limsup_{n\to\infty}\frac{\|f_{j,n}\|_{v_\alpha}(1-|z_n|^2)^\mu|\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+k+1}}. \quad (3.16)\end{aligned}$$

Here C_1,C_2 are some positive constants. Since $|\varphi(z_n)|\to 1$ as $n\to\infty$, it follows from (3.2), (3.13) and (3.16) that

$$\begin{aligned}\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} &\gtrsim \limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{j,n}\|_{\mathscr{B}^\mu}\\ &\geqslant \limsup_{n\to\infty}\frac{j!(1-|z_n|^2)^\mu\left|\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^{\alpha+j}}\end{aligned}$$

$$
\begin{aligned}
&-C_2 \limsup_{n\to\infty} \frac{\|f_{j,n}\|_{v_\alpha}(1-|z_n|^2)^\mu |\psi_k(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+k+1}} \\
&-C_1 \sum_{i=j+1}^{k} \|f_{j,n}\|_{v_\alpha} \limsup_{n\to\infty} \frac{(1-|z_n|^2)^\mu |\psi_i'(z_n)+\psi_{i-1}(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+i}} \\
\geqslant &\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{j!(1-|z|^2)^\mu \left|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)\right|}{(1-|\varphi(z)|^2)^{\alpha+j}} \\
&-C_2 \|f_{j,n}\|_{v_\alpha} \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu |\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}} \\
&-C_1 \|f_{j,n}\|_{v_\alpha} \sum_{i=j+1}^{k} \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu |\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+i}}.
\end{aligned}
$$

Thus, applying (3.6) and (3.12),

$$
\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu \left|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)\right|}{(1-|\varphi(z)|^2)^{\alpha+j}} \lesssim \|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu}.
$$

Therefore, for all $j=1,2,\ldots,k$,

$$
\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \gtrsim \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu \left|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)\right|}{(1-|\varphi(z)|^2)^{\alpha+j}} = B_j. \tag{3.17}
$$

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ with $|\varphi(z_n)|\to 1$ as $n\to\infty$ such that

$$
\lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu |\psi_0'(z)|}{(1-|\varphi(z)|^2)^\alpha} = \lim_{n\to\infty} \frac{(1-|z_n|^2)^\mu |\psi_0'(z_n)|}{(1-|\varphi(z_n)|^2)^\alpha}. \tag{3.18}
$$

For each n, define $f_{0,n}(z) = \frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{\alpha+1}}$, $z\in\mathbb{D}$. It is easy to see that $f_{0,n}\in H^\infty_\alpha$ and $\|f_{0,n}\|_{v_\alpha} \leqslant 2^{\alpha+1}$, and

$$
\left|f_{0,n}(\varphi(z_n))\right| = \frac{1}{(1-|\varphi(z_n)|^2)^\alpha}. \tag{3.19}
$$

Clearly, $\{f_{0,n}\}_{n\in\mathbb{N}}$ is a bounded sequence in H^∞_α which converges to zero uniformly on compact subsets of $\mathbb{D}$. So, similarly we have

$$
\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} = \inf_{\mathscr{K}} \|T^k_{\vec{\psi},\varphi} - \mathscr{K}\|_{H^\infty_\alpha\to\mathscr{B}^\mu} \geqslant \limsup_{n\to\infty} \|T^k_{\vec{\psi},\varphi} f_{0,n}\|_{\mathscr{B}^\mu}. \tag{3.20}
$$

Using Lemma 2.1 and (3.19), (3.20) we get

$$
\begin{aligned}
&\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \geqslant \limsup_{n\to\infty} \|T^k_{\vec{\psi},\varphi} f_{0,n}\|_{\mathscr{B}^\mu} \\
\geqslant &\limsup_{n\to\infty} (1-|z_n|^2)^\mu \left| \sum_{j=0}^{k} \left(\psi_j'(z_n) f_{0,n}^{(j)}(\varphi(z_n)) + \psi_j(z_n)\varphi'(z_n) f_{0,n}^{(j+1)}(\varphi(z_n)) \right) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \limsup_{n\to\infty}(1-|z_n|^2)^{\mu}|\psi_0'(z_n)||f_{0,n}(\varphi(z_n))| \\
&\quad -\sum_{j=1}^{k}\limsup_{n\to\infty}(1-|z_n|^2)^{\mu}\left|\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)\right|\left|f_{0,n}^{(j)}(\varphi(z_n))\right| \\
&\quad -\limsup_{n\to\infty}(1-|z_n|^2)^{\mu}\left|\psi_k(z_n)\varphi'(z_n)\right|\left|f_{0,n}^{(k+1)}(\varphi(z_n))\right| \\
&\geqslant \limsup_{n\to\infty}\frac{(1-|z_n|^2)^{\mu}|\psi_0'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha}} \\
&\quad -C_3\|f_{0,n}\|_{v_\alpha}\sum_{j=1}^{k}\limsup_{n\to\infty}\frac{(1-|z_n|^2)^{\mu}|\psi_j'(z_n)+\psi_{j-1}(z_n)\varphi'(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha+j}} \\
&\quad -C_4\|f_{0,n}\|_{v_\alpha}\limsup_{n\to\infty}\frac{(1-|z_n|^2)^{\mu}\left|\psi_k(z_n)\varphi'(z_n)\right|}{(1-|\varphi(z_n)|^2)^{\alpha+k+1}}. \qquad (3.21)
\end{aligned}
$$

Here C_3, C_4 are some positive constants. Since $|\varphi(z_n)| \to 1$ as $n \to \infty$, it follows from (3.2), (3.13), (3.18) and (3.21) that

$$
\begin{aligned}
&\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \geqslant \limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}f_{0,n}\|_{\mathscr{B}^\mu} \\
&\geqslant \lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_0'(z)|}{(1-|\varphi(z)|^2)^{\alpha}} - C_4\|f_{0,n}\|_{v_\alpha}\lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}} \\
&\quad -C_3\|f_{0,n}\|_{v_\alpha}\sum_{j=1}^{k}\lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}\left|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)\right|}{(1-|\varphi(z)|^2)^{\alpha+j}}.
\end{aligned}
$$

Hence, by (3.6) and (3.17),

$$
\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \gtrsim \lim_{r\to1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_0'(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = B_0. \qquad (3.22)
$$

Therefore, (3.6), (3.17) and (3.22) imply that

$$
\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \gtrsim \max_{0\leqslant i\leqslant k+1} B_i. \qquad (3.23)
$$

Finally, we prove that

$$
\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \lesssim \max_{0\leqslant i\leqslant k+1} A_i
$$

and

$$
\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \lesssim \max_{0\leqslant i\leqslant k+1} B_i.
$$

For $r \in [0,1)$, set $\mathscr{K}_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $(\mathscr{K}_r f)(z) = f_r(z) = f(rz)$, $f \in H(\mathbb{D})$. It is obvious that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator $\mathscr{K}_r$ is compact on H^∞_α and $\|\mathscr{K}_r\|_{H^\infty_\alpha\to H^\infty_\alpha} \leqslant 1$. Let $\{r_n\} \subset (0,1)$ be a sequence such that $r_n \to 1$ as $n \to \infty$. Then for every positive integer n, the operator $T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n}$: $H^\infty_\alpha \to \mathscr{B}^\mu$ is compact. By the definition of the essential norm, we get

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha \to \mathscr{B}^\mu} \leqslant \limsup_{n\to\infty} \|T^k_{\vec{\psi},\varphi} - T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n}\|_{H^\infty_\alpha \to \mathscr{B}^\mu}. \tag{3.24}$$

Therefore, we only need to prove that

$$\limsup_{n\to\infty} \|T^k_{\vec{\psi},\varphi} - T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n}\|_{H^\infty_\alpha \to \mathscr{B}^\mu} \lesssim \max_{0\leqslant i\leqslant k+1} A_i$$

and

$$\limsup_{n\to\infty} \|T^k_{\vec{\psi},\varphi} - T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n}\|_{H^\infty_\alpha \to \mathscr{B}^\mu} \lesssim \max_{0\leqslant i\leqslant k+1} B_i.$$

For any $f \in H^\infty_\alpha$ such that $\|f\|_{v_\alpha} \leqslant 1$, we consider

$$\begin{aligned}
&\|(T^k_{\vec{\psi},\varphi} - T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n})f\|_{\mathscr{B}^\mu} \\
=& |(T^k_{\vec{\psi},\varphi} - T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n})f(0)| + \|(T^k_{\vec{\psi},\varphi} - T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n})f\| \\
=& \Big| \sum_{j=0}^k \Big(\psi_j(0) f^{(j)}(\varphi(0)) - \psi_j(0) f^{(j)}_{r_n}(\varphi(0)) \Big) \Big| + \Big\| \sum_{j=0}^k \psi_j (f^{(j)} - f^{(j)}_{r_n}) \circ \varphi \Big\|.
\end{aligned} \tag{3.25}$$

It is obvious that

$$\limsup_{n\to\infty} \Big| \sum_{j=0}^k \Big(\psi_j(0) f^{(j)}(\varphi(0)) - \psi_j(0) f^{(j)}_{r_n}(\varphi(0)) \Big) \Big| = 0. \tag{3.26}$$

Now we consider

$$\begin{aligned}
&\limsup_{n\to\infty} \Big\| \sum_{j=0}^k \psi_j (f^{(j)} - f^{(j)}_{r_n}) \circ \varphi \Big\| \\
=& \limsup_{n\to\infty} \sup_{z\in\mathbb{D}} (1-|z|^2)^\mu \Big| \psi_0'(z)(f - f_{r_n})(\varphi(z)) + \psi_k(z)\varphi'(z) \Big(f^{(k+1)} - f^{(k+1)}_{r_n} \Big)(\varphi(z)) \\
&+ \sum_{j=1}^k \big(\psi_j'(z) + \psi_{j-1}(z)\varphi'(z) \big) \Big(f^{(j)} - f^{(j)}_{r_n} \Big)(\varphi(z)) \Big| \\
\leqslant& \limsup_{n\to\infty} \sup_{|\varphi(z)|\leqslant r_N} (1-|z|^2)^\mu |(f - f_{r_n})(\varphi(z))||\psi_0'(z)| \\
&+ \limsup_{n\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\mu |(f - f_{r_n})(\varphi(z))||\psi_0'(z)| \\
&+ \limsup_{n\to\infty} \sup_{|\varphi(z)|\leqslant r_N} (1-|z|^2)^\mu \sum_{j=1}^k \Big| \Big(f^{(j)} - f^{(j)}_{r_n} \Big)(\varphi(z)) \Big| |\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)| \\
&+ \limsup_{n\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\mu \sum_{j=1}^k \Big| \Big(f^{(j)} - f^{(j)}_{r_n} \Big)(\varphi(z)) \Big| |\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)|
\end{aligned}$$

$$+\limsup_{n\to\infty}\sup_{|\varphi(z)|\leqslant r_N}(1-|z|^2)^{\mu}\left|\left(f^{(k+1)}-f_{r_n}^{(k+1)}\right)(\varphi(z))\right||\psi_k(z)\varphi'(z)|$$

$$+\limsup_{n\to\infty}\sup_{|\varphi(z)|> r_N}(1-|z|^2)^{\mu}\left|\left(f^{(k+1)}-f_{r_n}^{(k+1)}\right)(\varphi(z))\right||\psi_k(z)\varphi'(z)|$$

$$=Q_1+Q_2+Q_3+Q_4+Q_5+Q_6, \tag{3.27}$$

where $N\in\mathbb{N}$ is large enough such that $r_n\geqslant\frac{1}{2}$ for all $n\geqslant N$,

$$Q_1:=\limsup_{n\to\infty}\sup_{|\varphi(z)|\leqslant r_N}(1-|z|^2)^{\mu}|(f-f_{r_n})(\varphi(z))||\psi_0'(z)|,$$

$$Q_2:=\limsup_{n\to\infty}\sup_{|\varphi(z)|> r_N}(1-|z|^2)^{\mu}|(f-f_{r_n})(\varphi(z))||\psi_0'(z)|,$$

$$Q_3:=\limsup_{n\to\infty}\sup_{|\varphi(z)|\leqslant r_N}(1-|z|^2)^{\mu}\sum_{j=1}^{k}\left|\left(f^{(j)}-f_{r_n}^{(j)}\right)(\varphi(z))\right||\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|,$$

$$Q_4:=\limsup_{n\to\infty}\sup_{|\varphi(z)|> r_N}(1-|z|^2)^{\mu}\sum_{j=1}^{k}\left|\left(f^{(j)}-f_{r_n}^{(j)}\right)(\varphi(z))\right||\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|,$$

$$Q_5:=\limsup_{n\to\infty}\sup_{|\varphi(z)|\leqslant r_N}(1-|z|^2)^{\mu}\left|\left(f^{(k+1)}-f_{r_n}^{(k+1)}\right)(\varphi(z))\right||\psi_k(z)\varphi'(z)|$$

and

$$Q_6:=\limsup_{n\to\infty}\sup_{|\varphi(z)|> r_N}(1-|z|^2)^{\mu}\left|\left(f^{(k+1)}-f_{r_n}^{(k+1)}\right)(\varphi(z))\right||\psi_k(z)\varphi'(z)|.$$

Since $T_{\vec{\psi},\varphi}^{k}:H_\alpha^\infty\to\mathscr{B}^\mu$ is bounded, from Theorem 2.1, we obtain

$$T_0:=\sup_{z\in\mathbb{D}}(1-|z|^2)^{\mu}|\psi_0'(z)|<\infty,$$

$$T_j:=\sup_{z\in\mathbb{D}}(1-|z|^2)^{\mu}|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|<\infty,\quad\text{for}\quad j=1,2,\dots,k$$

and

$$T_{k+1}:=\sup_{z\in\mathbb{D}}(1-|z|^2)^{\mu}|\psi_k(z)\varphi'(z)|<\infty.$$

Since $f_{r_n}^{(i)}\to f^{(i)}(i=0,1,\dots,k+1)$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$, we have

$$Q_1\leqslant T_0\limsup_{n\to\infty}\sup_{|w|\leqslant r_N}|f(w)-f_{r_n}(w)|=0, \tag{3.28}$$

$$Q_3\leqslant\sum_{j=1}^{k}T_j\limsup_{n\to\infty}\sup_{|w|\leqslant r_N}|f^{(j)}(w)-f_{r_n}^{(j)}(w)|=0 \tag{3.29}$$

and

$$Q_5 \leqslant T_{k+1} \limsup_{n\to\infty} \sup_{|w|\leqslant r_N} |f^{(k+1)}(w) - f_{r_n}^{(k+1)}(w)| = 0. \tag{3.30}$$

Next we consider Q_6. We have $Q_6 = \limsup_{n\to\infty} Q_{61}$, where

$$Q_{61} := \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\mu \left| \left(f^{(k+1)} - f_{r_n}^{(k+1)}\right)(\varphi(z)) \right| |\psi_k(z)\varphi'(z)|.$$

Using the fact that $\|f\|_{v_\alpha} \leqslant 1$ and Lemma 2.1, we have

$$\begin{aligned} Q_{61} &\leqslant C\|f - f_{r_n}\|_{v_\alpha} \sup_{|\varphi(z)|>r_N} \frac{(1-|z|^2)^\mu |\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}} \qquad (3.31) \\ &= \frac{C\|f - f_{r_n}\|_{v_\alpha}}{(k+1)!} \sup_{|\varphi(z)|>r_N} \frac{(k+1)!(1-|z|^2)^\mu |\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}} \\ &\lesssim \frac{\|f - f_{r_n}\|_{v_\alpha}}{(k+1)!} \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\mu |\psi_k(z)\varphi'(z)| |f_{k+1,\varphi(z)}^{(k+1)}(\varphi(z))| \\ &\lesssim \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\mu \left| \left(T_{\vec{\psi},\varphi}^k f_{k+1,\varphi(z)}\right)'(z) \right| \lesssim \sup_{|a|>r_N} \|T_{\vec{\psi},\varphi}^k f_{k+1,a}\|_{\mathscr{B}^\mu}. \qquad (3.32) \end{aligned}$$

Taking limit as $N \to \infty$ we obtain

$$Q_6 = \limsup_{n\to\infty} Q_{61} \lesssim \limsup_{|a|\to 1} \|T_{\vec{\psi},\varphi}^k f_{k+1,a}\|_{\mathscr{B}^\mu} = A_{k+1}. \tag{3.33}$$

From (3.31), we see that

$$Q_6 = \limsup_{n\to\infty} Q_{61} \lesssim \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{(1-|z|^2)^\mu |\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}} = B_{k+1}. \tag{3.34}$$

Next we consider Q_4. We have $Q_4 = \limsup_{n\to\infty} Q_{41}$, where

$$Q_{41} := \sum_{j=1}^{k} \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\mu \left| \left(f^{(j)} - f_{r_n}^{(j)}\right)(\varphi(z)) \right| |\psi_j'(z) + \psi_{j-1}(z)\varphi'(z)|.$$

Using the fact that $\|f\|_{v_\alpha} \leqslant 1$, Lemma 2.1 and (3.32), we have

$$\begin{aligned} &\sup_{|\varphi(z)|>r_N} (1-|z|^2)^\mu \left| \left(f^{(k)} - f_{r_n}^{(k)}\right)(\varphi(z)) \right| |\psi_k'(z) + \psi_{k-1}(z)\varphi'(z)| \\ \leqslant\ & \frac{C\|f - f_{r_n}\|_{v_\alpha}}{k!} \sup_{|\varphi(z)|>r_N} \frac{k!(1-|z|^2)^\mu |\psi_k'(z) + \psi_{k-1}(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k}} \\ \lesssim\ & \sup_{|\varphi(z)|>r_N} (1-|z|^2)^\mu \left| \left(T_{\vec{\psi},\varphi}^k f_{k,\varphi(z)}\right)'(z) - \psi_k(z)\varphi'(z) f_{k,\varphi(z)}^{(k+1)}(\varphi(z)) \right| \end{aligned}$$

$$
\begin{aligned}
&\lesssim \sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\left(\left|\left(T^k_{\vec{\psi},\varphi}f_{k,\varphi(z)}\right)'(z)\right|+|\psi_k(z)\varphi'(z)||f^{(k+1)}_{k,\varphi(z)}(\varphi(z))|\right)\\
&\leqslant \sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\left(\left|\left(T^k_{\vec{\psi},\varphi}f_{k,\varphi(z)}\right)'(z)\right|+C\|f_{k,\varphi(z)}\|_{v_\alpha}\frac{|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}\right)\\
&\leqslant \sup_{|\varphi(z)|>r_N}\sup_{|a|>r_N}(1-|z|^2)^{\mu}\left(\left|\left(T^k_{\vec{\psi},\varphi}f_{k,a}\right)'(z)\right|+C\left|\left(T^k_{\vec{\psi},\varphi}f_{k+1,a}\right)'(z)\right|\right)\\
&\lesssim \sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{k,a}\|_{\mathscr{B}^\mu}+\sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{k+1,a}\|_{\mathscr{B}^\mu}. \qquad (3.35)
\end{aligned}
$$

Further, fix $1\leqslant j\leqslant k-1$ and assume that

$$
\begin{aligned}
&\sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\left|\left(f^{(i)}-f^{(i)}_{r_n}\right)(\varphi(z))\right||\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)|\\
&\leqslant C\|f-f_{r_n}\|_{v_\alpha}\sup_{|\varphi(z)|>r_N}\frac{(1-|z|^2)^{\mu}|\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+i}}\\
&\lesssim \sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{i,a}\|_{\mathscr{B}^\mu}+\sum_{t=i+1}^{k+1}\sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{t,a}\|_{\mathscr{B}^\mu} \qquad (3.36)
\end{aligned}
$$

for each $i=j+1,\dots,k$. Now we establish (3.36) for $i=j$. Using the fact that $\|f\|_{v_\alpha}\leqslant 1$, Lemma 2.1, (3.32) and (3.36), we have

$$
\begin{aligned}
&\sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\left|\left(f^{(j)}-f^{(j)}_{r_n}\right)(\varphi(z))\right||\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|\\
&\leqslant \frac{C\|f-f_{r_n}\|_{v_\alpha}}{j!}\sup_{|\varphi(z)|>r_N}\frac{j!(1-|z|^2)^{\mu}|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}}\\
&=\frac{C\|f-f_{r_n}\|_{v_\alpha}}{j!}\sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\Big|\left(T^k_{\vec{\psi},\varphi}f_{j,\varphi(z)}\right)'(z)\\
&\quad-\sum_{i=j+1}^{k}\left(\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)\right)f^{(i)}_{j,\varphi(z)}(\varphi(z))-\psi_k(z)\varphi'(z)f^{(k+1)}_{j,\varphi(z)}(\varphi(z))\Big|\\
&\lesssim \sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\Big(\left|\left(T^k_{\vec{\psi},\varphi}f_{j,\varphi(z)}\right)'(z)\right|\\
&\quad+\sum_{i=j+1}^{k}|\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)||f^{(i)}_{j,\varphi(z)}(\varphi(z))|+|\psi_k(z)\varphi'(z)||f^{(k+1)}_{j,\varphi(z)}(\varphi(z))|\Big)\\
&\leqslant \sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\Big(\left|\left(T^k_{\vec{\psi},\varphi}f_{j,\varphi(z)}\right)'(z)\right|\\
&\quad+\sum_{i=j+1}^{k}C\|f_{j,\varphi(z)}\|_{v_\alpha}\frac{|\psi_i'(z)+\psi_{i-1}(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+i}}+C\|f_{j,\varphi(z)}\|_{v_\alpha}\frac{|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sup_{|\varphi(z)|>r_N}\ \sup_{|a|>r_N}(1-|z|^2)^{\mu}\Bigg(\left|\left(T^k_{\vec{\psi},\varphi}f_{j,a}\right)'(z)\right| \\
&\quad+\sum_{i=j+1}^{k}\left(\left|\left(T^k_{\vec{\psi},\varphi}f_{i,a}\right)'(z)\right|+\sum_{t=i+1}^{k+1}\left|\left(T^k_{\vec{\psi},\varphi}f_{t,a}\right)'(z)\right|\right)+\left|\left(T^k_{\vec{\psi},\varphi}f_{k+1,a}\right)'(z)\right|\Bigg) \\
&\lesssim \sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{j,a}\|_{\mathscr{B}^{\mu}}+\sum_{i=j+1}^{k+1}\sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{i,a}\|_{\mathscr{B}^{\mu}} \qquad (3.37)
\end{aligned}
$$

for every $j=1,2,\dots,k$. After a calculation, using (3.37), we have

$$
\begin{aligned}
Q_{41} &\leqslant \sum_{j=1}^{k} C\|f-f_{r_n}\|_{v_\alpha}\sup_{|\varphi(z)|>r_N}\frac{(1-|z|^2)^{\mu}|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}} \qquad (3.38)\\
&\lesssim \sum_{j=1}^{k}\sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{j,a}\|_{\mathscr{B}^{\mu}}+\sum_{j=1}^{k}\sum_{i=j+1}^{k+1}\sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{i,a}\|_{\mathscr{B}^{\mu}}. \qquad (3.39)
\end{aligned}
$$

Taking limit as $N\to\infty$ we obtain

$$
Q_4=\limsup_{n\to\infty}Q_{41}\lesssim \limsup_{|a|\to 1}\sum_{i=1}^{k+1}\|T^k_{\vec{\psi},\varphi}f_{i,a}\|_{\mathscr{B}^{\mu}}=\sum_{i=1}^{k+1}A_i\lesssim \max_{1\leqslant i\leqslant k+1}A_i. \qquad (3.40)
$$

From (3.38), we see that

$$
\begin{aligned}
Q_4=\limsup_{n\to\infty}Q_{41} &\lesssim \lim_{r\to 1}\sup_{|\varphi(z)|>r}\sum_{j=1}^{k}\frac{(1-|z|^2)^{\mu}|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}}\\
&\lesssim \max_{1\leqslant i\leqslant k+1}B_i. \qquad (3.41)
\end{aligned}
$$

Finally we consider Q_2. We have $Q_2=\limsup_{n\to\infty}Q_{21}$, where

$$
Q_{21}:=\sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}|(f-f_{r_n})(\varphi(z))||\psi_0'(z)|.
$$

Using the fact that $\|f\|_{v_\alpha}\leqslant 1$, Lemma 2.1, (3.32) and (3.37), we have

$$
\begin{aligned}
Q_{21} &\leqslant \|f-f_{r_n}\|_{v_\alpha}\sup_{|\varphi(z)|>r_N}\frac{(1-|z|^2)^{\mu}|\psi_0'(z)|}{(1-|\varphi(z)|^2)^{\alpha}} \qquad (3.42)\\
&=\|f-f_{r_n}\|_{v_\alpha}\sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\Bigg|\left(T^k_{\vec{\psi},\varphi}f_{0,\varphi(z)}\right)'(z)\\
&\qquad-\sum_{j=1}^{k}\left(\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)\right)f^{(j)}_{0,\varphi(z)}(\varphi(z))-\psi_k(z)\varphi'(z)f^{(k+1)}_{0,\varphi(z)}(\varphi(z))\Bigg|\\
&\leqslant \sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\Bigg(\left|\left(T^k_{\vec{\psi},\varphi}f_{0,\varphi(z)}\right)'(z)\right|\\
&\qquad+\sum_{j=1}^{k}|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)||f^{(j)}_{0,\varphi(z)}(\varphi(z))|+|\psi_k(z)\varphi'(z)||f^{(k+1)}_{0,\varphi(z)}(\varphi(z))|\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \sup_{|\varphi(z)|>r_N}(1-|z|^2)^{\mu}\Bigg(\Big|\Big(T^k_{\vec{\psi},\varphi}f_{0,\varphi(z)}\Big)'(z)\Big| \\
&\quad+\sum_{j=1}^{k}C\|f_{0,\varphi(z)}\|_{v_\alpha}\frac{|\psi_j'(z)+\psi_{j-1}(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}}+C\|f_{0,\varphi(z)}\|_{v_\alpha}\frac{|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}\Bigg) \\
&\lesssim \sup_{|\varphi(z)|>r_N}\sup_{|a|>r_N}(1-|z|^2)^{\mu}\Bigg(\Big|\Big(T^k_{\vec{\psi},\varphi}f_{0,a}\Big)'(z)\Big|+\Big|\Big(T^k_{\vec{\psi},\varphi}f_{k+1,a}\Big)'(z)\Big| \\
&\quad+\sum_{j=1}^{k}\Bigg(\Big|\Big(T^k_{\vec{\psi},\varphi}f_{j,a}\Big)'(z)\Big|+\sum_{i=j+1}^{k+1}\Big|\Big(T^k_{\vec{\psi},\varphi}f_{i,a}\Big)'(z)\Big|\Bigg)\Bigg) \\
&\lesssim \sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{0,a}\|_{\mathscr{B}^\mu}+\sum_{j=1}^{k+1}\sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{j,a}\|_{\mathscr{B}^\mu}+\sum_{j=1}^{k}\sum_{i=j+1}^{k+1}\sup_{|a|>r_N}\|T^k_{\vec{\psi},\varphi}f_{i,a}\|_{\mathscr{B}^\mu}.
\end{aligned}
$$

Taking limit as $N\to\infty$ we obtain

$$
Q_2=\limsup_{n\to\infty}Q_{21}\lesssim \limsup_{|a|\to 1}\sum_{i=0}^{k+1}\|T^k_{\vec{\psi},\varphi}f_{i,a}\|_{\mathscr{B}^\mu}=\sum_{i=0}^{k+1}A_i\lesssim \max_{0\leqslant i\leqslant k+1}A_i. \tag{3.43}
$$

From (3.42), we see that

$$
Q_2=\limsup_{n\to\infty}Q_{21}\lesssim \lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_0'(z)|}{(1-|\varphi(z)|^2)^{\alpha}}=B_0. \tag{3.44}
$$

Hence by (3.25), (3.26), (3.27), (3.28), (3.29), (3.30), (3.33), (3.40) and (3.43) we get

$$
\begin{aligned}
\limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}-T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n}\|_{H^\infty_\alpha\to\mathscr{B}^\mu} &= \limsup_{n\to\infty}\sup_{\|f\|_{v_\alpha}\leqslant 1}\|(T^k_{\vec{\psi},\varphi}-T_{\Psi,\varphi}\mathscr{K}_{r_n})f\|_{\mathscr{B}^\mu} \\
&= \limsup_{n\to\infty}\sup_{\|f\|_{v_\alpha}\leqslant 1}\Big\|\sum_{j=0}^{k}\psi_j(f^{(j)}-f^{(j)}_{r_n})\circ\varphi\Big\| \\
&\lesssim \max_{0\leqslant i\leqslant k+1}A_i.
\end{aligned} \tag{3.45}
$$

Similarly, by (3.25), (3.26), (3.27), (3.28), (3.29), (3.30), (3.34), (3.41) and (3.44) we get

$$
\limsup_{n\to\infty}\|T^k_{\vec{\psi},\varphi}-T^k_{\vec{\psi},\varphi}\mathscr{K}_{r_n}\|_{H^\infty_\alpha\to\mathscr{B}^\mu}\lesssim \max_{0\leqslant i\leqslant k+1}B_i. \tag{3.46}
$$

Therefore, by (3.24), (3.45) and (3.46), we obtain

$$
\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu}\lesssim \max_{0\leqslant i\leqslant k+1}A_i,\qquad \|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu}\lesssim \max_{0\leqslant i\leqslant k+1}B_i.
$$

This completes the proof. □

Now, motivated by [3], we give another characterization for the essential norm of the operator $T^k_{\vec{\psi},\varphi}:H^\infty_\alpha\to\mathscr{B}^\mu$.

THEOREM 3.2. *Let* $\min\{\alpha,\mu\}>0$, $k\in\mathbb{N}_0$, $\varphi\in S(\mathbb{D})$ *and* $\psi_j\in H(\mathbb{D})$, $j=0,1,\ldots,k$. *If the operator* $T^k_{\vec{\psi},\varphi}:H^\infty_\alpha\to\mathscr{B}^\mu$ *is bounded, then*

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu}\approx\max_{0\leqslant i\leqslant k+1}D_i,$$

where

$$D_0=\limsup_{n\to\infty}n^\alpha\|\psi_0'\varphi^{n-1}\|_{v_\mu},\quad D_{k+1}=\limsup_{n\to\infty}n^{\alpha+k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu};$$

$$D_j=\limsup_{n\to\infty}n^{\alpha+j}\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_\mu},\quad \textit{for all}\quad j=1,2,\ldots,k.$$

Proof. According to Theorem 3.1, we known that the boundedness of the operator $T^k_{\vec{\psi},\varphi}:H^\infty_\alpha\to\mathscr{B}^\mu$ is equivalent to the boundedness of the operators $\psi_0'C_\varphi:H^\infty_{v_\alpha}\to H^\infty_{v_\mu}$, $(\psi_{j-1}\varphi'+\psi_j')C_\varphi:H^\infty_{v_{\alpha+j}}\to H^\infty_{v_\mu}$ for $j=1.2.\ldots,k$, and $\psi_k\varphi'C_\varphi:H^\infty_{v_{\alpha+k+1}}\to H^\infty_{v_\mu}$.

The upper estimate. By Lemmas 2.4 and 3.1, we get

$$\begin{aligned}\|\psi_0'C_\varphi\|_{e,H^\infty_{v_\alpha}\to H^\infty_{v_\mu}}&=\limsup_{n\to\infty}\frac{\|\psi_0'\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_\alpha}}=\limsup_{n\to\infty}\frac{n^\alpha\|\psi_0'\varphi^{n-1}\|_{v_\mu}}{n^\alpha\|\xi^{n-1}\|_{v_\alpha}}\\&\approx\limsup_{n\to\infty}n^\alpha\|\psi_0'\varphi^{n-1}\|_{v_\mu},\end{aligned}$$

$$\begin{aligned}\|(\psi_{j-1}\varphi'+\psi_j')C_\varphi\|_{e,H^\infty_{v_{\alpha+j}}\to H^\infty_{v_\mu}}&=\limsup_{n\to\infty}\frac{\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_{\alpha+j}}}\\&=\limsup_{n\to\infty}\frac{n^{\alpha+j}\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_\mu}}{n^{\alpha+j}\|\xi^{n-1}\|_{v_{\alpha+j}}}\\&\approx\limsup_{n\to\infty}n^{\alpha+j}\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_\mu}\\&\quad\text{for}\quad j=1,2,\ldots,k\end{aligned}$$

and

$$\begin{aligned}\|\psi_k\varphi'C_\varphi\|_{e,H^\infty_{v_{\alpha+k+1}}\to H^\infty_{v_\mu}}&=\limsup_{n\to\infty}\frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_{\alpha+k+1}}}\\&=\limsup_{n\to\infty}\frac{n^{\alpha+k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu}}{n^{\alpha+k+1}\|\xi^{n-1}\|_{v_{\alpha+k+1}}}\\&\approx\limsup_{n\to\infty}n^{\alpha+k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu}.\end{aligned}$$

It follows that

$$\begin{aligned}\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} &\lesssim \|\psi_0'C_\varphi\|_{e,H^\infty_{v_\alpha}\to H^\infty_{v_\mu}} + \|\psi_k\varphi'C_\varphi\|_{e,H^\infty_{v_{\alpha+k+1}}\to H^\infty_{v_\mu}} \\ &\quad + \sum_{j=1}^{k}\|(\psi_{j-1}\varphi'+\psi_j')C_\varphi\|_{e,H^\infty_{v_{\alpha+j}}\to H^\infty_{v_\mu}} \\ &\approx \sum_{i=0}^{k+1} D_i \lesssim \max_{0\leqslant i\leqslant k+1} D_i.\end{aligned}$$

The lower estimate. From Theorem 3.1, Lemmas 2.4 and 3.1, we have

$$\begin{aligned}\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} &\gtrsim B_0 = \|\psi_0'C_\varphi\|_{e,H^\infty_{v_\alpha}\to H^\infty_{v_\mu}} = \limsup_{n\to\infty}\frac{\|\psi_0'\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_\alpha}} \\ &\approx \limsup_{n\to\infty} n^\alpha \|\psi_0'\varphi^{n-1}\|_{v_\mu},\end{aligned}$$

$$\begin{aligned}\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} &\gtrsim B_j = \|(\psi_{j-1}\varphi'+\psi_j')C_\varphi\|_{e,H^\infty_{v_{\alpha+j}}\to H^\infty_{v_\mu}} \\ &= \limsup_{n\to\infty}\frac{\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_{\alpha+j}}} \\ &\approx \limsup_{n\to\infty} n^{\alpha+j}\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_\mu} \\ &\quad \text{for} \quad j=1,2,\dots,k\end{aligned}$$

and

$$\begin{aligned}\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} &\gtrsim B_{k+1} = \|\psi_k\varphi'C_\varphi\|_{e,H^\infty_{v_{\alpha+k+1}}\to H^\infty_{v_\mu}} = \limsup_{n\to\infty}\frac{\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu}}{\|\xi^{n-1}\|_{v_{\alpha+k+1}}} \\ &\approx \limsup_{n\to\infty} n^{\alpha+k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu}.\end{aligned}$$

Therefore

$$\|T^k_{\vec{\psi},\varphi}\|_{e,H^\infty_\alpha\to\mathscr{B}^\mu} \gtrsim \max_{0\leqslant i\leqslant k+1} D_i.$$

This completes the proof of this theorem. □

From Theorems 3.1 and 3.2, we immediately get the following characterizations for the compactness of $T^k_{\vec{\psi},\varphi}: H^\infty_\alpha \to \mathscr{B}^\mu$.

COROLLARY 3.1. *Let* $\min\{\alpha,\mu\}>0$, $k\in\mathbb{N}_0$, $\varphi\in S(\mathbb{D})$ *and* $\psi_j\in H(\mathbb{D})$, $j=0,1,\dots,k$. *If* $T^k_{\vec{\psi},\varphi}: H^\infty_\alpha\to\mathscr{B}^\mu$ *is bounded, then the following statements are equivalent.*

(a) The operator $T^k_{\vec{\psi},\varphi}: H^\infty_\alpha\to\mathscr{B}^\mu$ *is compact.*

(b) $\lim_{|\varphi(a)|\to 1}\|T^k_{\vec{\psi},\varphi}f_{j,\varphi(a)}\|_{\mathscr{B}^\mu}=0$, *for* $j=0,1,\dots,k+1$.

(c)

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_0'(z)|}{(1-|\varphi(z)|^2)^{\alpha}}=0;$$

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_{j-1}(z)\varphi'(z)+\psi_j'(z)|}{(1-|\varphi(z)|^2)^{\alpha+j}}=0,\quad \textit{for}\quad j=1,2,\dots,k;$$

$$\lim_{r\to 1}\sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^{\mu}|\psi_k(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k+1}}=0.$$

(d)

$$\limsup_{n\to\infty} n^{\alpha}\|\psi_0'\varphi^{n-1}\|_{v_\mu}=0;\quad \limsup_{n\to\infty} n^{\alpha+k+1}\|\psi_k\varphi'\varphi^{n-1}\|_{v_\mu}=0;$$

$$\limsup_{n\to\infty} n^{\alpha+j}\|(\psi_{j-1}\varphi'+\psi_j')\varphi^{n-1}\|_{v_\mu}=0,\quad \textit{for}\quad j=1,2,\dots,k.$$

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