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WEIGHTED HARDY INEQUALITY WITH TWO–DIMENSIONAL RECTANGULAR OPERATOR: THE CASE $q < p$

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Abstract. A criterion is obtained for the boundedness of the two–dimensional rectangular integration operator from a weighted Lebesgue space $L_v^p(\mathbb{R}_+^2)$ to $L_w^q(\mathbb{R}_+^2)$ for $1 < q < p < \infty$, which is a supplement to E. Sawyer's theorem [8] and its extension [9] given for $1 < p \leqslant q < \infty$.

1. Introduction

A weight is a locally integrable function on $\mathbb{R}_+^2 := (0,\infty)^2$, positive almost everywhere (a.e.). We study the Hardy integral operator

$$(I_2 f)(x,y):\ = \int_0^x \int_0^y f(s,t)\,ds\,dt, \qquad (x,y) \in \mathbb{R}_+^2,$$

in weighted Lebesgue spaces. For a real parameter $p > 1$ and a weight v the Lebesgue space $L_v^p(\mathbb{R}_+^2)$ consists of all measurable functions f on $\mathbb{R}_+^2$ satisfying

$$\|f\|_{p,v} := \left(\int_{\mathbb{R}_+^2} |f(x,y)|^p v(x,y)\,dx\,dy\right)^{\frac{1}{p}} < \infty.$$

The dual to I_2 operator is

$$(I_2^* f)(x,y):\ = \int_x^\infty \int_y^\infty f(s,t)\,ds\,dt, \qquad (x,y) \in \mathbb{R}_+^2.$$

We begin with the remarkable characterisation by E.T. Sawyer given in 1985 for the weighted two–dimensional Hardy inequality [8].

THEOREM 1. [8, Theorem 1A] *Let $1 < p \leqslant q < \infty$. Suppose v and w are weights. Denote $p' := p/(p-1)$ and $\sigma := v^{1-p'}$. The inequality*

$$\|(I_2 f)\|_{q,w} \leqslant C\|f\|_{p,v} \tag{1}$$

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holds for all measurable non-negative functions f on $\mathbb{R}^2_+$ if and only if

$$A_1 := A_1(p,q) := \sup_{(u,z)\in\mathbb{R}^2_+} \left[(I_2^* w)(u,z)\right]^{\frac{1}{q}} \left[(I_2\sigma)(u,z)\right]^{\frac{1}{p'}} < \infty,$$

$$A_2 := A_2(p,q) := \sup_{(u,z)\in\mathbb{R}^2_+} \left[\left(I_2\left[(I_2\sigma)^q w\right]\right)(u,z)\right]^{\frac{1}{q}} \left[(I_2\sigma)(u,z)\right]^{-\frac{1}{p}} < \infty,$$

$$A_3 := A_3(p,q) := \sup_{(u,z)\in\mathbb{R}^2_+} \left[\left(I_2^*\left[(I_2^* w)^{p'}\sigma\right]\right)(u,z)\right]^{\frac{1}{p'}} \left[(I_2^* w)(u,z)\right]^{-\frac{1}{q'}} < \infty,$$

and $C \approx A_1 + A_2 + A_3$ with equivalence constants depending on parameters p and q.

The Sawyer theorem above gives an explicit boundedness criterion in terms of v and w for $I_2 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ in the case $1 < p \leqslant q < \infty$. For $p < q$ this result was refined in [9] by providing an easier characterisation of (1) by the only functional $A := A_1$.

THEOREM 2. ([9, Theorem 2]) *Let $1 < p < q < \infty$, and suppose v and w are weights. Put $\alpha := \alpha(p,q) := \frac{p^2(q-1)}{q-p}$, $\alpha' = \alpha(q',p')$ and denote*

$$\mathbb{C}_{s,t} := 3^{3q}\left[\left(\frac{2^4}{3}\right)^q \max\left\{s, 2q(q')^{\frac{q}{p'}}\right\}\left(\frac{2^{p-1}}{2^{p-1}-1}\right)^{\frac{q}{p}} + 3^{\frac{1}{p}+\frac{1}{q'}} t^{\frac{1}{p'}}\right].$$

The inequality (1) holds if and only if $A < \infty$. Besides, the following estimate is true for the best constant C in (1):

$$A \leqslant C \leqslant \mathbb{C}_{\alpha,\alpha'} A. \tag{2}$$

Since $\lim_{p\to q-0} \alpha(p,q) = \lim_{p\to q-0} \alpha'(q',p') = \infty$ then the sufficient part of Theorem 2 and the right hand side of the inequality (2) has blow–up effect.

If (1) holds then $A_1 \leqslant C$, actually, for all $p,q > 1$. Therefore, $(I_2\sigma)(u,z) < \infty$ and $(I_2^* w)(u,z) < \infty$ at any $(u,z) \in \mathbb{R}^2_+$, and we may and shall assume these properties throughout the paper. In particular, $\sigma \in L^1_{\text{loc}}(\mathbb{R}^2_+)$ and $w \in L^1_{\text{loc}}(\mathbb{R}^2_+)$. From $(I_2\sigma)(u,z) < \infty$ it follows that $\left(\int_0^u \sigma(s,z)\,ds\right)\left(\int_0^z \sigma(u,t)\,dt\right) < \infty$ for almost all $(u,z) \in \mathbb{R}^2_+$. Analogously, $\left(\int_x^\infty w(s,y)\,ds\right)\left(\int_y^\infty w(x,t)\,dt\right) < \infty$ a.e. on $\mathbb{R}^2_+$.

The case $q < p$ was also discussed in [9]. In particular, the authors provided separate necessary and sufficient conditions for the validity of (1).

THEOREM 3. ([9, Theorem 3]) *Let $1 < q < p < \infty, 1/r := 1/q - 1/p$, and v, w be weights. If the inequality (1) is true then $B < \infty$, where*

$$B := B(p,q) := \left(\int_{\mathbb{R}^2_+} d_y\left[(I_2\sigma)(x,y)\right]^{\frac{r}{p'}} d_x\left(-\left[(I_2^* w)(x,y)\right]^{\frac{r}{q}}\right)\right)^{\frac{1}{r}}$$

$$= \left(\int_{\mathbb{R}^2_+} d_x\left[(I_2\sigma)(x,y)\right]^{\frac{r}{p'}} d_y\left(-\left[(I_2^* w)(x,y)\right]^{\frac{r}{q}}\right)\right)^{\frac{1}{r}}$$

$$= \left(\int_{\mathbb{R}^2_+} \left[(I_2\sigma)(x,y) \right]^{\frac{r}{p'}} d_x d_y \left[(I_2^* w)(x,y) \right]^{\frac{r}{q}} \right)^{\frac{1}{r}}$$

$$= \left(\int_{\mathbb{R}^2_+} \left[(I_2^* w)(x,y) \right]^{\frac{r}{q}} d_x d_y \left[(I_2\sigma)(x,y) \right]^{\frac{r}{p'}} \right)^{\frac{1}{r}}.$$

The inequality (1) *holds if*

$$B_v := \left(\int_{\mathbb{R}^2_+} \sigma(u,z) \left[\left(I_2^* \left[(I_2\sigma)^{q-1} w \right] \right)(u,z) \right]^{\frac{r}{q}} du\, dz \right)^{\frac{1}{r}} < \infty.$$

Moreover, $B \lesssim C \lesssim B_v$.

History and other results related to multi–dimensional Hardy inequalities may be found in [1, 2, 3, 4, 6, 11, 12, 13].

In this paper we continue the study of the problem in the case $1 < q < p < \infty$ and provide a criterion for I_2 to be bounded from $L^p_v(\mathbb{R}^2_+)$ to $L^q_w(\mathbb{R}^2_+)$ under some conditions on v and w. More precisely, we require the existence of parameters $\gamma \in [q/p, 1)$ and $\gamma^* \in [p'/q', 1)$ such that for almost all $(x,y) \in \mathbb{R}^2_+$

$$\frac{\partial^2 \left(\left[(I_2\sigma)(x,y) \right]^{\gamma} \right)}{\partial x \partial y} = \gamma \left[(I_2\sigma)(x,y) \right]^{\gamma-2} \times \left((I_2\sigma)(x,y)\sigma(x,y) - (1-\gamma) \left(\int_0^x \sigma(s,y)\, ds \right) \left(\int_0^y \sigma(x,t)\, dt \right) \right) \geqslant 0 \quad (3)$$

and

$$\frac{\partial^2 \left(\left[(I_2^* w)(x,y) \right]^{\gamma^*} \right)}{\partial x \partial y} = \gamma^* \left[(I_2^* w)(x,y) \right]^{\gamma^*-2} \times \left((I_2^* w)(x,y) w(x,y) - (1-\gamma^*) \left(\int_x^{\infty} w(s,y)\, ds \right) \left(\int_y^{\infty} w(x,t)\, dt \right) \right) \geqslant 0. \quad (4)$$

For instance, product type weights [7] are covered by (3) and (4) provided $\gamma > 0$, $\gamma^* > 0$. Examples of non–product weights satisfying (3) and (4) are given at the end of § 3.

Our main result is Theorem 4. We complement it by the compactness criterion for $I_2 : L^p_v(\mathbb{R}^2_+) \to L^q_w(\mathbb{R}^2_+)$ if $q < p$ (see Theorem 6). Analogs of Theorems 4 and 6 are also valid for the dual operator I_2^* and mixed Hardy operators (see [8, Remark 1] for details). Some important sufficient boundedness conditions for the Hardy operator are found in Theorem 5 to connect our results with Saywer's theorem in the case $p = q$.

Throughout the work, the notation of the form $\Phi \lesssim \Psi$ means that the relation $\Phi \leqslant c\Psi$ holds with some constant $c > 0$, independent of Φ and Ψ. We write $\Phi \approx \Psi$ in the case of $\Phi \lesssim \Psi \lesssim \Phi$. The symbol $\mathbb{Z}$ is used for integers. The characteristic function of the subset $E \subset \mathbb{R}^n_+$ is denoted by χ_E. Symbols $:=$ and $=:$ are used to define new values.

2. The E. Saywer partitioning scheme and technical statements

Denote $1/r := 1/q - 1/p$ and

$$\beta(p,q,c) := 3\max\left\{\frac{3^{3q+1}}{4}, \frac{3^{\frac{q}{p}+2q}}{2^{\frac{q}{p}}}, \frac{3^{2q-\frac{q}{p}+2-c}}{4c}\left[\frac{2\cdot 3^{3c}}{1-c} - 3^{c-2}\right]\right\}$$

for some $0 < q/p \leqslant c < 1$. We define also $B_1(p,q) = B(p,q) = B_w = B^*_\sigma$, where

$$\begin{aligned} B_u^r :=& \frac{r^2}{pq}\int_{\mathbb{R}^2_+} u(x,t)\left(\int_0^t \left[(I_2\sigma)(x,y)\right]^{\frac{r}{p'}}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}-1}\left(\int_x^\infty w(s,y)\,ds\right)dy\right)dx\,dt \\ &+ \frac{r}{q}\int_{\mathbb{R}^2_+} u(x,y)\left[(I_2\sigma)(x,y)\right]^{\frac{r}{p'}}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}dx\,dy, \\ (B_u^*)^r :=& \frac{r^2}{pq}\int_{\mathbb{R}^2_+} u(x,t)\left(\int_t^\infty \left[(I_2\sigma)(x,y)\right]^{\frac{r}{q'}-1}\left(\int_0^x \sigma(s,y)\,ds\right)\left[(I_2^*w)(x,y)\right]^{\frac{r}{q}}dy\right)dx\,dt \\ &+ \frac{r}{q}\int_{\mathbb{R}^2_+} u(x,y)\left[(I_2\sigma)(x,y)\right]^{\frac{r}{q'}}\left[(I_2^*w)(x,y)\right]^{\frac{r}{q}}dx\,dy, \end{aligned}$$

and

$$\begin{aligned} B_2^r :=& B_2^r(p,q) := \int_{\mathbb{R}^2_+}\left[(I_2\sigma)(x,y)\right]^{-\frac{r}{p}} d_x d_y\left[\left(I_2\left[(I_2\sigma)^q w\right]\right)(x,y)\right]^{\frac{r}{q}} = \mathbb{B}_w^r, \\ B_3^r :=& B_3^r(p,q) := \int_{\mathbb{R}^2_+}\left[I_2^*w(x,y)\right]^{-\frac{r}{q'}} d_x d_y\left[\left(I_2^*\left[(I_2^*w)^{p'}\sigma\right]\right)(x,y)\right]^{\frac{r}{p'}} = [\mathbb{B}_\sigma^*]^r, \end{aligned}$$

where

$$\begin{aligned} \mathbb{B}_u^r =& \frac{r^2}{pq}\int_{\mathbb{R}^2_+}\left[(I_2\sigma)(s,y)\right]^q u(s,y)\left(\int_s^\infty \left[(I_2\sigma)(x,y)\right]^{-\frac{r}{p}}\left[\left(I_2\left[(I_2\sigma)^q w\right]\right)(x,y)\right]^{\frac{r}{p}-1}\right. \\ &\times\left.\left(\int_0^y \left[(I_2\sigma)(x,t)\right]^q w(x,t)\,dt\right)dx\right)ds\,dy \\ &+ \frac{r}{q}\int_{\mathbb{R}^2_+}\left[(I_2\sigma)(x,y)\right]^{q-\frac{r}{p}} u(x,y)\left[\left(I_2\left[(I_2\sigma)^q w\right]\right)(x,y)\right]^{\frac{r}{p}}dx\,dy, \end{aligned}$$

$$\begin{aligned} [\mathbb{B}_u^*]^r :=& \frac{r^2}{p'q'}\int_{\mathbb{R}^2_+}\left[(I_2^*w)(s,y)\right]^{p'} u(s,y) \\ &\times\left(\int_0^s \left[(I_2^*w)(x,y)\right]^{-\frac{r}{q'}}\left[\left(I_2^*\left[(I_2^*w)^{p'}\sigma\right]\right)(x,y)\right]^{\frac{r}{q'}-1}\right. \\ &\times\left.\left(\int_y^\infty \left[(I_2^*w)(x,t)\right]^{p'}\sigma(x,t)\,dt\right)dx\right)ds\,dy \\ &+ \frac{r}{p'}\int_{\mathbb{R}^2_+}\left[(I_2^*w)(x,y)\right]^{p'-\frac{r}{q'}} u(x,y)\left[\left(I_2^*\left[(I_2^*w)^{p'}\sigma\right]\right)(x,y)\right]^{\frac{r}{q'}}dx\,dy. \end{aligned}$$

Our proof is based on E. Saywer's scheme of partitioning of $\mathbb{R}^2_+$ and several auxiliary technical statements (see Lemmas 1 – 4).

To describe *the E. Saywer scheme* [8, Theorem 1A] we reduce our consideration to the subclass $M \subset L^p_v(\mathbb{R}^2_+)$ of all functions $f \geqslant 0$ bounded on $\mathbb{R}^2_+$ with compact supports contained in the set $\{I_2\sigma > 0\}$. Further, we fix $f \in M$ and define the domains

$$\Omega_k := \left\{ I_2 f > 3^k \right\}, \qquad k \in \mathbb{Z}.$$

Then, by our assumptions on f, there exists $K \in \mathbb{Z}$ such that $\Omega_k \neq \varnothing$ for $k \leqslant K$, $\Omega_k = \varnothing$ for $k > K$, $\bigcup_{k\in\mathbb{Z}} \Omega_k = \mathbb{R}^2_+$ and

$$3^k < (I_2 f)(x,y) \leqslant 3^{k+1}, \qquad k \leqslant K, \qquad (x,y) \in (\Omega_k \setminus \Omega_{k+1}).$$

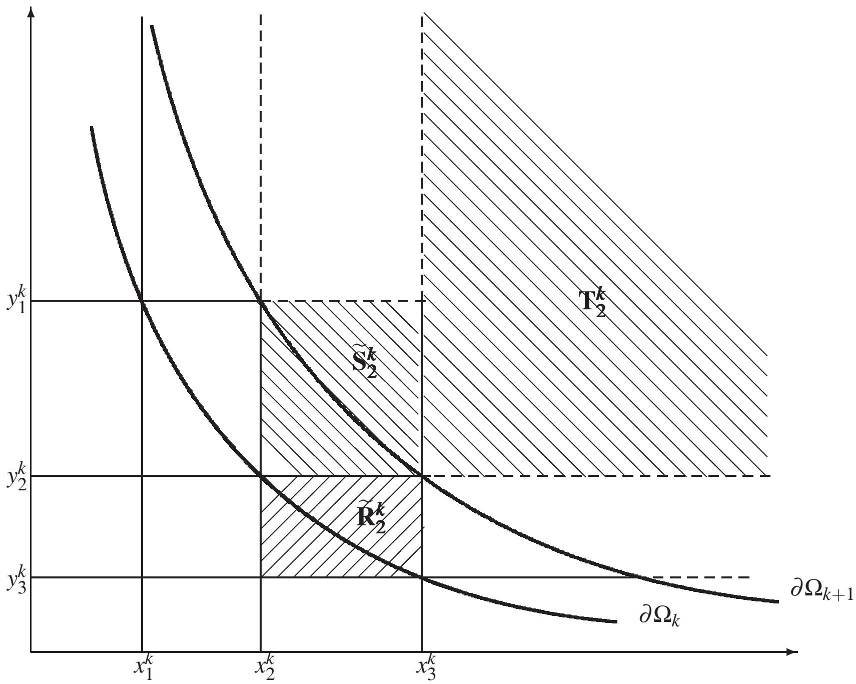

Fig. 1

We write

$$\int_{\mathbb{R}^2_+} (I_2 f)^q w = \sum_{k \leqslant K-2} \int_{\Omega_{k+2}\setminus\Omega_{k+3}} (I_2 f)^q w \leqslant 3^{3q} \sum_{k \leqslant K-2} 3^{kq} |\Omega_{k+2} \setminus \Omega_{k+3}|_w,$$

where $|\Omega_{k+2} \setminus \Omega_{k+3}|_w := \int_{\Omega_{k+2}\setminus\Omega_{k+3}} w$ and $\Omega_K \setminus \Omega_{K+1} = \Omega_K$, since Ω_{K+1} is empty.

Next, we introduce rectangles. For a fixed k such that $\Omega_{k+1} \neq \varnothing$ we choose points (x^k_j, y^k_j), $1 \leqslant j \leqslant N = N_k$, lying on the boundary $\partial\Omega_k$ in such a way to have

(x_j^k, y_{j-1}^k) belonging to $\partial\Omega_{k+1}$ for $2 \leqslant j \leqslant N$ and $\Omega_{k+1} \subset \bigcup_{j=1}^N S_j^k$, where S_j^k is a rectangle of the form $(x_j^k, \infty) \times (y_j^k, \infty)$. We also define rectangles $\widetilde{S}_j^k = (x_j^k, x_{j+1}^k) \times (y_j^k, y_{j-1}^k)$ for $1 \leqslant j \leqslant N$ and $R_j^k = (0, x_{j+1}^k) \times (0, y_j^k)$, $\widetilde{R}_j^k = (x_j^k, x_{j+1}^k) \times (y_{j+1}^k, y_j^k)$ and $T_j^k = (x_{j+1}^k, \infty) \times (y_j^k, \infty)$ for $1 \leqslant j \leqslant N-1$. Put $y_0^k = x_{N+1}^k = \infty$ (see Figure 1).

Now we choose the sets $E_j^k \subset T_j^k$ so that $E_j^k \cap E_i^k = \varnothing$ for $j \neq i$ and $\bigcup_j E_j^k = (\Omega_{k+2} \setminus \Omega_{k+3}) \cap \left(\bigcup_j T_j^k\right)$. Since $\Omega_{k+2} \setminus \Omega_{k+3} \subset \Omega_{k+1} \subset \left(\bigcup_j T_j^k\right) \cup \left(\bigcup_j \widetilde{S}_j^k\right)$, then

$$3^{-3q} \int_{\mathbb{R}_+^2} (I_2 f)^q w \leqslant \sum_{k,j} 3^{kq} \left|E_j^k\right|_w + \sum_{k,j} 3^{kq} \left|\widetilde{S}_j^k \cap (\Omega_{k+2} \setminus \Omega_{k+3})\right|_w =: I + II. \tag{5}$$

We are ready to state technical lemmas, which provide upper estimates on

$$\mathbb{V}_{(a,b)\times(c,d)}(\sigma, w) := \int_a^b \int_c^d w(x,y) \left[(I_2\sigma)(x,y)\right]^q dx\,dy$$

and

$$\mathbb{W}_{(a,b)\times(c,d)}(w, \sigma) := \int_a^b \int_c^d \sigma(x,y) \left[(I_2^* w)(x,y)\right]^{p'} dx\,dy$$

for some $0 \leqslant a < b \leqslant \infty$ and $0 \leqslant c < d \leqslant \infty$. We shall use notation $w_{\mathbb{V}} := w\chi_{(a,b)\times(c,d)}$.

Notice that Lemmas 1 and 2 have the same auxiliary meaning for obtaining our main result as Lemmas 1 and 2 had in [9], where the case $p < q$ was characterised.

LEMMA 1. *Let $0 \leqslant a < b < \infty$, $0 \leqslant c < d < \infty$ and $1 < q < p < \infty$. Suppose that the weight $\sigma := v^{1-p'}$ satisfies the condition (3) for almost all $(x,y) \in \mathbb{R}_+^2$. Then*

$$\mathbb{V}_{(a,b)\times(c,d)}(\sigma, w) \leqslant \beta(p,q,\gamma) \left[(I_2\sigma)(b,d)\right]^{\frac{q}{p}} B^q_{w\chi_{(a,b)\times(c,d)}}.$$

Proof. We apply a scheme of partitioning $(a,b)\times(c,d) \subset \mathbb{R}_+^2 = \bigcup_k \Omega_k$ of the E. Sawyer type for $f = \sigma$, where $\Omega_k := \left\{(x,y) \in \mathbb{R}_+^2 : (I_2\sigma)(x,y) > 3^k\right\}$.

According to that scheme, we define $R_j^k = (0, x_{j+1}^k) \times (0, y_j^k)$, $S_j^k = (x_j^k, \infty) \times (y_j^k, \infty)$, $\widetilde{S}_j^k = (x_j^k, x_{j+1}^k) \times (y_j^k, y_{j-1}^k)$ and observe that $|R_j^k|_\sigma = 3^{k+1}$. Then

$$\mathbb{V}_{(a,b)\times(c,d)}(\sigma, w) = \sum_k \int_{\Omega_{k+2}\setminus\Omega_{k+3}} (I_2\sigma)^q w_{\mathbb{V}} \leqslant 3^{2q} \sum_k 3^{(k+1)q} \left|\Omega_{k+2} \setminus \Omega_{k+3}\right|_{w_{\mathbb{V}}},$$

where for each k

$$\left|\Omega_{k+2} \setminus \Omega_{k+3}\right|_{w_{\mathbb{V}}} = \sum_{j=1}^{N_k} \left[\left|E_j^k\right|_{w_{\mathbb{V}}} + \left|\widetilde{S}_j^k \cap (\Omega_{k+2} \setminus \Omega_{k+3})\right|_{w_{\mathbb{V}}}\right].$$

More precisely, $E_j^k = T_j^k \cap (\Omega_{k+2} \setminus \Omega_{k+3}) := (x_{j+1}^k, \infty) \times (y_j^k, \infty) \cap (\Omega_{k+2} \setminus \Omega_{k+3})$. We put z_j^k as intersection of $y = y_j^k$ and $\partial\Omega_{k+3}$ (or $z_j^k = b$ if there is no intersection) and suppose that $E_j^k \subset (x_{j+1}^k, z_j^k) \times (y_j^k, y_{j-1}^k)$.

All the pairs (k,j), where $k \in \mathbb{Z}$ and $j = 1,\dots,N_k$, can be split into two groups. We say that $(k,j) \in \mathbb{I}$ if $\widetilde{S}_j^k \cap (\Omega_{k+2} \setminus \Omega_{k+3}) \neq \varnothing$. The rest (k,j) are assigned to the group $\mathbb{II}$. We have

$$\begin{aligned}\mathbb{V}_{(a,b)\times(c,d)}(\sigma,w) &\leqslant 3^{2q}\sum_{k,j} 3^{(k+1)q}\Big[\big|E_j^k\big|_{w_{\mathbb{V}}} + \big|\widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3})\big|_{w_{\mathbb{V}}}\Big] \\ &= 3^{3q}\sum_{(k,j)\in\mathbb{I}} 3^{kq}\Big[\big|E_j^k\big|_{w_{\mathbb{V}}} + \big|\widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3})\big|_{w_{\mathbb{V}}}\Big] \\ &\quad + \sum_{(k,j)\in\mathbb{II}} 3^{(k+3)q}\big|E_j^k\big|_{w_{\mathbb{V}}}. \end{aligned} \tag{6}$$

To estimate the sum over $(k,j) \in \mathbb{I}$ we denote $D_j^k := \widetilde{S}_j^k \setminus \Omega_{k+3}$ and observe that

$$\begin{aligned} 3^{k+2} &\leqslant \big(I_2(\chi_{D_j^k} f)\big)(x,y) + (I_2 f)(x_j^k, y_{j-1}^k) + (I_2 f)(x_{j+1}^k, y_j^k) - (I_2 f)(x_j^k, y_j^k) \\ &= \big(I_2(\chi_{D_j^k} f)\big)(x,y) + 2\cdot 3^{k+1} - 3^k \quad \text{if} \quad (x,y) \in \widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3}), \end{aligned} \tag{7}$$

from which it follows with $f = \sigma$:

$$\big(I_2(\chi_{D_j^k}\sigma)\big)(x,y) \geqslant 4\cdot 3^k \quad \text{for} \quad (x,y) \in \widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3}). \tag{8}$$

Further, on the strength of Hölder's inequality,

$$\begin{aligned} 4\cdot 3^k \big|\widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3})\big|_{w_{\mathbb{V}}} &\leqslant \int_{\widetilde{S}_j^k \cap (\Omega_{k+2}\setminus\Omega_{k+3})} \big(I_2(\chi_{D_j^k}\sigma)\big)(x,y) w_{\mathbb{V}}(x,y)\,dx\,dy \\ &\leqslant \int_{D_j^k} \Big(\int_{x_j^k}^x \int_{y_j^k}^y \sigma\Big) w_{\mathbb{V}}(x,y)\,dx\,dy \\ &= \int_{D_j^k} \sigma(s,t)\big(I_2^*(w_{\mathbb{V}}\chi_{D_j^k})\big)(s,t)\,ds\,dt \\ &\leqslant \Big(\int_{D_j^k}\sigma\Big)^{\frac{q}{p}} \Big(\int_{D_j^k} \sigma(s,t)\big[\big(I_2^*(w_{\mathbb{V}}\chi_{D_j^k})\big)(s,t)\big]^{\frac{r}{q}}\,ds\,dt\Big)^{\frac{q}{r}}. \end{aligned}$$

Besides, by integration by parts on $\widetilde{S}_j^k$,

$$\begin{aligned} & 3^{\frac{rk}{q'}} \int_{D_j^k} \sigma(s,t)\big[\big(I_2^*(w_{\mathbb{V}}\chi_{D_j^k})\big)(s,t)\big]^{\frac{r}{q}}\,ds\,dt \\ =& 3^{\frac{rk}{q'}} \int_{\widetilde{S}_j^k} \chi_{D_j^k}(s,t)\sigma(s,t)\big[\big(I_2^*(w_{\mathbb{V}}\chi_{D_j^k})\big)(s,t)\big]^{\frac{r}{q}}\,ds\,dt \\ =& \frac{r3^{\frac{rk}{q'}}}{q} \int_{\widetilde{S}_j^k} \Big(\int_{x_j^k}^s \sigma(x,t)\chi_{D_j^k}(x,t)\,dx\Big)\Big(\int_t^\infty w_{\mathbb{V}}\chi_{D_j^k}(s,y)\,dy\Big)\big[\big(I_2^*(w_{\mathbb{V}}\chi_{D_j^k})\big)(s,t)\big]^{\frac{r}{p}}\,ds\,dt \\ \leqslant& \frac{r3^{\frac{rk}{q'}}}{q} \int_{\widetilde{S}_j^k} \Big(\int_{x_j^k}^s \sigma(x,t)\chi_{D_j^k}(x,t)\,dx\Big)\Big(\int_t^\infty w_{\mathbb{V}}\chi_{D_j^k}(s,y)\,dy\Big)\big[(I_2^* w)(s,t)\big]^{\frac{r}{p}}\,ds\,dt \end{aligned}$$

$$
\begin{aligned}
&=\frac{r3^{\frac{rk}{q'}}}{q}\int_{\widetilde{S}_j^k}\Big(\int_t^\infty w_{\mathbb{V}}\chi_{D_j^k}(s,y)\,dy\Big)\big[(I_2^*w)(s,t)\big]^{\frac{r}{p}}\,ds\,d_t\Big(\int_{x_j^k}^s\int_{y_j^k}^t\sigma\chi_{D_j^k}\Big)\\
&=\frac{r3^{\frac{rk}{q'}}}{q}\int_{\widetilde{S}_j^k}\Big\{\frac{r}{p}\Big(\int_t^\infty w_{\mathbb{V}}\chi_{D_j^k}(s,y)\,dy\Big)\Big(\int_s^\infty w(x,t)\,dx\Big)\big[(I_2^*w)(s,t)\big]^{\frac{r}{p}-1}\\
&\quad+w_{\mathbb{V}}(s,t)\chi_{D_j^k}(s,t)\big[(I_2^*w)(s,t)\big]^{\frac{r}{p}}\Big\}\Big(\int_{y_j^k}^t\int_{x_j^k}^s\sigma\chi_{D_j^k}\Big)\,ds\,dt\\
&\leqslant\frac{r}{q}\int_{\widetilde{S}_j^k}\Big\{\frac{r}{p}\Big(\int_t^\infty w_{\mathbb{V}}\chi_{D_j^k}(s,y)\,dy\Big)\Big(\int_s^\infty w(x,t)\,dx\Big)\big[(I_2^*w)(s,t)\big]^{\frac{r}{p}-1}\\
&\quad+w_{\mathbb{V}}(s,t)\chi_{D_j^k}(s,t)\big[(I_2^*w)(s,t)\big]^{\frac{r}{p}}\Big\}\big[(I_2\sigma)(s,t)\big]^{\frac{r}{p'}}\,ds\,dt=B^r_{w_{\mathbb{V}}\chi_{D_j^k}}.
\end{aligned}
$$

Therefore, on the strength of the estimate $\sum_{k,j}\chi_{D_j^k}\leqslant\sum_k\chi_{\Omega_k\setminus\Omega_{k+3}}\leqslant 3$ and by Hölder's inequality with p/q and r/q,

$$
\begin{aligned}
\sum_{(k,j)\in\mathbb{I}}3^{kq}\big|\widetilde{S}_j^k\cap(\Omega_{k+2}\setminus\Omega_{k+3})\big|_{w_{\mathbb{V}}}&\leqslant\frac{1}{4}\sum_{k,j}\Big(\int_{D_j^k}\sigma\Big)^{\frac{q}{p}}B^q_{w_{\mathbb{V}}\chi_{D_j^k}}\\
&\leqslant\frac{3}{4}B^q_{w\chi_{(a,b)\times(c,d)}}\big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}.
\end{aligned}\tag{9}
$$

To evaluate $\sum_{(k,j)\in\mathbb{I}}3^{kq}\big|E_j^k\big|_{w_{\mathbb{V}}}$ we shall exploit (8) with some $(\bar{x}_j^k,\bar{y}_j^k)\in\partial\Omega_{k+2}$. Since $\widetilde{S}_j^k$ are disjoint for a fixed k, then

$$
\sum_{k,j}\chi_{\{(x_j^k,\bar{x}_j^k)\times(y_j^k,\bar{y}_j^k)\}}\leqslant\sum_k\chi_{\Omega_k\setminus\Omega_{k+2}}\leqslant 2.\tag{10}
$$

We also write

$$
\begin{aligned}
|E_j^k|_{w_{\mathbb{V}}}^{\frac{r}{q}}&=\Big(\int_{x_{j+1}^k}^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}\chi_{E_j^k}w_{\mathbb{V}}\Big)^{\frac{r}{q}}=\int_{x_{j+1}^k}^{z_j^k}d_x\Big(-\Big[\int_x^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}\chi_{E_j^k}w_{\mathbb{V}}\Big]^{\frac{r}{q}}\Big)\\
&=\frac{r}{q}\int_{x_{j+1}^k}^{z_j^k}\Big[\int_x^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}\chi_{E_j^k}w_{\mathbb{V}}\Big]^{\frac{r}{p}}\Big(\int_{y_j^k}^{y_{j-1}^k}\chi_{E_j^k}(x,t)w_{\mathbb{V}}(x,t)\,dt\Big)dx\\
&\leqslant\frac{r}{q}\int_{x_{j+1}^k}^{z_j^k}\big[(I_2^*w)(x,y_j^k)\big]^{\frac{r}{p}}\Big(\int_{y_j^k}^{y_{j-1}^k}\chi_{E_j^k}(x,t)w_{\mathbb{V}}(x,t)\,dt\Big)dx\\
&\leqslant\frac{r}{q}\int_{x_{j+1}^k}^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}d_y\Big(-\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}}\Big(\int_y^{y_{j-1}^k}\chi_{E_j^k}(x,t)w_{\mathbb{V}}(x,t)\,dt\Big)\Big)dx
\end{aligned}
$$

$$=\frac{r}{q}\int_{x_{j+1}^k}^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}\bigg\{\frac{r}{p}\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}-1}\Big(\int_x^\infty w(s,y)\,ds\Big)$$
$$\times\Big(\int_y^{y_{j-1}^k}\chi_{E_j^k}(x,t)w_{\mathbb{V}}(x,t)\,dt\Big)+\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}}\chi_{E_j^k}(x,y)w_{\mathbb{V}}(x,y)\bigg\}dx\,dy$$
$$=\frac{r^2}{pq}\int_{x_{j+1}^k}^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}\chi_{E_j^k}(x,t)w_{\mathbb{V}}(x,t)\bigg(\int_{y_j^k}^t\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}-1}\Big(\int_x^\infty w(s,y)\,ds\Big)\,dy\bigg)dt$$
$$+\frac{r}{q}\int_{x_{j+1}^k}^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}}\chi_{E_j^k}(x,y)w_{\mathbb{V}}(x,y)dx\,dy. \tag{11}$$

Therefore,

$$3^{\frac{r(k+1)}{p'}}|E_j^k|_{w_{\mathbb{V}}}^{\frac{r}{q}}\leqslant 3^{\frac{r(k+1)}{p'}}\frac{r^2}{pq}\int_{E_j^k}w_{\mathbb{V}}(x,t)\bigg(\int_{y_j^k}^t\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}-1}\Big(\int_x^\infty w(s,y)\,ds\Big)\,dy\bigg)dt$$
$$+3^{\frac{r(k+1)}{p'}}\frac{r}{q}\int_{E_j^k}\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}}w_{\mathbb{V}}(x,y)\,dx\,dy$$
$$\leqslant\frac{r^2}{pq}\int_{E_j^k}w_{\mathbb{V}}(x,t)\bigg(\int_{y_j^k}^t\big[(I_2\sigma)(x,y)\big]^{\frac{r}{p'}}\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}-1}\Big(\int_x^\infty w(s,y)\,ds\Big)\,dy\bigg)dt$$
$$+\frac{r}{q}\int_{E_j^k}\big[(I_2\sigma)(x,y)\big]^{\frac{r}{p'}}\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}}w_{\mathbb{V}}(x,y)\,dx\,dy\leqslant B^r_{w_{\mathbb{V}}\chi_{E_j^k}}. \tag{12}$$

Thus, on the strength of (10) and by Hölder's inequality,

$$\sum_{(k,j)\in\mathbb{I}}3^{(k+1)q}|E_j^k|_{w_{\mathbb{V}}}\leqslant\Big(\frac{3}{4}\Big)^{\frac{q}{p}}\sum_{k,j}B^q_{w_{\mathbb{V}}\chi_{E_j^k}}\bigg(\int_{x_j^k}^{\bar{x}_j^k}\int_{y_j^k}^{\bar{y}_j^k}\sigma\bigg)^{\frac{q}{p}}\leqslant\Big(\frac{3}{2}\Big)^{\frac{q}{p}}\big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}B^q_{w_{\mathbb{V}}}.$$

By combining this with (9), we obtain

$$3^{3q}\sum_{(k,j)\in\mathbb{I}}3^{kq}\Big[|E_j^k|_{w_{\mathbb{V}}}+\big|\widetilde{S}_j^k\cap(\Omega_{k+2}\setminus\Omega_{k+3})\big|_{w_{\mathbb{V}}}\Big]$$
$$\leqslant\frac{3^{3q+1}}{4}\big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}B^q_{w_{\mathbb{V}}}+\frac{3^{\frac{q}{p}+2q}}{2^{\frac{q}{p}}}\big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}B^q_{w_{\mathbb{V}}}. \tag{13}$$

Consider the sum over $(k,j)\in\mathbb{III}$ in (6). One can write for any $(\rho,\tau)\in\mathbb{R}^2_+$

$$\big[(I_2\sigma)(\rho,\tau)\big]^{-1}=\int_\rho^\infty\int_\tau^\infty\bigg\{2\big[(I_2\sigma)(x,y)\big]^{-3}\Big(\int_0^x\sigma(s,y)\,ds\Big)\Big(\int_0^y\sigma(x,t)\,dt\Big)$$
$$-\big[(I_2\sigma)(x,y)\big]^{-2}\sigma(x,y)\bigg\}dx\,dy=:\int_\rho^\infty\int_\tau^\infty\Phi(x,y)\,dx\,dy \tag{14}$$

provided $(I_2\sigma)(\infty,\tau)=(I_2\sigma)(\rho,\infty)=(I_2\sigma)(\infty,\infty)=\infty$, that can be achieved by letting

$$\sigma(s,t)=\sigma_n(s,t):=\sigma(s,t)\chi_{(0,n]^2}(s,t)+\chi_{\mathbb{R}^2_+\setminus(0,n]^2}(s,t),\qquad n>\max\{b,d\}.$$

It holds $3^k = (I_2\sigma)(x_j^k, y_j^k)$, therefore, $3^{-k} = \big[(I_2\sigma)(x_j^k, y_j^k)\big]^{-1}$. We have

$$(x_j^k,\infty)\times(y_j^k,\infty) = \widetilde{S}_j^k \cup \{(x_j^k,\infty)\times(y_{j-1}^k,\infty)\}\cup\{(x_{j+1}^k,\infty)\times(y_j^k,\infty)\} \setminus\{(x_{j+1}^k,\infty)\times(y_{j-1}^k,\infty)\}.$$

Notice that $3^{-(k+2)} < \int_{x_{j+1}^k}^\infty \int_{y_{j-1}^k}^\infty \Phi < 3^{-(k+1)}$ because $(k,j)\in\mathbb{III}$. Therefore, there exists $0<\varepsilon_j^k<1$ such that $\int_{x_{j+1}^k}^\infty \int_{y_{j-1}^k}^\infty \Phi = 3^{-(k+1+\varepsilon_j^k)}$.

We can write, by virtue of (14), that

$$\begin{aligned} 3^{-k} &= \int_{x_j^k}^\infty \int_{y_j^k}^\infty \Phi = \int_{\widetilde{S}_j^k}\Phi + \int_{x_j^k}^\infty \int_{y_{j-1}^k}^\infty \Phi + \int_{x_{j+1}^k}^\infty \int_{y_j^k}^\infty \Phi - \int_{x_{j+1}^k}^\infty \int_{y_{j-1}^k}^\infty \Phi \\ &= \int_{\widetilde{S}_j^k}\Phi + 2\cdot 3^{-(k+1)} - 3^{-(k+1+\varepsilon_j^k)}. \end{aligned}$$

Thus, for $(k,j)\in\mathbb{III}$

$$\int_{\widetilde{S}_j^k}\Phi = 3^{-k} - 2\cdot 3^{-(k+1)} + 3^{-(k+1+\varepsilon_j^k)} = 3^{-k}\big(1/3 + 3^{-(1+\varepsilon_j^k)}\big) > \frac{4}{9}\cdot 3^{-k}. \tag{15}$$

We have

$$\begin{aligned} &\int_{\widetilde{S}_j^k}\big[(I_2\sigma)(x,y)\big]^{-2}\sigma(x,y)\,dx\,dy \qquad (16)\\ &= \int_{\widetilde{S}_j^k}\bigg(\int_y^\infty d_t\Big(-\big[(I_2\sigma)(x,t)\big]^{-2}\Big)\bigg)\sigma(x,y)\,dx\,dy \\ &= 2\int_{\widetilde{S}_j^k}\bigg(\int_y^\infty \big[(I_2\sigma)(x,t)\big]^{-3}\Big(\int_0^x \sigma(s,t)\,ds\Big)\,dt\bigg)\sigma(x,y)\,dx\,dy \\ &= 2\int_{\widetilde{S}_j^k}\bigg(\Big[\int_y^{y_{j-1}^k} + \int_{y_{j-1}^k}^\infty\Big]\big[(I_2\sigma)(x,t)\big]^{-3}\Big(\int_0^x \sigma(s,t)\,ds\Big)\,dt\bigg)\sigma(x,y)\,dx\,dy \\ &= 2\int_{x_j^k}^{x_{j+1}^k}\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{-3}\Big(\int_0^x \sigma(s,t)\,ds\Big)\Big(\int_{y_j^k}^t \sigma(x,y),dy\Big)\,dx\,dt \\ &\quad + \int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{-2}\Big(\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\Big)\,dx. \qquad (17) \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\widetilde{S}_j^k}\Phi &= 2\int_{x_j^k}^{x_{j+1}^k}\Big(\int_0^{y_j^k}\sigma(x,y),dy\Big)\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{-3}\Big(\int_0^x \sigma(s,t)\,ds\Big)\,dx\,dt \\ &\quad - \int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{-2}\Big(\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\Big)\,dx. \end{aligned} \tag{18}$$

By combining (16)–(18), we obtain in view of (15),

$$\begin{aligned}
\frac{4}{9}\cdot 3^{\gamma(k+3)} &< 3^{(1+\gamma)k+3\gamma}\int_{\widetilde{S}_j^k}\Phi\\
&=2\cdot 3^{(1+\gamma)k+3\gamma}\int_{x_j^k}^{x_{j+1}^k}\Big(\int_0^{y_j^k}\sigma(x,y),dy\Big)\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{-3}\Big(\int_0^x\sigma(s,t)\,ds\Big)\,dx\,dt\\
&\quad -3^{(1+\gamma)k+3\gamma}\int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{-2}\Big(\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\Big)\,dx. \qquad (19)
\end{aligned}$$

It holds

$$\begin{aligned}
&2\cdot 3^{(1+\gamma)k+3\gamma}\int_{x_j^k}^{x_{j+1}^k}\Big(\int_0^{y_j^k}\sigma(x,y),dy\Big)\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{-3}\Big(\int_0^x\sigma(s,t)\,ds\Big)\,dx\,dt\\
&\leqslant 2\cdot 3^{3\gamma}\int_{x_j^k}^{x_{j+1}^k}\Big(\int_0^{y_j^k}\sigma(x,y),dy\Big)\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{\gamma-2}\Big(\int_0^x\sigma(s,t)\,ds\Big)\,dx\,dt
\end{aligned}$$

and

$$\begin{aligned}
&3^{(1+\gamma)k+3\gamma}\int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{-2}\Big(\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\Big)\,dx\\
&\geqslant 3^{\gamma-2}\int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{\gamma-1}\Big(\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\Big)\,dx.
\end{aligned}$$

Besides,

$$\begin{aligned}
&2\cdot 3^{3\gamma}\int_{x_j^k}^{x_{j+1}^k}\Big(\int_0^{y_j^k}\sigma(x,y)\,dy\Big)\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{\gamma-2}\Big(\int_0^x\sigma(s,t)\,ds\Big)\,dx\,dt\\
&=2\cdot 3^{3\gamma}\int_{\widetilde{S}_j^k}\big[(I_2\sigma)(x,t)\big]^{\gamma-2}\Big(\int_0^x\sigma(s,t)\,ds\Big)\Big(\int_0^t\sigma(x,y)\,dy\Big)\,dx\,dt\\
&\quad -2\cdot 3^{3\gamma}\int_{x_j^k}^{x_{j+1}^k}\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{\gamma-2}\Big(\int_0^x\sigma(s,t)\,ds\Big)\Big(\int_{y_j^k}^t\sigma(x,y)\,dy\Big)\,dx\,dt, \qquad (20)
\end{aligned}$$

where

$$\begin{aligned}
&\int_{x_j^k}^{x_{j+1}^k}\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{\gamma-2}\Big(\int_0^x\sigma(s,t)\,ds\Big)\Big(\int_{y_j^k}^t\sigma(x,y)\,dy\Big)\,dx\,dt\\
&=\int_{x_j^k}^{x_{j+1}^k}\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\int_y^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{\gamma-2}\Big(\int_0^x\sigma(s,t)\,ds\Big)\,dt\,dx\,dy\\
&=\frac{1}{1-\gamma}\int_{\widetilde{S}_j^k}\sigma(x,y)\big[(I_2\sigma)(x,y)\big]^{\gamma-1}\,dx\,dy\\
&\quad -\frac{1}{1-\gamma}\int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{\gamma-1}\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\,dx.
\end{aligned}$$

Therefore, we can continue (20) as follows:

$$
\begin{aligned}
&2\cdot 3^{3\gamma}\int_{x_j^k}^{x_{j+1}^k}\Big(\int_0^{y_j^k}\sigma(x,y)\,dy\Big)\int_{y_j^k}^{y_{j-1}^k}\big[(I_2\sigma)(x,t)\big]^{\gamma-2}\Big(\int_0^x\sigma(s,t)\,ds\Big)\,dx\,dt\\
&=2\cdot 3^{3\gamma}\int_{\widetilde{S}_j^k}\big[(I_2\sigma)(x,t)\big]^{\gamma-2}\Big(\int_0^x\sigma(s,t)\,ds\Big)\Big(\int_0^t\sigma(x,y)\,dy\Big)\,dx\,dt\\
&\quad-\frac{2\cdot 3^{3\gamma}}{1-\gamma}\int_{\widetilde{S}_j^k}\sigma(x,y)\big[(I_2\sigma)(x,y)\big]^{\gamma-1}\,dx\,dy\\
&\quad+\frac{2\cdot 3^{3\gamma}}{1-\gamma}\int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{\gamma-1}\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\,dx\\
&=\frac{2\cdot 3^{3\gamma}}{1-\gamma}\int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{\gamma-1}\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\,dx\\
&\quad-\frac{2\cdot 3^{3\gamma}}{\gamma(1-\gamma)}\int_{\widetilde{S}_j^k}d_xd_y\big[(I_2\sigma)(x,y)\big]^{\gamma}.
\end{aligned}
$$

From this and (19),

$$
\begin{aligned}
\frac{4}{9}\cdot 3^{\gamma(k+3)}<&\Big[\frac{2\cdot 3^{3\gamma}}{1-\gamma}-3^{\gamma-2}\Big]\int_{x_j^k}^{x_{j+1}^k}\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{\gamma-1}\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\,dx\\
&-\frac{2\cdot 3^{3\gamma}}{\gamma(1-\gamma)}\int_{\widetilde{S}_j^k}d_xd_y\big[(I_2\sigma)(x,y)\big]^{\gamma},
\end{aligned}
$$

where $\int_{\widetilde{S}_j^k}d_xd_y\big[(I_2\sigma)(x,y)\big]^{\gamma}\geqslant 0$ on the strength of (3). It also holds

$$
\begin{aligned}
\int_{x_j^k}^{x_{j+1}^k}&\big[(I_2\sigma)(x,y_{j-1}^k)\big]^{\gamma-1}\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\,dx\\
&\leqslant\int_{x_j^k}^{x_{j+1}^k}\Big(\int_{x_j^k}^{x}\int_{y_j^k}^{y_{j-1}^k}\sigma\Big)^{\gamma-1}\int_{y_j^k}^{y_{j-1}^k}\sigma(x,y)\,dy\,dx\\
&=\frac{1}{\gamma}\Big(\int_{x_j^k}^{x_{j+1}^k}\int_{y_j^k}^{y_{j-1}^k}\sigma\Big)^{\gamma}=\frac{1}{\gamma}|\widetilde{S}_j^k|_\sigma^\gamma.
\end{aligned}
$$

Therefore,

$$
3^{\gamma(k+3)}<\frac{9}{4\gamma}\Big[\frac{2\cdot 3^{3\gamma}}{1-\gamma}-3^{\gamma-2}\Big]|\widetilde{S}_j^k|_\sigma^\gamma. \tag{21}
$$

Now we can evaluate the sum over $(k,j)\in\mathbb{III}$ in (6). To this end we write, by making use of (11) and (12),

$$\sum_{(k,j)\in\mathbb{III}} 3^{(k+3)q}\big|E_j^k\big|_{w_{\mathbb{V}}} \leqslant \sum_{(k,j)\in\mathbb{III}} 3^{(k+3)q}$$
$$\times\bigg(\frac{r^2}{pq}\int_{x_{j+1}^k}^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}\chi_{E_j^k}(x,t)w_{\mathbb{V}}(x,t)\bigg(\int_{y_j^k}^{t}\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}-1}\Big(\int_x^\infty w(s,y)\,ds\Big)\,dy\bigg)dt$$
$$+\frac{r}{q}\int_{x_{j+1}^k}^{z_j^k}\int_{y_j^k}^{y_{j-1}^k}\big[(I_2^*w)(x,y)\big]^{\frac{r}{p}}\chi_{E_j^k}(x,y)w_{\mathbb{V}}(x,y)\,dx\,dy\bigg)^{\frac{q}{r}}$$
$$\leqslant 3^{\frac{2q}{p'}}\sum_{(k,j)\in\mathbb{III}} 3^{(k+3)\frac{q}{p}}B^q_{w_{\mathbb{V}}\chi_{E_j^k}}.$$

According to (21) and in view of $3^{(k+2)(q/p-\gamma)}\leqslant\big|\widetilde{S}_j^k\big|_\sigma^{q/p-\gamma}$, $(k,j)\in\mathbb{III}$,

$$3^{\frac{2q}{p'}}\sum_{(k,j)\in\mathbb{III}} 3^{\frac{q(k+3)}{p}}B^q_{w_{\mathbb{V}}\chi_{E_j^k}} = 3^{\frac{2q}{p'}}\sum_{(k,j)\in\mathbb{III}} 3^{(k+3)\gamma}3^{(k+3)(\frac{q}{p}-\gamma)}B^q_{w_{\mathbb{V}}\chi_{E_j^k}}$$
$$\leqslant 3^{\frac{2q}{p'}}\frac{9}{4\gamma}\Big[\frac{2\cdot 3^{3\gamma}}{1-\gamma}-3^{\gamma-2}\Big]\sum_{(k,j)\in\mathbb{III}} 3^{(k+3)(\frac{q}{p}-\gamma)}\big|\widetilde{S}_j^k\big|_\sigma^\gamma B^q_{w_{\mathbb{V}}\chi_{E_j^k}}$$
$$\leqslant 3^{\frac{2q}{p'}+\frac{q}{p}-\gamma}\frac{9}{4\gamma}\Big[\frac{2\cdot 3^{3\gamma}}{1-\gamma}-3^{\gamma-2}\Big]\sum_{(k,j)\in\mathbb{III}}\big|\widetilde{S}_j^k\big|_\sigma^{\frac{q}{p}}B^q_{w_{\mathbb{V}}\chi_{E_j^k}}.$$

Therefore, by Hölder's inequality with r/q and p/q,

$$\sum_{(k,j)\in\mathbb{III}} 3^{(k+3)q}\big|E_j^k\big|_{w_{\mathbb{V}}} \leqslant \frac{3^{2q-\frac{q}{p}+2-\gamma}}{4\gamma}\Big[\frac{2\cdot 3^{3\gamma}}{1-\gamma}-3^{\gamma-2}\Big]\big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}B^q_{w_{\mathbb{V}}}.$$

By combining this with (13) we approach the required estimate through the inequality

$$\mathbb{V}_{(a,b)\times(c,d)}(\sigma,w)\leqslant\frac{3^{3q+1}}{4}B^q_{w\chi_{(a,b)\times(c,d)}}\big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}$$
$$+\max\bigg\{\frac{3^{\frac{q}{p}+2q}}{2^{\frac{q}{p}}},\frac{3^{2q-\frac{q}{p}+2-\gamma}}{4\gamma}\Big[\frac{2\cdot 3^{3\gamma}}{1-\gamma}-3^{\gamma-2}\Big]\bigg\}$$
$$\times\big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}B^q_{w\chi_{(a,b)\times(c,d)}}.\quad\square$$

A similar to Lemma 1 statement holds with the (inner) integral of w.

LEMMA 2. *Let $0<a<b\leqslant\infty$, $0<c<d\leqslant\infty$ and $1<q<p<\infty$. Assume that the weight w satisfies the condition (4) a.e. on $\mathbb{R}^2_+$. Then*

$$\mathbb{W}_{(a,b)\times(c,d)}(w,\sigma)\leqslant\beta(q',p',\gamma^*)\big[(I_2^*w)(a,c)\big]^{\frac{p'}{q'}}\big(B^*_{\sigma\chi_{(a,b)\times(c,d)}}\big)^{p'}.$$

For proving some complementary results we shall use the following

LEMMA 3. *Let* $0 \leqslant a < b < \infty$, $0 \leqslant c < d < \infty$ *and* $1 < q < p < \infty$. *Then*

$$\mathbb{V}_{(a,b)\times(c,d)}(\sigma,w) \leqslant \big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}} \mathbb{B}^q_{w\chi_{(a,b)\times(c,d)}}.$$

Proof. We can write

$$\begin{aligned}
\mathbb{V}^{\frac{r}{q}}_{(a,b)\times(c,d)}(\sigma,w) &= \frac{r}{q}\int_c^d \left(\int_a^b\int_c^y (I_2\sigma)^q w\right)^{\frac{r}{p}} \left(\int_a^b \big[(I_2\sigma)(s,y)\big]^q w(s,y)\,ds\right)dy \\
&\leqslant \frac{r}{q}\int_c^d \Big[\Big(I_2\big[(I_2\sigma)^q w\big]\Big)(b,y)\Big]^{\frac{r}{p}} \left(\int_a^b \big[(I_2\sigma)(s,y)\big]^q w(s,y)\,ds\right)dy \\
&= \frac{r}{q}\int_c^d\int_a^b d_x\bigg\{\Big[\Big(I_2\big[(I_2\sigma)^q w\big]\Big)(x,y)\Big]^{\frac{r}{p}} \\
&\quad\times\left(\int_a^x \big[(I_2\sigma)(s,y)\big]^q w(s,y)\,ds\right)\bigg\}dy \\
&= \frac{r^2}{pq}\int_a^b\int_c^d \Big[\Big(I_2\big[(I_2\sigma)^q w\big]\Big)(x,y)\Big]^{\frac{r}{p}-1} \left(\int_a^x \big[(I_2\sigma)(s,y)\big]^q w(s,y)\,ds\right) \\
&\quad\times\left(\int_0^y \big[(I_2\sigma)(x,t)\big]^q w(x,t)\,dt\right)dx\,dy \\
&\quad+\frac{r}{q}\int_a^b\int_c^d \big[(I_2\sigma)(x,y)\big]^q w(x,y)\Big[\Big(I_2\big[(I_2\sigma)^q w\big]\Big)(x,y)\Big]^{\frac{r}{p}}dx\,dy \\
&= \frac{r^2}{pq}\int_a^b\int_c^d \big[(I_2\sigma)(s,y)\big]^q w(s,y)\left(\int_s^b \Big[\Big(I_2\big[(I_2\sigma)^q w\big]\Big)(x,y)\Big]^{\frac{r}{p}-1}\right. \\
&\quad\times\left.\left(\int_0^y \big[(I_2\sigma)(x,t)\big]^q w(x,t)\,dt\right)dx\right)dy \\
&\quad+\frac{r}{q}\int_a^b\int_c^d \big[(I_2\sigma)(x,y)\big]^q w(x,y)\Big[\Big(I_2\big[(I_2\sigma)^q w\big]\Big)(x,y)\Big]^{\frac{r}{p}}dx\,dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{V}_{(a,b)\times(c,d)}(\sigma,w)\big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}\big[(I_2\sigma)(b,d)\big]^{-\frac{q}{p}} \\
&\leqslant \big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}\Bigg[\frac{r^2}{pq}\int_a^b\int_c^d \big[(I_2\sigma)(s,y)\big]^q w(s,y) \\
&\quad\times\left(\int_s^b \big[(I_2\sigma)(x,y)\big]^{-\frac{r}{p}}\Big[\Big(I_2\big[(I_2\sigma)^q w\big]\Big)(x,y)\Big]^{\frac{r}{p}-1}\right. \\
&\quad\times\left.\left(\int_0^y \big[(I_2\sigma)(x,t)\big]^q w(x,t)\,dt\right)dx\right)dy \\
&\quad+\frac{r}{q}\int_a^b\int_c^d \big[(I_2\sigma)(x,y)\big]^{q-\frac{r}{p}} w(x,y)\Big[\Big(I_2\big[(I_2\sigma)^q w\big]\Big)(x,y)\Big]^{\frac{r}{p}}dx\,dy\Bigg]^{\frac{q}{r}} \\
&\leqslant \big[(I_2\sigma)(b,d)\big]^{\frac{q}{p}}\mathbb{B}^q_{w\chi_{(a,b)\times(c,d)}}. \qquad \square
\end{aligned}$$

Analogous to Lemma 3 statement holds for $\mathbb{W}_{(a,b)\times(c,d)}(w,\sigma)$.

LEMMA 4. *Let* $0<a<b\leqslant\infty$, $0<c<d\leqslant\infty$ *and* $1<q<p<\infty$. *Then*

$$\mathbb{W}_{(a,b)\times(c,d)}(w,\sigma)\leqslant\left[(I_2^*w)(a,c)\right]^{\frac{p'}{q'}}\left(\mathbb{B}^*_{\sigma\chi_{(a,b)\times(c,d)}}\right)^{p'}.$$

3. Main result

Let $\beta^*:=\beta(q',p',\gamma^*)$ and

$$\mathbf{C}_{t,s}:=3^{3q}\left[4\left(\frac{2}{3}\right)^q\max\left\{t,2q(p')^{q-1}\left(\frac{q}{r}\right)^{\frac{q}{r}}\right\}\left(\frac{2^{p-1}}{2^{p-1}-1}\right)^{\frac{q}{p}}+\frac{3}{4}s^{\frac{1}{p'}}\left(\frac{3^{q-1}}{3^{q-1}-1}\right)^{\frac{1}{q'}}\right].$$

The main result of the work is the following.

THEOREM 4. *Let* $1<q<p<\infty$. *Assume that weights* σ *and* w *satisfy a.e. on* $\mathbb{R}^2_+$ *the conditions* (3) *and* (4), *respectively. Then the inequalty* (1) *holds if and only if* $B<\infty$. *Moreover,*

$$2^{-\frac{1}{p'}}\left(\frac{q}{r}\right)^{\frac{1}{q}}\left(\frac{p'}{r}\right)^{\frac{1}{p'}}B\leqslant C\leqslant\mathbf{C}_{\beta,\beta^*}B. \tag{22}$$

Proof. (*Sufficiency*) Similarly to how it was done in E. Sawyer's paper [8] for the case $1<p\leqslant q<\infty$, we show that the conditions of the theorem are sufficient, limiting ourselves to proving the inequality (1) on the subclass M (see § 2). Then the inequality (1) for arbitrary $0\leqslant f\in L^p_v(\mathbb{R}^2_+)$ follows by the standard arguments.

Suppose $B<\infty$ and fix $f\in M$. By analogy with the proof of [8, Theorem 1A], we apply the E. Saywer scheme of partitioning of $\mathbb{R}^2_+$, where

$$\Omega_k:=\left\{(x,y)\in\mathbb{R}^2_+:(I_2f)(x,y)>3^k\right\}.$$

To estimate II in (5) we denote $D^k_j:=\widetilde{S}^k_j\setminus\Omega_{k+3}$ and, similarly to (7), we have

$$\left(I_2(\chi_{D^k_j}f)\right)(x,y)\geqslant 4\cdot 3^k\quad\text{if}\quad(x,y)\in\widetilde{S}^k_j\cap(\Omega_{k+2}\setminus\Omega_{k+3}).$$

Therefore, one can write, by applying Hölder's inequality,

$$\begin{aligned}4\left|\widetilde{S}^k_j\cap(\Omega_{k+2}\setminus\Omega_{k+3})\right|_w&\leqslant 3^{-k}\int_{\widetilde{S}^k_j\cap(\Omega_{k+2}\setminus\Omega_{k+3})}\left(I_2(\chi_{D^k_j}f)\right)(x,y)w(x,y)\,dx\,dy\\&\leqslant 3^{-k}\int_{D^k_j}\left(\int_{x^k_j}^x\int_{y^k_j}^y f\right)w(x,y)\,dx\,dy\\&=3^{-k}\int_{D^k_j}f(s,t)\left(I_2^*(w\chi_{D^k_j})\right)(s,t)\,ds\,dt\\&\leqslant 3^{-k}\left(\int_{D^k_j}f^pv\right)^{\frac{1}{p}}\left(\int_{D^k_j}\sigma(s,t)\left[(I_2^*w)(s,t)\right]^{p'}ds\,dt\right)^{\frac{1}{p'}}.\end{aligned}$$

By applying Lemma 2 to $(a,b)\times(c,d)=\widetilde{S}_j^k$ with $\sigma\chi_{D_j^k}$ instead of σ, we obtain that

$$\mathbb{W}_{\widetilde{S}_j^k}(w,\sigma\chi_{D_j^k})=\int_{D_j^k}\sigma(s,t)\big[(I_2^*w)(s,t)\big]^{p'}dsdt\leqslant\beta^*|S_j^k|_w^{\frac{p'}{q'}}(B^*_{\sigma\chi_{D_j^k}})^{p'}.$$

Then it follows from Hölder's inequality with q and q' that

$$4(\beta^*)^{-\frac{1}{p'}}\cdot II\leqslant\sum_{k,j}3^{k(q-1)}\Big(\int_{D_j^k}f^pv\Big)^{\frac{1}{p}}B^*_{\sigma\chi_{D_j^k}}|S_j^k|_w^{\frac{1}{q'}}$$
$$\leqslant\Big(\sum_{k,j}3^{kq}|S_j^k|_w\Big)^{\frac{1}{q'}}\Big[\sum_{k,j}\Big(\int_{D_j^k}f^pv\Big)^{\frac{q}{p}}(B^*_{\sigma\chi_{D_j^k}})^q\Big]^{\frac{1}{q}}.$$

On the strength of [8, (2.6)] we have $\sum_{j=1}^{N_k}\chi_{S_j^k}\leqslant 3^{-k}\chi_{\Omega_k}I_2f$ for all k. Then

$$\begin{aligned}\sum_{k,j}3^{kq}|S_j^k|_w&=\sum_k 3^{kq}\sum_{j=1}^{N_k}\int_{\mathbb{R}^2_+}\chi_{S_j^k}w\\&=\sum_k 3^{kq}\int_{\mathbb{R}^2_+}\Big(\sum_{j=1}^{N_k}\chi_{S_j^k}\Big)w\\&\leqslant\sum_k 3^{k(q-1)}\int_{\mathbb{R}^2_+}\chi_{\Omega_k}(I_2f)w=\sum_k 3^{k(q-1)}\sum_{m\geqslant k}\int_{\mathbb{R}^2_+}\chi_{\Omega_m\setminus\Omega_{m+1}}(I_2f)w\\&=\sum_m 3^{m(q-1)}\int_{\mathbb{R}^2_+}\chi_{\Omega_m\setminus\Omega_{m+1}}(I_2f)w\sum_{k\leqslant m}3^{(k-m)(q-1)}\\&\leqslant\frac{3^{q-1}}{3^{q-1}-1}\sum_m 3^{m(q-1)}\int_{\mathbb{R}^2_+}\chi_{\Omega_m\setminus\Omega_{m+1}}(I_2f)w\end{aligned}$$

and, therefore,

$$\sum_{k,j}3^{kq}|S_j^k|_w\leqslant\frac{3^{q-1}}{3^{q-1}-1}\sum_m\int_{\Omega_m\setminus\Omega_{m+1}}(I_2f)^qw=\frac{3^{q-1}}{3^{q-1}-1}\int_{\mathbb{R}^2_+}(I_2f)^qw.$$

Hölder's inequality with p/q, r/q and the estimate $\sum_{k,j}\chi_{D_j^k}\leqslant\sum_k\chi_{\Omega_k\setminus\Omega_{k+3}}\leqslant 3$ entail

$$\sum_{k,j}\Big(\int_{D_j^k}f^pv\Big)^{\frac{q}{p}}(B^*_{\sigma\chi_{D_j^k}})^q\leqslant\Big(\sum_{k,j}\int_{D_j^k}f^pv\Big)^{\frac{q}{p}}\Big(\sum_{k,j}(B^*_{\sigma\chi_{D_j^k}})^r\Big)^{\frac{q}{r}}\leqslant 3\Big(\int_{\mathbb{R}^2_+}f^pv\Big)^{\frac{q}{p}}(B^*_\sigma)^q.$$

Thus,

$$II\leqslant\frac{3}{4}(\beta^*)^{\frac{1}{p'}}\Big(\frac{3^{q-1}}{3^{q-1}-1}\Big)^{\frac{1}{q'}}B\Big(\int_{\mathbb{R}^2_+}f^pv\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{R}^2_+}(I_2f)^qw\Big)^{\frac{1}{q'}}.\qquad(23)$$

To estimate I in (5), similarly to the proof of [8, Theorem 1A, pp. 8–9], we put $g\sigma:=f$ and write:

$$3^qI=\sum_{k,j}3^{(k+1)q}|E_j^k|_w=\sum_{k,j}|E_j^k|_w\Big(\int_{R_j^k}f\Big)^q=\sum_{k,j}|E_j^k|_w|R_j^k|_\sigma^q\Big(\frac{1}{|R_j^k|_\sigma}\int_{R_j^k}g\sigma\Big)^q.\qquad(24)$$

For an integer l, by Γ_l we denote the set of pairs (k,j) such that $\left|E_j^k\right|_w > 0$ and

$$2^l < \frac{1}{\left|R_j^k\right|_\sigma} \int_{R_j^k} g\sigma \leqslant 2^{l+1}, \qquad (k,j) \in \Gamma_l$$

and observe that $\Gamma_{l'} \cap \Gamma_{l''} = \varnothing,\ l' \neq l''$. For fixed l the family $\{U_i^l\}_{i=1}^{i(l)}$ consists of maximal rectangles from the collection $\{R_j^k\}_{(k,j)\in\Gamma_l}$, that is, each R_j^k with $(k,j) \in \Gamma_l$ is contained in some U_i^l (or coincides with it). In [8, p. 8] it was shown that $\widetilde{U}_i^l$ are disjoint for fixed l, where we denote $\widetilde{U}_i^l = \widetilde{R}_i^l$ if $U_i^l = R_i^l$.

Let χ_i^l be the characteristic function of the union of the sets E_j^k over all $(k,j) \in \Gamma_l$ such that $R_j^k \subset U_i^l$. Further, following [8, (2.13)], we arrive to

$$\begin{aligned}\sum_{(k,j)\in\Gamma_l} \left|E_j^k\right|_w \left|R_j^k\right|_\sigma^q &= \sum_{i=1}^{i(l)} \sum_{(k,j):\, R_j^k \subset U_i^l} \int_{E_j^k} w\big[\big(I_2(\chi_{U_i^l}\sigma)\big)(x_{j+1}^k, y_j^k)\big]^q \\ &\leqslant \sum_{i=1}^{i(l)} \int_{\mathbb{R}_+^2} \chi_i^l w\big[I_2(\chi_{U_i^l}\sigma)\big]^q. \end{aligned} \tag{25}$$

Let $U_i^l = (0,a_i^l) \times (0,b_i^l)$. By analogy with [8, (2.8)], we need to confirm that

$$\int_{\mathbb{R}_+^2} \chi_i^l w\big[I_2(\chi_{U_i^l}\sigma)\big]^q \leqslant 4\max\Big\{\beta, 2q(p')^{q-1}\Big(\frac{q}{r}\Big)^{\frac{q}{r}}\Big\} B_{w\chi_i^l}^q \left|U_i^l\right|_\sigma^{\frac{q}{p}}. \tag{26}$$

We have $\mathbb{R}_+^2 = U_i^l \cup \big\{(a_i^l,\infty)\times(b_i^l,\infty)\big\} \cup \big\{(0,a_i^l]\times[b_i^l,\infty)\big\} \cup \big\{[a_i^l,\infty)\times(0,b_i^l]\big\}$. On U_i^l, on the strength of Lemma 1,

$$\int_{U_i^l} \chi_i^l w(I_2\sigma)^q = \mathbb{V}_{U_i^l}(\sigma, \chi_i^l w) \leqslant \beta \cdot B_{\chi_i^l w}^q \left|U_i^l\right|_\sigma^{\frac{q}{p}}.$$

On the rectangle $(a_i^l,\infty)\times(b_i^l,\infty)$

$$\int_{(a_i^l,\infty)\times(b_i^l,\infty)} \chi_i^l w \left|U_i^l\right|_\sigma^q = \left|U_i^l\right|_\sigma^q \int_{(a_i^l,\infty)\times(b_i^l,\infty)} \chi_i^l w,$$

and, since

$$\begin{aligned}&\Big(\int_{(a_i^l,\infty)\times(b_i^l,\infty)} \chi_i^l w\Big)^{\frac{r}{q}} \\ &= \int_{a_i^l}^\infty d_x\Big(-\big[\big(I_2^*(\chi_i^l w)\big)(x,b_i^l)\big]^{\frac{r}{q}}\Big) \\ &= \frac{r}{q}\int_{a_i^l}^\infty \big[\big(I_2^*(\chi_i^l w)\big)(x,b_i^l)\big]^{\frac{r}{p}} \Big(\int_{b_i^l}^\infty \chi_i^l(x,t)w(x,t)\,dt\Big)dx \\ &\leqslant \frac{r}{q}\int_{a_i^l}^\infty \big[(I_2^* w)(x,b_i^l)\big]^{\frac{r}{p}} \Big(\int_{b_i^l}^\infty \chi_i^l(x,t)w(x,t)\,dt\Big)dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{r}{q}\int_{a_i^l}^{\infty}\int_{b_i^l}^{\infty} d_y\left(-\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}\left(\int_y^{\infty}\chi_i^l(x,t)w(x,t)\,dt\right)\right)dx\\
&= \frac{r}{q}\int_{a_i^l}^{\infty}\int_{b_i^l}^{\infty}\left\{\frac{r}{p}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}-1}\left(\int_x^{\infty}w(s,y)\,ds\right)\left(\int_y^{\infty}\chi_i^l(x,t)w(x,t)\,dt\right)\right.\\
&\quad\left. + \left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}\chi_i^l(x,y)w(x,y)\right\}dx\,dy\\
&\quad = \frac{r^2}{pq}\int_{a_i^l}^{\infty}\int_{b_i^l}^{\infty}\chi_i^l(x,t)w(x,t)\left(\int_{b_i^l}^{t}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}-1}\left(\int_x^{\infty}w(s,y)\,ds\right)dy\right)dt\,dx\\
&\quad + \frac{r}{q}\int_{a_i^l}^{\infty}\int_{b_i^l}^{\infty}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}\chi_i^l(x,y)w(x,y)\,dx\,dy,
\end{aligned}
$$

then

$$
\begin{aligned}
&\int_{(a_i^l,\infty)\times(b_i^l,\infty)}\chi_i^l w|U_i^l|_\sigma^q\\
&\leqslant |U_i^l|_\sigma^q\left(\frac{r}{q}\int_{a_i^l}^{\infty}\int_{b_i^l}^{\infty}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}\chi_i^l(x,y)w(x,y)\,dx\,dy\right.\\
&\quad\left. + \frac{r^2}{pq}\int_{a_i^l}^{\infty}\int_{b_i^l}^{\infty}\chi_i^l(x,t)w(x,t)\left(\int_{b_i^l}^{t}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}-1}\left(\int_x^{\infty}w(s,y)\,ds\right)dy\right)dt\right)^{\frac{q}{r}}dx\\
&\leqslant |U_i^l|_\sigma^{\frac{q}{p}}\left(\frac{r}{q}\int_{a_i^l}^{\infty}\int_{b_i^l}^{\infty}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}\left[(I_2\sigma)(x,y)\right]^{\frac{r}{p'}}\chi_i^l(x,y)w(x,y)\,dx\,dy\right.\\
&\quad + \frac{r^2}{pq}\int_{a_i^l}^{\infty}\int_{b_i^l}^{\infty}\chi_i^l(x,t)w(x,t)\left(\int_{b_i^l}^{t}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}-1}\left[(I_2\sigma)(x,y)\right]^{\frac{r}{p'}}\right.\\
&\quad\left.\left.\times\left(\int_x^{\infty}w(s,y)\,ds\right)dy\right)dx\,dt\right)^{\frac{q}{r}} \leqslant |U_i^l|_\sigma^{\frac{q}{p}}B^q_{\chi_i^l w}.
\end{aligned}
$$

In the first of the two mixed cases — $(0,a_i^l]\times[b_i^l,\infty)$ and $[a_i^l,\infty)\times(0,b_i^l]$ — we obtain, by applying the criteria for the validity of the one–dimensional weighted Hardy inequality for $f^p(x)=\int_0^{b_i^l}\sigma(x,y)\,dy$ (see [5, § 1.3.2]):

$$
\begin{aligned}
&q^{-1}(p')^{1-q}\int_{(0,a_i^l)\times(b_i^l,\infty)}\chi_i^l w|U_i^l|_\sigma^q\\
&=q^{-1}(p')^{1-q}\int_{(0,a_i^l)\times(b_i^l,\infty)}\chi_i^l(x,y)w(x,y)\left[(I_2\sigma)(x,b_i^l)\right]^q\,dx\,dy\\
&=q^{-1}(p')^{1-q}\int_0^{a_i^l}\left(\int_{b_i^l}^{\infty}\chi_i^l(x,y)w(x,y)\,dy\right)\left[(I_2\sigma)(x,b_i^l)\right]^q\,dx\\
&\leqslant\left(\int_0^{a_i^l}\left[(I_2^*\chi_i^l w)(x,b_i^l)\right]^{\frac{r}{p}}\left[(I_2\sigma)(x,b_i^l)\right]^{\frac{r}{p'}}\left(\int_{b_i^l}^{\infty}\chi_i^l(x,t)w(s,t)\,dt\right)dx\right)^{\frac{q}{r}}|U_i^l|_\sigma^{\frac{q}{\sigma}}\\
&\leqslant\left(\int_0^{a_i^l}\left[(I_2^*w)(x,b_i^l)\right]^{\frac{r}{p}}\left[(I_2\sigma)(x,b_i^l)\right]^{\frac{r}{p'}}\left(\int_{b_i^l}^{\infty}\chi_i^l(x,t)w(x,t)\,dt\right)dx\right)^{\frac{q}{r}}|U_i^l|_\sigma^{\frac{q}{p}}
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\int_0^{a_i^l}\left[(I_2\sigma)(x,b_i^l)\right]^{\frac{r}{p'}}\int_{b_i^l}^{\infty}d_y\left[-\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}\left(\int_y^{\infty}\chi_i^l(x,t)w(x,t)\,dt\right)\right]dx\right)^{\frac{q}{r}}|U_i^l|_\sigma^{\frac{q}{p}}\\
&=\left(\int_0^{a_i^l}\left[(I_2\sigma)(x,b_i^l)\right]^{\frac{r}{p'}}\left[\int_{b_i^l}^{\infty}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}\chi_i^l(x,y)w(x,y)\,dy\right.\right.\\
&\quad\left.\left.+\frac{r}{p}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}-1}\left(\int_x^{\infty}w(s,y)\,ds\right)\left(\int_y^{\infty}\chi_i^l(x,t)w(x,t)\,dt\right)\right]dx\right)^{\frac{q}{r}}|U_i^l|_\sigma^{\frac{q}{p}}\\
&\leqslant|U_i^l|_\sigma^{\frac{q}{p}}\left(\int_0^{a_i^l}\int_{b_i^l}^{\infty}\left[\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}}\left[(I_2\sigma)(x,y)\right]^{\frac{r}{p'}}\chi_i^l(x,y)w(x,y)\,dy\right.\right.\\
&\quad+\frac{r}{p}\left[(I_2^*w)(x,y)\right]^{\frac{r}{p}-1}\\
&\quad\left.\left.\times\left[(I_2\sigma)(x,y)\right]^{\frac{r}{p'}}\left(\int_x^{\infty}w(s,y)\,ds\right)\left(\int_y^{\infty}\chi_i^l(x,t)w(x,t)\,dt\right)\right]dx\,dy\right)^{\frac{q}{r}}\leqslant|U_i^l|_\sigma^{\frac{q}{p}}B^q_{\chi_i^l w}.
\end{aligned}
$$

The second mixed case can be estimated in a similar way. So, (26) is proven. Continuing (25), we obtain, using [8, (2.11)] and Hölder's inequality with $r/q,\ p/q$:

$$
\begin{aligned}
&\sum_{(k,j)\in\Gamma_l}|E_j^k|_w|R_j^k|_\sigma^q\\
&\leqslant 4\max\left\{\beta,2q(p')^{q-1}\left(\frac{q}{r}\right)^{\frac{q}{r}}\right\}\sum_i B^q_{\chi_i^l w}|U_i^l|_\sigma^{\frac{q}{p}}\\
&\leqslant 4\max\left\{\beta,2q(p')^{q-1}\left(\frac{q}{r}\right)^{\frac{q}{r}}\right\}\sum_i B^q_{\chi_i^l w}\left(2^{-l}\int_{\widetilde{U}_i^l\cap\{g>2^{l-3}\}}g\sigma\right)^{\frac{q}{p}}\\
&\leqslant 4\max\left\{\beta,2q(p')^{q-1}\left(\frac{q}{r}\right)^{\frac{q}{r}}\right\}\left(\sum_i B^r_{\chi_i^l w}\right)^{\frac{q}{r}}\left(\sum_i 2^{-l}\int_{\widetilde{U}_i^l\cap\{g>2^{l-3}\}}g\sigma\right)^{\frac{q}{p}}\\
&\leqslant 4\max\left\{\beta,2q(p')^{q-1}\left(\frac{q}{r}\right)^{\frac{q}{r}}\right\}2^{-lq/p}B^q_{\chi_l w}\left(\int_{\{g>2^{l-3}\}}g\sigma\right)^{\frac{q}{p}},
\end{aligned}
$$

where $\chi_l=\sum_{\{k,j:\,\cup_{(k,j)\in\Gamma_l}\}}\chi_{E_j^k}$. The last estimate is valid due to the fact that for fixed l the rectangles $\widetilde{U}_i^l$ do not intersect (see [8, p. 8]). Combining it with (24), we obtain, taking into account the relation

$$
\sum_l 2^{l(p-1)}\chi_{\{g>2^{l-3}\}}\leqslant\frac{2^{p-1}}{2^{p-1}-1}g^{p-1}\qquad\text{for}\quad p>1,
$$

Hölder's inequality with r/q and p/q and the fact that all E_j^k are disjoint:

$$
\begin{aligned}
I&\leqslant\left(\frac{2}{3}\right)^q\sum_l 2^{lq}\sum_{(k,j)\in\Gamma_l}|E_j^k|_w|R_j^k|_\sigma^q\\
&\leqslant 4\left(\frac{2}{3}\right)^q\max\left\{\beta,2q(p')^{q-1}\left(\frac{q}{r}\right)^{\frac{q}{r}}\right\}\sum_l 2^{lq}B^q_{\chi_l w}\left(2^{-l}\int_{\{g>2^{l-3}\}}g\sigma\right)^{\frac{q}{p}}
\end{aligned}
$$

$$\leqslant 4\left(\frac{2}{3}\right)^{q}\max\left\{\beta, 2q(p')^{q-1}\left(\frac{q}{r}\right)^{\frac{q}{r}}\right\}\left(\sum_{l} B^{r}_{\chi_l w}\right)^{\frac{q}{r}}\left(\sum_{l} 2^{l(p-1)}\int_{\{g>2^{l-3}\}} g\sigma\right)^{\frac{q}{p}}$$

$$\leqslant 4\left(\frac{2}{3}\right)^{q}\max\left\{\beta, 2q(p')^{q-1}\left(\frac{q}{r}\right)^{\frac{q}{r}}\right\}\left(\frac{2^{p-1}}{2^{p-1}-1}\right)^{\frac{q}{p}} B^{q}_{w}\left(\int_{\mathbb{R}^2_+} f^p v\right)^{\frac{q}{p}}. \tag{27}$$

Combining (27) with (23) we arrive at the required upper bound. In detail, the estimate

$$\int_{\mathbb{R}^2_+}(I_2 f)^q w \leqslant \mathbf{C}\left(\int_{\mathbb{R}^2_+} f^p v\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^2_+}(I_2 f)^q w\right)^{\frac{1}{q'}} + \mathbf{C}^q\left(\int_{\mathbb{R}^2_+} f^p v\right)^{\frac{q}{p}}$$

follows from (5) combined with (23) and (27), where $\mathbf{C} := B_w \cdot \mathbf{C}_{\beta,\beta^*}$.

(*Necessity*) The lower bound for C was established in [9, Theorem 3]. □

REMARK 1. (i) The necessity part of Theorem 4 and, therefore, the lower bound for C, is valid for all weights independently of the conditions (3) and (4).

(ii) Similarly to the case $p<q$, the sufficient part of Theorem 4 and the right hand side of the inequality (22) has blow–up effect because

$$\lim_{q\to p-0}\beta(p,q,\gamma) = \lim_{q\to p-0}\beta^*(q',p',\gamma^*) = \infty.$$

Below we give examples of weight functions satisfying (3) and (4).

EXAMPLE 1. A function $\sigma(x,y) = (x+y)^\tau$ satisfies (3). One can take $\frac{1}{\tau+2} \leqslant \gamma < 1$. Indeed, since $\int_0^x (s+y)^\tau\,ds = \frac{(x+y)^{\tau+1}-y^{\tau+1}}{\tau+1}$, $\int_0^y (x+t)^\tau\,dt = \frac{(x+y)^{\tau+1}-x^{\tau+1}}{\tau+1}$ and $(I_2\sigma)(x,y) = \frac{1}{(\tau+1)(\tau+2)}\left[(x+y)^{\tau+2}-x^{\tau+2}-y^{\tau+2}\right]$ then, to satisfy (3), we must have

$$(I_2\sigma)(x,y)\sigma(x,y) \geqslant (1-\gamma)\left(\int_0^x \sigma(s,y)\,ds\right)\left(\int_0^y \sigma(x,t)\,dt\right)$$

that is

$$\frac{(x+y)^\tau\left[(x+y)^{\tau+2}-x^{\tau+2}-y^{\tau+2}\right]}{\tau+2} \geqslant \frac{1-\gamma}{\tau+1}\left[(x+y)^{\tau+1}-y^{\tau+1}\right]\left[(x+y)^{\tau+1}-x^{\tau+1}\right].$$

This is the same as

$$J := 1-\left(\frac{x}{x+y}\right)^{\tau+2}-\left(\frac{y}{x+y}\right)^{\tau+2} \geqslant \frac{(1-\gamma)(\tau+2)}{\tau+1}\left[1-\left(\frac{y}{x+y}\right)^{\tau+1}\right]\left[1-\left(\frac{x}{x+y}\right)^{\tau+1}\right]. \tag{28}$$

Suppose that $y \leqslant x$. Then

$$\begin{aligned} J &= 1 - \frac{x}{x+y}\left(\frac{x}{x+y}\right)^{\tau+1} - \frac{y}{x+y}\left(\frac{y}{x+y}\right)^{\tau+1} \\ &= 1 - \left(\frac{x}{x+y}\right)^{\tau+1} + \frac{y}{x+y}\left[\left(\frac{x}{x+y}\right)^{\tau+1} - \left(\frac{y}{x+y}\right)^{\tau+1}\right] \\ &\geqslant 1 - \left(\frac{x}{x+y}\right)^{\tau+1}, \end{aligned}$$

and (28) follows from $1 \geqslant \frac{(1-\gamma)(\tau+2)}{\tau+1}$. Thus, $\gamma \geqslant \frac{1}{\tau+2}$ with $\tau > -1$.

EXAMPLE 2. A weight $w(x,y) = (x+y)^{-\rho}$ satisfies (4) for $\rho > 2$, since $\int_x^\infty w(s,y)\, ds = \int_y^\infty w(x,t)\, dt = \frac{(x+y)^{1-\rho}}{\rho-1}$ and $(I_2^* w)(x,y) = \frac{(x+y)^{2-\rho}}{(\rho-1)(\rho-2)}$, that is, automatically,

$$(I_2^* w)(x,y) w(x,y) \geqslant (1-\gamma^*)\left(\int_x^\infty w(s,y)\, ds\right)\left(\int_y^\infty w(x,t)\, dt\right).$$

4. A sufficient boundedness condition and the compactness criterion

In this part we find an independent of Theorem 4 sufficient condition for the validity of (1), which consists of three functionals B_i, $i = 1,2,3$ tending to A_i as $q \uparrow p$.

THEOREM 5. *Let $1 < q < p < \infty$. The inequality (1) holds if $\sum_{i=1}^3 B_i(p,q) < \infty$, besides, $C \leqslant \mathbf{C}_{1,1} \sum_{i=1}^3 B_i(p,q)$.*

Proof. The proof is analogous to Theorem 4. The only difference is that instead of Lemmas 1 and 2 one should apply the estimates from Lemmas 3 and 4. □

Observe that the lower estimate on C found by E. Sawyer in Theorem 1 is valid for all $1 < p,q < \infty$. This fact entails in combination with Theorem 5 the inequality:

$$\frac{1}{3}\sum_{i=1}^{3} A_i(p,q) \leqslant C \leqslant \mathbf{C}_{1,1} \sum_{i=1}^{3} B_i(p,q). \tag{29}$$

Moreover,

$$\lim_{q \uparrow p} B_i(p,q) = A_i(p,p), \quad i = 1,2,3,$$

and, unlike the sufficient part of Theorem 4, there is no blow–up effect in the right hand side of the inequality (29) (see also [9, Remark 1]).

In conclusion, let us state the compactness criterion for $I_2\colon\ : L_v^p(\mathbb{R}_+^2) \to L_w^q(\mathbb{R}_+^2)$.

THEOREM 6. *Let $1 < q < p < \infty$. Assume that weights σ and w satisfy a.e. on $\mathbb{R}_+^2$ the conditions (3) and (4), respectively. Then the Hardy operator $I_2\colon L_v^p(\mathbb{R}_+^2) \to L_w^q(\mathbb{R}_+^2)$ is compact if and only if $B < \infty$.*

Details of the proof of Theorem 6 can be found in [10, Theorem 5]. Compactness criteria for $I_2\colon\ : L_v^p(\mathbb{R}_+^2) \to L_w^q(\mathbb{R}_+^2)$ in the case $1 < p \leqslant q$ were found in [10].

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